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New Sixth-Order Improvements of the Jarratt Method

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Abstract. In this paper, we construct some improvements of the Jarratt method for solving non-linear equations. A new sixth-order method are developed and numerical examples are given to support that the method obtained can compete with other sixth-order iterative methods.

1. Introduction

A large number of problems in engineering, applied mathematics, economics and also in the physical sciences are solved by finding the solution of nonlinear equation $f(x) = 0$. We consider iterative methods to find a simple root x^*, i.e., $f(x^*) = 0$ and $f'(x^*) \neq 0$, of a nonlinear equation $f(x) = 0$ that uses f and f' but not the higher derivatives of f.

The best known iterative method for the calculation of x^* is Newton's method defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

where x_0 is an initial approximation sufficiently close to x^*. This method is quadratically convergent [7].

To improve the local order of convergence, many modified methods have been proposed. The Jarratt method [2] is given by

$$x_{n+1} = x_n - J_f(x_n)\frac{f(x_n)}{f'(x_n)},$$

where $J_f(x_n) = \frac{3f'(y_n)+f'(x_n)}{6f'(y_n)-2f'(x_n)}$ and $y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}$.

For the sequence $\{x_n\}_0^\infty$ generated by an iterative method, if there exist positive constants λ and p such that

$$\lim_{n\to\infty}\frac{|x_{n+1}-x^*|}{|x_n - x^*|^p} = \lambda$$

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then the method is said to converge to x^* with the local order of convergence p or we say that the method has the local order of convergence p [4]. When considering a practical utility of any method, the study of its efficiency is needed. The efficiency of a method may be measured by the efficiency index introduced by Ostrowski [7], which is defined by

$$I = p^{\frac{1}{d}},$$

where p is the order of the method and d is the number of the function-evaluations per step. The efficiency index of Newton's method is 1.414.

A systematic treatment of iterative methods, both old and new, are provided in [7, 8]. Many researchers developed modifications of Newton's method or Newton-like methods in a number of ways to improve the order of convergence of Newton's method at the expense of additional evaluations of functions and/or derivatives mostly at the point iterated by the method. All these modifications are targeted at increasing the local order of convergence with a view of increasing their efficiency index.

Recently, some variants of Jarratt method with sixth-order convergence have been developed in [3], [6] and [9], which improve the local order of convergence of Jarratt method by an additional evaluation of the function. And we get some variants of Jarratt method with twelfth-order convergence in [5]. From a practical point of view, it is interesting to improve the order of convergence of the known methods. Motivated and inspired by the ongoing research with the iterative methods, in this paper we are concerned with the iterative methods improving the Jarratt method, and present a new interesting family of methods. By analysis of convergence we prove that the local order of convergence of the proposed method is six, and by illustration we demonstrate their performance in comparison with other methods of the same order.

2. Iterative methods and convergence analysis

The Jarratt method which has fourth-order convergence, is given by

$$x_{n+1} = x_n - J_f(x_n)\frac{f(x_n)}{f'(x_n)}, \tag{1}$$

where $J_f(x_n) = \frac{3f'(y_n)+f'(x_n)}{6f'(y_n)-2f'(x_n)}$ and $y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}$. Wang et al. [9] improved the Jarratt method as follows:

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)},$$

where

$$z_n = x_n - J_f(x_n)\frac{f(x_n)}{f'(x_n)}. \tag{2}$$

Using approximation of $f'(z_n)$ they obtained the new method

$$x_{n+1} = z_n - \frac{\gamma f'(x_n) + (\alpha+\beta-\gamma)f'(y_n)}{\alpha f'(x_n) + \beta f'(y_n)} \frac{f(z_n)}{f'(x_n)}. \tag{3}$$

Now we improve the above method (3) by using the method of undetermined coefficients. To get some approximation of $f'(z_n)$ we set

$$f'(z_n) \simeq f'(x_n) \frac{\alpha f'(x_n) + \beta f'(y_n) + Af(x_n) + Bf(z_n)}{\gamma f'(x_n) + (\alpha+\beta-\gamma)f'(y_n) + Cf(x_n) + Df(z_n)}. \tag{4}$$

