KYUNGPOOK Math. J. 50(2010), 1-5

The Factor Domains that Result from Uppers to Prime Ideals in Polynomial Rings

David Earl Dobbs
Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37996-0612
e-mail : dobbs@math.utk.edu

Abstract. Let P be a prime ideal of a commutative unital ring R; X an indeterminate; $D := R/P$; L the quotient field of D; F an algebraic closure of L; $\alpha \in L[X]$ a monic irreducible polynomial; ξ any root of α in F; and $Q = \langle P, \alpha \rangle$, the upper to P with respect to α. Then $R[X]/Q$ is R-algebra isomorphic to $D[\xi]$; and is R-isomorphic to an overring of D if and only if $\deg(\alpha) = 1$.

1. Introduction

All rings and algebras considered in this note are commutative with $1 \neq 0$; all subrings/subalgebras and algebra homomorphisms are unital; and X denotes an indeterminate over the ambient coefficient ring(s). Our main concern here is the notion of an *upper*, which was implicit in a brief passage [5, page 25] introducing the basic facts about the Krull dimension of a polynomial ring $R[X]$; made explicit, with suggestive and helpful notation, in case R is a (commutative integral) domain, in [6, pages 706–708]; and generalized to the case of an arbitrary coefficient ring R in [3, pages 291–292]. The definition of an "upper to P" depends on the following data. Let P be a prime ideal of a commutative unital ring R; X an indeterminate; $D := R/P$; L the quotient field of D; and $\alpha \in L[X]$ a monic irreducible polynomial. Then the *upper to P with respect to α* is defined to be $\langle P, \alpha \rangle := \{h \in R[X] \mid$ the canonical image of h in $D[X]$ is divisible by α in $L[X]\}$. As the passages cited above show, the "upper" concept is important because, if P is a prime ideal of a ring R, the prime ideals Q of the polynomial ring $R[X]$ such that $Q \cap R = P$ are of two kinds: either $Q = P^* := PR[X]$ or $Q = \langle P, \alpha \rangle$ for some monic irreducible polynomial $\alpha \in L[X]$. Since it is easy to see that $R[X]/P^*$ is R-algebra isomorphic to $(R/P)[X]$, the question arises as to the nature of the factor domains of the form $R[X]/\langle P, \alpha \rangle$. We answer this question in Theorem 2.2 (a) below. The answer is elegant and its proof is elementary. Using the above notation, we show in Theorem 2.2 that $R[X]/\langle P, \alpha \rangle$ is R-algebra isomorphic to $D[\xi]$, where ξ denotes any given

Received September 12, 2009; accepted January 27, 2010.
2000 Mathematics Subject Classification:Primary 13A15; Secondary 13G05, 13B25, 13B30.
Key words and phrases: Commutative ring, prime ideal, polynomial ring, upper, integral domain, factor ring, degree.

root of α in an algebraic closure of L. In this way, we see that factor domains with respect to uppers give another way of describing a class of domains that has been the subject of considerable attention in a number of classical contexts (cf. [7, Theorem], [8, Proposition 3.11], [1, Theorem]).

2. Results

Our first result explains how our basic question reduces to working with coefficient rings that are domains and uppers to 0.

Lemma 2.1. *Let P be a prime ideal of a ring R; $D := R/P$; L the quotient field of D; and $\alpha \in L[X]$ a monic irreducible polynomial. Then $R[X]/\langle P, \alpha\rangle$ is R-algebra isomorphic to $D[X]/\langle 0, \alpha\rangle$.*

Proof. The canonical projection $R \to R/P$ extends to a surjective R-algebra homomorphism $h : R[X] \to D[X]$ that send X to X; thus, $h(\sum_{i=0}^{n} r_i X^i) = \sum_{i=0}^{n} (r_i + P)X^i$ for any polynomial $\sum_{i=0}^{n} r_i X^i \in R[X]$. Composing h with a canonical projection, we obtain a surjective R-algebra homomorphism $g : R[X] \to D[X]/\langle 0, \alpha\rangle$, satisfying $g(\sum_{i=0}^{n} r_i X^i) = \sum_{i=0}^{n} (r_i + P)X^i + \langle 0, \alpha\rangle$. Clearly, $\ker(g) = \langle P, \alpha\rangle$, and so the assertion follows from the First Isomorphism Theorem for R-algebras. □

We next present our main result. Recall that if D is a domain with quotient field L, then an *overring* of D is any D-subalgebra of L (that is, any subring of L that contains D).

