

The Leading Finite Type Coefficients of the Links–Gould Polynomial of a Link

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ABSTRACT. We show that the Links–Gould polynomial of a link has finite type coefficients in a multivariate series expansion, and express the leading coefficients in terms of the linking numbers of a link.

1. Introduction

The Links–Gould polynomial [11] is a quantum invariant which is derived from the one-parameter family of four dimensional representations of the quantum superalgebra $U_q[gl(2|1)]$. D. De Wit, L. H. Kauffman and J. R. Links [4] gave an explicit form of the R -matrix for the invariant, and showed that it is a powerful invariant through its evaluation. The invariant is complete for all prime knots of up to 10 crossings [3] and for the Kanenobu knots [8], [9]. The Links–Gould polynomial is not only a powerful invariant. It is also a two-variable generalization of the Alexander–Conway polynomial [7]. In this paper, we show that the Links–Gould polynomial has finite type coefficients in a multivariate series expansion with respect to symmetrical variables, where an invariant is finite type [2], [14] if it vanishes for singular knots with finite singularities. Furthermore, we determine the leading coefficients in terms of the linking numbers of links. This result is analogous to that on the first coefficient of the Conway polynomial of a link due to Hoste [5].

For an ordered oriented r -component link $L = K_1 \cup \cdots \cup K_r$, we denote by $\lambda_{i,j}(L)$ the linking number of K_i and K_j . Let Φ be a graph with r vertices p_1, \dots, p_r and $e(i, j)$ edges joining p_i and p_j . We define the invariant Λ_Φ by

$$\Lambda_\Phi(L) = \prod_{i < j} \lambda_{i,j}(L)^{e(i,j)}.$$

Let \mathcal{C}_r be the set of cycle graphs with r vertices p_1, \dots, p_r , where a cycle graph is a connected graph which forms one cycle. Then $\sum_{\Phi \in \mathcal{C}_r} \Lambda_\Phi(L)$ is well-defined for an unordered oriented r -component link L . For example,

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$$\begin{aligned}
\sum_{\Phi \in \mathcal{C}_2} \Lambda_{\Phi}(L) &= \lambda_{1,2}(L)\lambda_{2,1}(L), \\
\sum_{\Phi \in \mathcal{C}_3} \Lambda_{\Phi}(L) &= \lambda_{1,2}(L)\lambda_{2,3}(L)\lambda_{3,1}(L), \\
\sum_{\Phi \in \mathcal{C}_4} \Lambda_{\Phi}(L) &= \lambda_{1,2}(L)\lambda_{2,3}(L)\lambda_{3,4}(L)\lambda_{4,1}(L) + \lambda_{1,2}(L)\lambda_{2,4}(L)\lambda_{4,3}(L)\lambda_{3,1}(L) \\
&\quad + \lambda_{1,3}(L)\lambda_{3,2}(L)\lambda_{2,4}(L)\lambda_{4,1}(L).
\end{aligned}$$

We denote by $LG(L; t_0, t_1)$ the Links–Gould polynomial of an oriented link L . Let $a_0(L)$ be the finite type invariant of type 0, defined by

$$a_0(L) = \begin{cases} 1 & \text{if } L \text{ is a knot,} \\ 0 & \text{otherwise.} \end{cases}$$

In [7], we showed that $LG(L; t_0, t_1) \in \mathbb{Z}[t_0^{\pm 1}, t_1^{\pm 1}]$ and the equalities

$$LG(L; t_0, 1) = LG(L; 1, t_1) = a_0(L).$$

Then the Links–Gould polynomial is expressed in the following form:

$$LG(L; t_0, t_1) = a_0(L) + \sum_{i,j=1}^{\infty} a_{i,j}(L)(t_0 - 1)^i(t_1 - 1)^j$$

in $\mathbb{Z}[[t_0 - 1, t_1 - 1]]$, where $a_{i,j}(L) \in \mathbb{Z}$.