Expand the terms $f'(z_n)$, $f'(y_n)$ and $f(y_n)$ about the point x_n up to second derivatives and collect terms. Then we get the system of equations for the unknowns $A, \cdots, D$ by comparing the coefficients of the derivatives of f at x_n.

$$\begin{cases} \delta D = \delta B \\ \gamma\delta + (\alpha+\beta-\gamma)\epsilon + (\alpha+\beta-\gamma)\delta + \delta^2 D = \beta\epsilon \\ A + B = C + D \\ \delta C + \delta D = 0 \\ (\alpha+\beta-\gamma)\delta\epsilon = 0, \end{cases}$$

where $\delta = z_n - x_n$ and $\epsilon = y_n - x_n$. This system has the solution

$$A = C = \frac{\gamma\delta - \beta\epsilon}{\delta^2}, B = D = -\frac{\gamma\delta - \beta\epsilon}{\delta^2}.$$

So we have the following iterative fomula

$$x_{n+1} = z_n - \frac{\gamma\delta^2 f'(x_n) + (\gamma\delta - \beta\epsilon)[f(x_n) - f(z_n)]}{\alpha\delta^2 f'(x_n) + \beta\delta^2 f'(y_n) + (\gamma\delta - \beta\epsilon)[f(x_n) - f(z_n)]} \frac{f(z_n)}{f'(x_n)}, \tag{5}$$

where $\gamma = \alpha + \beta$, and z_n is defined by (2).

Theorem 2.1. *Assume that the function $f : D \subset R \to R$ for an open interval D has a simple root $x^* \in D$. If $f(x)$ is sufficiently smooth in the neighborhood of the root x^*, then the family of method given by (5), for $\gamma = \alpha + \beta$, is of order six.*

Proof. Using Taylor expansion and taking into account $f(x^*) = 0$, we have

$$f(x_n) = f'(x^*)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + O(e_n^5)], \tag{6}$$

where $e_n = x_n - x^*$ and $c_k = \frac{1}{k!}\frac{f^{(k)}(x^*)}{f'(x^*)}, k = 2, 3, \ldots$. And so we get

$$f'(x_n) = f'(x^*)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + O(e_n^4)]. \tag{7}$$

Dividing (6) by (7) gives us

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 + O(e_n^5). \tag{8}$$

Expanding $f'(y_n)$ about x^*, we have

$$f'(y_n) = f'(x^*)\left[1+\frac{2}{3}c_2e_n+\frac{1}{3}(4c_2^2+c_3)e_n^2-\left(\frac{8}{3}c_2^3-4c_2c_3-\frac{4}{27}c_4\right)e_n^3+O(e_n^4)\right]. \tag{9}$$

From (6), (7) and (9) we have

$$J_f(x_n)\frac{f(x_n)}{f'(x_n)} = e_n-\left(c_2^3-c_2c_3+\frac{1}{9}c_4\right)e_n^4+O(e_n^5). \tag{10}$$

From (10) we get

$$z_n-x^* = \left(c_2^3-c_2c_3+\frac{1}{9}c_4\right)e_n^4+O(e_n^5). \tag{11}$$

Expanding $f(z_n)$ about $x*$, we have

$$f(z_n) = f'(x^*)[(z_n-x^*)+O((z_n-x^*)^2)]. \tag{12}$$

Dividing (12) by (7) gives us

$$\frac{f(z_n)}{f'(x_n)} = [1-2c_2e_n+(4c_2^2-3c_3)e_n^2+O(e_n^3)][(z_n-x^*)+O((z_n-x^*)^2)]. \tag{13}$$