Theorem 2.2. *Let P be a prime ideal of a ring R; $D := R/P$; L the quotient field of D; F an algebraic closure of L; and $\alpha \in L[X]$ a monic irreducible polynomial. Then:*

(a) *$R[X]/\langle P, \alpha\rangle$ is R-algebra isomorphic to $D[\xi]$ for each root ξ of α in F.*

(b) *$R[X]/\langle P, \alpha\rangle$ is R-algebra isomorphic to an overring of D if and only if $\deg(\alpha) = 1$.*

Proof. By Lemma 2.1, we can replace R with D and also replace P with 0. In other words, we can assume, without loss of generality, that R is a domain and $P = 0$.

(a) As in the proof of Lemma 2.1, we obtain an explicit surjective R-algebra homomorphism $h : R[X] \to D[X]$, $\sum_{i=0}^{n} r_i X^i \mapsto \sum_{i=0}^{n} (r_i + P)X^i$. Note that $\ker(h) = PR[X] =: P^* \subseteq \langle P, \alpha\rangle$. Moreover, h carries the set $\langle P, \alpha\rangle$ onto the set $S := \{g \in D[X] \mid \alpha | g \text{ in } L[X]\}$. (In fact, S is a prime ideal of $D[X]$.) It then follows from a standard homomorphism theorem that $R[X]/\langle P, \alpha\rangle$ and $D[X]/S$ are isomorphic as R-algebras. Therefore, it suffices to prove that $D[X]/S \cong D[\xi]$ as R-algebras (where ξ denotes any given root of α in F).

To simplify matters, let us use the above reduction, so that R is a domain and P is the prime ideal 0 of R. Our task is to show that $R[X]/S \cong R[\xi]$ as R-algebras. But since α is the minimum polynomial of ξ over L, it follows that $S = R[X] \cap \alpha L[X]$ is the kernel of the surjective R-algebra (evaluation) homomorphism $e : R[X] \to R[\xi]$ that sends X to ξ. Hence, the required isomorphism follows by applying the First Isomorphism Theorem for R-algebras to e.

(b) Choose a root ξ of α in F. Recall that we have reduced to the case $R = D$ and $P = 0$. Thus, $R[X]/\langle P, \alpha\rangle$ is R-algebra isomorphic to an overring of D if and only if there is an injective R-algebra homomorphism $R[X]/\langle 0, \alpha\rangle \to L$; that is, by (a), if and only if there is an injective R-algebra homomorphism $g : R[\xi] \to L$.

Assume first that such g exists. We will show that $\deg(\alpha) = 1$. Taking a common denominator for the coefficients of α, we can write $\alpha = \beta/r$ for some $\beta \in R[X]$ and some nonzero element $r \in R$. Consequently, $\beta(\xi) = r\alpha(\xi) = r \cdot 0 = 0$. Also, since g is an R-algebra homomorphism, we see that $\beta(g(\xi)) = g(\beta(\xi))$. Thus, $\eta := g(\xi) \in L$ satisfies

$$\alpha(\eta) = \alpha(g(\xi)) = \frac{1}{r}\beta(g(\xi)) = \frac{1}{r}g(\beta(\xi)) = \frac{1}{r}g(0) = \frac{1}{r} \cdot 0 = 0;$$

that is, η is a root of α in L. Since α is irreducible in $L[X]$, it follows that $\deg(\alpha) = 1$.

Conversely, suppose that $\deg(\alpha) = 1$. Then $\alpha = X - \delta$ for some $\delta \in L$. Note that δ is a root of α. Therefore, by (a), $R[X]/\langle P, \alpha\rangle$ is R-algebra isomorphic to $R[\delta]$, which is an overring of R (that is, of D). □

The following is a useful restatement of Theorem 2.2 (a).