Theorem 1. *The coefficient $a_{i,j}(L)$ is a finite type invariant of type $i + j$. Let r be the number of components of L . For $i + j < r$, we have $a_{i,j}(L) = 0$. Furthermore, for $i + j = r$, the leading coefficients $a_{i,j}(L)$ are given by*

$$a_{i,j}(L) = \begin{cases} -\sum_{C \in \mathcal{C}_2} \Lambda_C(L) & \text{if } i = j = 1, \\ -2\binom{r-2}{i-1} \sum_{C \in \mathcal{C}_r} \Lambda_C(L) & \text{otherwise.} \end{cases}$$

2. Preliminaries

We recall the definition of the Links–Gould polynomial. Let V be a four dimensional vector space with a basis $\{e_i\}_{i=1}^4$, and let V^* be its dual. We denote the dual basis by $\{e_i^*\}_{i=1}^4$:

$$e_i^*(e_j) = \delta_j^i,$$

where δ_j^i is the Kronecker symbol. We define $R : V \otimes V \rightarrow V \otimes V$ by

$$\begin{aligned}
& t_0 e_{11}^{11} - (e_{22}^{22} + e_{33}^{33}) + t_1 e_{44}^{44} + (t_0 - 1)(e_{21}^{21} + e_{31}^{31}) + (t_0 - 1)(1 - t_1)e_{41}^{41} \\
& + (t_1 - 1)(e_{42}^{42} + e_{43}^{43}) + (t_0 t_1 - 1)e_{23}^{23} + (e_{41}^{14} + e_{14}^{41}) - t_0^{1/2} t_1^{1/2} (e_{32}^{23} + e_{23}^{32}) \\
& + t_0^{1/2} (e_{21}^{12} + e_{12}^{21} + e_{31}^{13} + e_{13}^{31}) + t_1^{1/2} (e_{42}^{24} + e_{24}^{42} + e_{43}^{34} + e_{34}^{43}) \\
& - t_0^{1/2} t_1^{1/2} ((t_0 - 1)(1 - t_1))^{1/2} (e_{41}^{23} + e_{23}^{41}) + ((t_0 - 1)(1 - t_1))^{1/2} (e_{41}^{32} + e_{32}^{41}),
\end{aligned}$$



Figure 1:

where the map $e_{j_1 j_2}^{i_1 i_2} : V \otimes V \rightarrow V \otimes V$ is defined by

$$e_{j_1 j_2}^{i_1 i_2}(e_{k_1} \otimes e_{k_2}) = \delta_{k_1}^{j_1} \delta_{k_2}^{j_2} e_{i_1} \otimes e_{i_2}.$$

We define $n : V \otimes V^* \rightarrow \mathbb{C}$ and $\tilde{n} : V^* \otimes V \rightarrow \mathbb{C}$ by

$$n(e_i \otimes e_j^*) = \delta_j^i \quad \text{and} \quad \tilde{n}(e_i^* \otimes e_j) = \mu_i \delta_j^i,$$

where $(\mu_1, \mu_2, \mu_3, \mu_4) = (t_0, -t_1^{-1}, -t_0, t_1^{-1})$. We define $u : \mathbb{C} \rightarrow V \otimes V^*$ and $\tilde{u} : \mathbb{C} \rightarrow V^* \otimes V$ by

$$u(1) = \sum_{i=1}^4 \mu_i^{-1} e_i \otimes e_i^* \quad \text{and} \quad \tilde{u}(1) = \sum_{i=1}^4 e_i^* \otimes e_i.$$

Any oriented tangle diagram can be expressed up to isotopy as a diagram composed from the elementary tangle diagrams shown in Figure 1. Furthermore any oriented tangle diagram can be expressed up to isotopy as a *sliced diagram* which is such a diagram sliced by horizontal lines such that each domain between adjacent horizontal lines has either a single crossing or a single critical point.

We associate the maps $\text{id}_V, \text{id}_{V^*}, R, R^{-1}, n, \tilde{n}, u,$ and \tilde{u} to elementary oriented tangle diagrams as described in Figure 2. Corresponding to any oriented tangle diagram D , we may then obtain a linear map $[D]$ as the composition of tensor products of copies of the linear maps associated to the elementary tangle diagrams within D . For example,

$$\left[\begin{array}{c} \uparrow \\ \text{Diagram of a crossing} \end{array} \right] = (\text{id}_V \otimes n)(R \otimes \text{id}_{V^*})(\text{id}_V \otimes u).$$

Let T be a $(1, 1)$ -tangle represented by a diagram D_T . We denote by \widehat{T} the closure of T . The Links–Gould polynomial of the link \widehat{T} is defined by the following identity:

$$[D_T] = LG(\widehat{T}; t_0, t_1) \text{id}_V.$$

For the details we refer the reader to [4], [13].