From the equations (1), (8), (9), (10) and (12) we have

$$\begin{aligned}&\gamma\delta^2f'(x_n)+(\gamma\delta-\beta\epsilon)[f(x_n)-f(z_n)]\\&=f'(x^*)\{\frac{2}{3}\beta e_n^2+\gamma c_2e_n^3+\left(\frac{2}{3}\beta c_2^2+\left(2\gamma-\frac{2}{3}\beta\right)c_3\right)e_n^4\\&+\left[-2\beta c_2^3+\frac{10}{3}\beta c_2c_3+\left(3\gamma-\frac{38}{27}\beta\right)c_4\right]e_n^5+O(e_n^6)\}\end{aligned} \tag{14}$$

and

$$\begin{aligned}&\alpha\delta^2f'(x_n)+\beta\delta^2f'(y_n)+(\gamma\delta-\beta\epsilon)[f(x_n)-f(z_n)]\\&=f'(x^*)\left\{\frac{2}{3}\beta e_n^2+\left(\gamma-\frac{4}{3}\beta\right)c_2e_n^3+\left(2\beta c_2^2+\left(2\gamma-\frac{10}{3}\beta\right)c_3\right)e_n^4\right.\\&\left.+\left[-\frac{14}{3}\beta c_2^3+\frac{22}{3}c_2c_3+\left(3\gamma-\frac{142}{27}\beta\right)c_4\right]e_n^5+O(e_n^6)\right\}.\end{aligned} \tag{15}$$

Dividing (14) by (15) gives us

$$\begin{aligned}&\frac{\gamma\delta^2f'(x_n)+(\gamma\delta-\beta\epsilon)[f(x_n)-f(z_n)]}{\alpha\delta^2f'(x_n)+\beta\delta^2f'(y_n)+(\gamma\delta-\beta\epsilon)[f(x_n)-f(z_n)]}\\&=1+2c_2e_n+\left[4c_3+\left(2-\frac{3\gamma}{\beta}\right)c_2^2\right]e_n^2+O(e_n^3).\end{aligned} \tag{16}$$

From (11), (13) and (16), we have

$$(17)\qquad \begin{aligned} e_{n+1} &= z_n - x^* - \tfrac{\gamma\delta^2 f'(x_n)+(\gamma\delta-\beta\epsilon)[f(x_n)-f(z_n)]}{\alpha\delta^2 f'(x_n)+\beta\delta^2 f'(y_n)+(\gamma\delta-\beta\epsilon)[f(x_n)-f(z_n)]}\tfrac{f(z_n)}{f'(x_n)} \\ &= \left[\left(\tfrac{3\gamma}{\beta}-2\right)c_2^2 - c_3\right]\left[c_2^3 - c_2c_3 + \tfrac{1}{9}c_4\right]e_n^6 + O(e_n^7). \end{aligned}$$

The equation (17) means that the family of method given by (5) is of sixth-order. □

3. Numerical Examples

We present some numerical test results for various iterative methods in the following tables. The following methods were compared: the Newton method (NM), the Jarratt method (JM), the method of Kou et al. ([6]) (KM), the method of Wang et al. ([9], $\alpha = 1$ and $\beta = -3$) (WM), the method of Chun ([3]) (CM) and our new proposed method ($\alpha = 1$ and $\beta = -10$) (PM).

All computations were done using Mathematica Ver. 5.1 using 150 digit floating point arithmetics (Digits:=150). We accept an approximate solution rather than the exact root, depending on the precision (ϵ) of the computer. We use the following stopping criteria for computer programs: $|f_k(x_{n+1})| < \epsilon$, and so, when the stopping criterion is satisfied, x_{n+1} is taken as the exact root x^* computed. We used $\epsilon = 10^{-150}$.