Corollary 2.3. *Let P be a prime ideal of a ring R; $D := R/P$; L the quotient field of D; and F an algebraic closure of L. Then, up to R-algebra isomorphism, the rings of the form $R[X]/Q$, where Q ranges over the set of uppers to P, are the same as the rings of the form $D[\xi]$, where ξ ranges over* (*the set of elements of*) *F.*

Proof. In view of Theorem 2.2 (a), it remains only to show that if $\xi \in F$, then there exists some α, a monic irreducible polynomial in $L[X]$, such that $D[\xi]$ is R-algebra isomorphic to $R[X]/\langle P, \alpha\rangle$. Choose $\alpha \in L[X]$ to be the minimum polynomial of ξ over L. Then an application of Theorem 2.2 (a) completes the proof. □

It is known that if P is a prime ideal of a ring R and α, β are distinct monic polynomials that are each irreducible over the quotient field of R/P, then $\langle P, \alpha\rangle$ and $\langle P, \beta\rangle$ are unequal and, in fact, incomparable under inclusion (cf. [3, Lemma 2.1 (a)]). In view of Corollary 2.3, this raises the following question. If (using the above notation) ξ and η are elements of an algebraic closure of the quotient field of R/P such that $D[\xi]$ and $D[\eta]$ are isomorphic as R-algebras, must it be the case that $D[\xi] = D[\eta]$? We will answer this question in Remark 2.4 (a) and a related question in Remark 2.4 (b).

Remark 2.4. (a) We proceed to answer the above question, assuming for simplicity that R is a domain and $P = 0$. Let R be a domain with quotient field L, let F be an algebraic closure of L, and let ξ and η be elements of F such that $R[\xi]$ and $R[\eta]$ are isomorphic as R-algebras. Then, if ξ and η are each elements of L, then $R[\xi] = R[\eta]$. However, if at least one of ξ, η does not belong to F, then it need not be the case that $R[\xi] = R[\eta]$.

To prove the first assertion, assume that ξ and η are each elements of L. Then $R[\xi]$ and $R[\eta]$ are R-algebra isomorphic overrings of R (that are inside the same quotient field of R). Under these conditions, it is known (see the first paragraph of [4, Remark 2.8 (a)]) that these overrings must coincide.

Finally, we will give an example where $R[\xi]$ and $R[\eta]$ are distinct but R-algebra isomorphic R-subalgebras of F. Take $R := \mathbb{Z}$, $P := 0$, and view $F \subseteq \mathbb{C}$. Choose $a, b \in \mathbb{Q}$ $(=L)$ with $2a \notin \mathbb{Z}$ and $b \neq 0$. Let $\xi := a + bi$ and $\eta := a - bi$ (where, as usual, $i := \sqrt{-1} \in \mathbb{C}$). It is easy to check that ξ and η have the same minimum polynomial over L, namely, $\alpha := X^2 - 2aX + a^2 + b^2$. Therefore, by Theorem 2.2 (a), $R[\xi]$ and $R[\eta]$ are each R-algebra isomorphic to $R[X]/\langle P, \alpha\rangle$ (and, hence, R-algebra isomorphic to each other). However, $R[\xi] \neq R[\eta]$, since the condition $2a \notin \mathbb{Z}$ ensures that

$$\eta = a - bi = 2a + (-1)(a + bi) = 2a + (-1)\xi \in (\mathbb{Q} + \mathbb{Q}\xi) \setminus (\mathbb{Z} + \mathbb{Z}\xi) = L[\xi] \setminus R[\xi].$$

(b) We next answer another question that is raised by Corollary 2.3 (and also motivated by (a)). With P a prime ideal of a ring R and α, β monic irreducible polynomials over the quotient field of R/P such that $R[X]/\langle P, \alpha\rangle \cong R[X]/\langle P, \beta\rangle$ as R-algebras, can it be the case that $\alpha \neq \beta$? In view of the above results, we can rephrase this question as follows. If R is a domain with quotient field L and α, β are distinct monic irreducible polynomials over L, can it be the case that $R[\xi]$ and $R[\eta]$ are isomorphic R-algebras, where ξ and η are roots of α and β, respectively, in a given algebraic closure of L?