3. A family of singular links

A *singular link* is an immersion of disjoint circles into S^3 , which has transverse

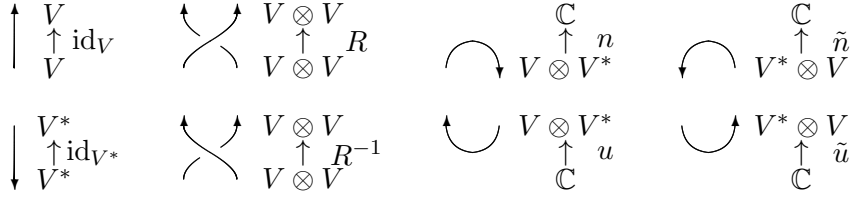


Figure 2:

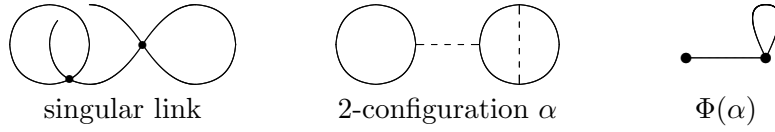


Figure 3:

double points and no other singularities. Let v be an isotopy invariant for oriented links, which takes values in an abelian group. We may extend v to a singular link invariant via the *Vassiliev skein relation*:

$$v \left(\begin{array}{c} \nearrow \searrow \\ \times \\ \nwarrow \nearrow \end{array} \right) = v \left(\begin{array}{c} \nearrow \searrow \\ \nearrow \searrow \end{array} \right) - v \left(\begin{array}{c} \nwarrow \nearrow \\ \nwarrow \nearrow \end{array} \right).$$

A singular link invariant v is called a *finite type invariant of type d* if $v(L) = 0$ for any singular link L with more than d singular points.

For a finite type invariant v_i of type d_i ($i = 1, 2$), the product $v_1 v_2$ is a finite type invariant of type $d_1 + d_2$. We remark that $(v_1 v_2)(L)$ is defined as follows: If L has a singular point, we use the Vassiliev skein relation. Then we use the equality $(v_1 v_2)(L) = v_1(L) v_2(L)$ for an oriented singular link L without singular points. For a graph Φ with d edges, the invariant Λ_Φ is a finite type invariant of type d .

A d -*configuration* is d pairs of $2d$ points on disjoint circles. A singular link with d singular points *respects* a d -configuration if each singular point represents a pair of the d -configuration. In Figure 3, we give an example of a d -configuration and a singular link respecting it. For any d -configuration, there exists a singular link respecting it. The configuration given in Figure 4 is called *inadmissible*. A configuration is called *admissible* if it is not inadmissible. For the details we refer the reader to [1], [2].

For a d -configuration α with r circles S_1, \dots, S_r , we define a graph $\Phi(\alpha)$ with r vertices p_1, \dots, p_r and d edges by deforming the circle S_k into the vertex p_k and a chord joining S_i to S_j into an edge joining p_i to p_j (see Figure 3). Let $\alpha_i^{d,r}; 1 \leq i \leq f_{d,r}$ be a list of the all distinct admissible d -configurations with r circles S_1, \dots, S_r . We set

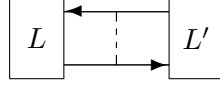


Figure 4:

$$\begin{aligned} \mathcal{A}^{d,r} &:= \{\alpha_i^{d,r} \mid 1 \leq i \leq f_{d,r}\}, \\ \mathcal{A}_\infty^{d,r} &:= \{\alpha \in \mathcal{A}^{d,r} \mid \alpha \text{ is disconnected}\}, \\ \mathcal{A}_0^{d,r} &:= \{\alpha \in \mathcal{A}^{d,r} \mid \alpha \text{ is a spanning tree}\}, \\ \mathcal{A}_l^{d,r} &:= \{\alpha \in \mathcal{A}^{d,r} \mid \alpha \text{ is a connected graph with one cycle of length } l\}, \end{aligned}$$

for $l \geq 2$. We remark that $\mathcal{A}_0^{d,r} = \emptyset$ if $d \neq r - 1$ and that $\mathcal{A}_l^{d,r} = \emptyset$ if $d \neq r$. For example, we have