We used the following test functions and display the computed approximate zero x^*.

$$\begin{aligned} f_1(x) &= \sqrt{x} - \frac{1}{x} - 3, \quad x^* = 9.6335955628326951924063127091908162 6\\ f_2(x) &= e^x + x - 20, \quad x^* = 2.8424389537844470678165859401509500 7\\ f_3(x) &= \ln x + \sqrt{x} - 5, \quad x^* = 8.3094326942315717953469556826920686 1\\ f_4(x) &= x^3 - x^2 - 1, \quad x^* = 1.4655712318767680266567312252199391 0 \end{aligned}$$

Table 1. f_1, $x_0 = 1.0$

n	NM	JM	KM	WM	CM	PM
0	1.60	6.63	6.63	17.83	6.63	5.48
1	0.50	1.02	0.22	7.28	1.02	6.09e-3
2	4.48e-2	1.84e-3	1.48e-8	0.68	1.87e-3	5.19e-18
3	3.60e-4	3.07e-14	1.82e-51	1.73e-6	7.50e-18	4.92e-108
4	2.32e-8	2.38e-57	0	8.78e-41	3.11e-104	0
5	9.67e-17	0		0	0	
6	1.68e-33					
7	5.04e-67					
8	4.65e-134					
9	0					

Table 2. f_2, $x_0 = 0.0$

n	NM	JM	KM	WM	CM	PM
0	13349.23	101.65	76.52	88.28	76.91	21.65
1	4907.05	5.24	8.79	1.85	8.22	1.27e-2
2	1801.71	2.58e-3	6.16e-4	1.53e-7	4.74e-4	8.63e-23
3	658.90	2.63e-16	1.54e-28	6.59e-50	3.58e-29	0
4	238.27	2.83e-68	0	0	0	
5	83.41	0				
6	26.61					
7	6.51					
8	0.76					
9	1.45e-2					
10	5.43e-6					
11	7.67e-13					
12	1.53e-26					
13	6.08e-54					
14	9.63e-109					
15	0					

Table 3. f_3, $x_0 = 1.0$

n	NM	JM	KM	WM	CM	PM
0	1.79	0.63	0.68	8.35e-2	0.64	5.64e-2
1	0.40	1.42e-4	2.28e-6	2.04e-12	1.78e-4	6.51e-13
2	2.28e-2	3.90e-19	4.69e-39	4.36e-76	4.00e-25	1.50e-78
3	7.50e-5	2.21e-77	0	0	0	0
4	8.12e-10	0				
5	9.52e-20					
6	1.31e-39					
7	2.47e-79					
8	0					

The numerical results presented in the above tables show that the proposed methods in this contribution have at least equal performance as compared with the other methods of the same order. Thus, the new methods can compete with other six-order methods in literature.

Table 4. f_4, $x_0 = 0.5$

n	NM	JM	KM	WM	CM	PM
0	81.00	9.22	div	6.40	4.57	67.31
1	24.17	1.34		1.04	1.00	3.35
2	7.37	1.18		4.44	17807.87	0.43
3	2.44	1.22		1.14	6192.32	4.01e-6
4	1.10	1.32		46.87	2155.78	1.80e-36
5	0.82	1.84		1.80	750.55	0
6	0.77	67.67		2.26e-3	259.62	
7	9.43e-2	5.60		2.45e-19	87.02	
8	2.25e-3	1.00		4.05e-115	26.11	
9	1.39e-6	135722.91		0	5.78	
10	5.33e-13	10361.85			1.06	
11	7.83e-26	790.39			8.21	
12	1.69e-51	59.81			0.22	
13	7.83e-103	4.15			8.66e-8	
14	0	0.11			1.88e-46	
15		1.67e-6			0	
16		1.12e-25				
17		2.28e-102				
18		0				

4. Conclusion

In this paper, we presented a new sixth-order family of methods for solving nonlinear equations. We observed from numerical examples that the proposed method have at least equal performance as compared with the other methods of the same order. And the practical utility of our method is good, the above-mentioned sixth-order methods requires two functions and three first derivative evaluations per iteration to improve the order of convergence so that, the efficiency of our method measured by the efficiency index, introduced by Ostrowski [7], is 1.431.

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