The answer is in the affirmative. We illustrate this next in the most trivial case possible, namely, where R is an algebraically closed field (so that, using the earlier notation, $R = D = L = F$ and, of course, $P = 0$). Under this assumption, choose distinct elements $\lambda, \mu \in R$; and put $\alpha := X - \lambda$, $\beta := X - \mu \in R[X]$. Take ξ and η to be roots of α and β, respectively, in R; that is, $\xi = \lambda$ and $\eta = \mu$. Of course, $R[\xi] = R$ is R-algebra isomorphic to $R = R[\eta]$, although $\alpha \neq \beta$.

(c) We next pursue the idea from (b) of considering the degenerate case where the coefficient ring R is a field. This brings to mind a classic result often referred to as the theorem of Cauchy, Kronecker and Steinitz. This result gives the usual construction of a field extension K of a given field k such that K contains a root of a given irreducible polynomial $f \in k[X]$. Related analysis shows that if ξ, η are each roots of f in K, then the fields $k[\xi]$ and $k[\eta]$ are isomorphic k-algebras (since each is k-algebra isomorphic to $k[X]/fk[X]$). Note that Theorem 2.2 (a) implies a generalization of this classical fact, in that the coefficient ring can be an arbitrary domain. Indeed, Theorem 2.2 (a) shows that if D is a domain with quotient field L and $f \in L[X]$ is a monic irreducible polynomial with roots $\xi, \eta, \ldots$ in some algebraic closure of L, then $D[\xi] \cong D[\eta]$ as D-algebras (since each is D-algebra isomorphic to $D[X]/\langle 0, f\rangle$).

(d) Some special cases (or variants thereof) of Theorem 2.2 (a) have appeared in the literature. For instance, [1, Lemma 1] states (when paraphrased using the above notation) that if R is an integrally closed domain with quotient field L and the element ξ is integral over R, then $R[X]/\alpha R[X] \cong R[\xi]$ as R-algebras, where $\alpha \in R[X]$ denotes the minimum polynomial of ξ over L. This is a special case of

Theorem 2.2 (a) since the extra assumptions on R and ξ in [1, Lemma 1] ensure that $\alpha R[X] = \alpha L[X] \cap R[X] = \langle 0, \alpha \rangle$. Of course, the domains R that we considered above need not be integrally closed; the algebraic elements ξ that we considered above need not be integral; and so the monic irreducible polynomials $\alpha \in L[X]$ that we considered need not have all their coefficients in R. One should note that [1, Lemma 1] is slightly more general that suggested above, in that it permits the integral element ξ to be taken from an arbitrary L-algebra (which is not necessarily contained in an algebraic closure of L).

Finally, we note that [2, Lemma 4.4] can be viewed as a weak version of Theorem 2.2 (a). Indeed, using the notation of Theorem 2.2 (a), we can paraphrase [2, Lemma 4.4] to say that if R is a domain and $\langle P, \alpha \rangle$ is an upper to some prime ideal P of R, then the domain $R[X]/\langle P, \alpha \rangle$ is algebraic over $D := R/P$. However, we believe that the stronger conclusions in Theorem 2.2 (a) and Corollary 2.3 have not appeared earlier in the literature.

References

[1] T. Albu, *On a paper of Uchida concerning simple finite extensions of Dedekind domains*, Osaka J. Math., **16**(1979), 65–69.

[2] A. Bouvier, D. E. Dobbs and M. Fontana, *Universally catenarian integral domains*, Adv. Math., **72**(1988), 211–238.

[3] D. E. Dobbs and M. Fontana, *Universally incomparable ring-homomorphisms*, Bull. Austral. Math. Soc., **29**(1984), 289–302.

[4] D. E. Dobbs and J. Shapiro, *A classification of the minimal ring extensions of an integral domain*, J. Algebra, **305**(2006), 185–193.

[5] I. Kaplansky, Commutative Rings, rev. ed., Univ. Chicago Press, Chicago, 1974.

[6] S. McAdam, *Going down in polynomial rings*, Can. J. Math., **23**(1971), 704–711.

[7] K. Uchida, *When is* $\mathbb{Z}[\alpha]$ *the ring of integers?*, Osaka J. Math., **14**(1977), 155–157.

[8] H. Uda, *Incomparability in ring extensions*, Hiroshima Math. J., **9**(1979), 451–463.