$$\begin{aligned} \mathcal{A}^{0,2} &= \mathcal{A}_\infty^{0,2} = \left\{ \begin{array}{c} \cdot \\ \cdot \end{array} \right\}, & \mathcal{A}^{1,2} &= \mathcal{A}_0^{1,2} = \left\{ \begin{array}{c} | \\ | \end{array} \right\}, \\ \mathcal{A}^{2,2} &= \mathcal{A}_\infty^{2,2} \cup \mathcal{A}_2^{2,2} = \left\{ \begin{array}{c} \infty \\ \cdot \\ \infty \end{array} \right\} \cup \left\{ \begin{array}{c} \diamond \end{array} \right\}, \\ \mathcal{A}^{0,3} &= \mathcal{A}_\infty^{0,3}, & \mathcal{A}^{1,3} &= \mathcal{A}_\infty^{1,3}, \\ \mathcal{A}^{2,3} &= \mathcal{A}_\infty^{2,3} \cup \mathcal{A}_0^{2,3}, & \mathcal{A}_0^{2,3} &= \left\{ \begin{array}{c} \nearrow \\ \searrow \end{array} \right\}, \\ \mathcal{A}^{3,3} &= \mathcal{A}_\infty^{3,3} \cup \mathcal{A}_2^{3,3} \cup \mathcal{A}_3^{3,3}, \\ \mathcal{A}_2^{3,3} &= \left\{ \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} \right\}, & \mathcal{A}_3^{3,3} &= \left\{ \begin{array}{c} \triangle \end{array} \right\}. \end{aligned}$$

Let θ_1 and $\theta_2(n)$ be the singular links as shown in Figure 5. For $\alpha \in \mathcal{A}_\infty^{d,r}$, we choose an ordered split singular link $M(\alpha)$ respecting α . For $\alpha \in \mathcal{A}_0^{d,r}$, let $M(\alpha)$ be the connected sum of d copies of θ_1 such that $M(\alpha)$ respects α . For $\alpha \in \mathcal{A}_l^{d,r}$, let $M(\alpha)$ be the connected sum of $\theta_2(l)$ and $d - l$ copies of θ_1 such that $M(\alpha)$ respects α .

We set

$$\begin{aligned} \mathcal{M}^{d,r} &:= \{M(\alpha) \mid \alpha \in \mathcal{A}^{d,r}\}, & \mathcal{M}_\infty^{d,r} &:= \{M(\alpha) \mid \alpha \in \mathcal{A}_\infty^{d,r}\}, \\ \mathcal{M}_0^{d,r} &:= \{M(\alpha) \mid \alpha \in \mathcal{A}_0^{d,r}\}, & \mathcal{M}_l^{d,r} &:= \{M(\alpha) \mid \alpha \in \mathcal{A}_l^{d,r}\}, \\ \widetilde{\mathcal{M}}^{d,r} &:= \mathcal{M}^{0,r} \cup \dots \cup \mathcal{M}^{d,r}, & \widetilde{\mathcal{M}}_\infty^{d,r} &:= \mathcal{M}_\infty^{0,r} \cup \dots \cup \mathcal{M}_\infty^{d,r}. \end{aligned}$$

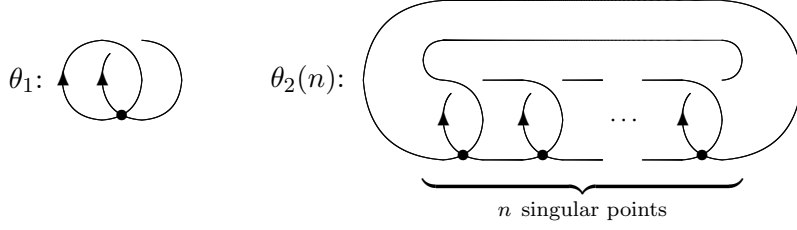


Figure 5:

Then

$$\widetilde{\mathcal{M}}^{d,r} = \begin{cases} \widetilde{\mathcal{M}}_{\infty}^{d,r} & \text{if } d \leq r-2, \\ \widetilde{\mathcal{M}}_{\infty}^{r-1,r} \cup \mathcal{M}_0^{r-1,r} & \text{if } d = r-1, \\ \widetilde{\mathcal{M}}_{\infty}^{r,r} \cup \mathcal{M}_0^{r-1,r} \cup \mathcal{M}_2^{r,r} \cup \cdots \cup \mathcal{M}_r^{r,r} & \text{if } d = r. \end{cases}$$

Put $\Phi(M(\alpha)) := \Phi(\alpha)$ for $\alpha \in \mathcal{A}^{d,r}$. We have the following lemma by direct calculation.

Lemma 2. *Let C be a cycle graph of length $r \geq 3$. For $M \in \widetilde{\mathcal{M}}^{r,r}$, we have*

$$\Lambda_C(M) = \begin{cases} 1 & \text{if } \Phi(M) = C, \\ 0 & \text{otherwise.} \end{cases}$$

4. Proof of theorem 1

The Links–Gould polynomial satisfies the following skein relations [6]:

$$\begin{aligned} LG \left(\begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \right) + t_0 t_1 LG \left(\begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \right) &= (s+1) LG \left(\begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \right) + (s+t_0 t_1) LG \left(\begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \right), \\ LG \left(\begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \right) - LG \left(\begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \right) &= (1-s) LG \left(\begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \right) + (s-1) LG \left(\begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \right), \end{aligned}$$

where $s = -(t_0 - 1)(t_1 - 1)$. Put $T_i = t_i - 1$ for $i = 0, 1$. By using these skein relations, we obtain the following skein relation:

$$(1) \quad LG \left(\begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array} \right) = (T_0 + T_1 - T_0 T_1) LG \left(\begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \right) + (T_0 T_1^2 + T_0^2 T_1) LG \left(\begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \right) \\ \quad - (2T_0 T_1 + T_0 T_1^2 + T_0^2 T_1 + T_0^2 T_1^2) LG \left(\begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \right).$$

We denote by $L_1 \# L_2$ a connected sum of L_1 and L_2 . Then we have

$$(2) \quad LG(L_1 \# L_2) = LG(L_1) LG(L_2).$$

Lemma 3. *For $n \geq 2$, we have*

$$\begin{aligned} LG(\theta_1) &= -T_0T_1, \\ LG(\theta_2(n)) &= -2(T_0 + T_1)^{n-2}T_0T_1 + \sum_{i+j>n} a_{i,j}(\theta_2(n))T_0^iT_1^j. \end{aligned}$$

Proof. By the definition of the Links–Gould polynomial, we have

$$\begin{aligned} LG(\theta_1) &= -T_0T_1, \\ LG(\theta_2(1)) &= 0, \\ LG(\theta_2(2)) &= -2T_0T_1 - T_0T_1^2 - T_0^2T_1 - T_0^2T_1^2, \\ LG(\theta_2(3)) &= -2(T_0 + T_1)T_0T_1 + \sum_{i+j>3} a_{i,j}(\theta_2(3))T_0^iT_1^j, \\ LG(\theta_2(n)) &= (T_0 + T_1 - T_0T_1)LG(\theta_2(n-1)) + (T_0T_1^2 + T_0^2T_1)LG(\theta_2(n-2)) \\ &\quad + (2 + T_0 + T_1 + T_0T_1)(-1)^{n-1}T_0^{n-1}T_1^{n-1}, \end{aligned}$$

for $n \geq 3$, where the last equality follows from the skein relation (1) and the equality (2).

We show the second equality in Lemma 3 by induction on n . Let $m \geq 4$. we suppose the equality for $n < m$. Then we have

$$\begin{aligned} LG(\theta_2(m)) &= (T_0 + T_1 - T_0T_1)LG(\theta_2(m-1)) + (T_0T_1^2 + T_0^2T_1)LG(\theta_2(m-2)) \\ &\quad + (2 + T_0 + T_1 + T_0T_1)(-1)^{m-1}T_0^{m-1}T_1^{m-1} \\ &= -2(T_0 + T_1)^{m-2}T_0T_1 + \sum_{i+j>m} a_{i,j}(\theta_2(m))T_0^iT_1^j. \quad \square \end{aligned}$$

An invariant for unordered links is that for ordered links by forgetting the order of an ordered link.

Lemma 4. *For $M \in \mathcal{M}_{\infty}^{d,r}$, we have*

$$a_{i,j}(M) = 0.$$

For $M \in \mathcal{M}_0^{r-1,r}$, we have

$$a_{i,j}(M) = \begin{cases} (-1)^{r-1} & \text{if } i + j = 2r - 2, \\ 0 & \text{otherwise.} \end{cases}$$

For $M \in \mathcal{M}_l^{r,r}$ ($l \geq 2$), we have

$$a_{i,j}(M) = \begin{cases} -2\binom{r-2}{i-1} & \text{if } i + j = r = l, \\ 0 & \text{if } i + j < 2r - l. \end{cases}$$

Proof. The Links–Gould polynomial vanishes for a split link, and so is the coefficient: $a_{i,j}(M) = 0$ for $M \in \mathcal{M}_\infty^{d,r}$.

For $M \in \mathcal{M}_0^{r-1,r}$, by Lemma 3 and the equality (2), we have $LG(M) = LG(\theta_1)^{r-1} = (-1)^{r-1} T_0^{r-1} T_1^{r-1}$, which implies

$$a_{i,j}(M) = \begin{cases} (-1)^{r-1} & \text{if } i + j = 2r - 2, \\ 0 & \text{otherwise.} \end{cases}$$

For $M \in \mathcal{M}_l^{r,r}$ ($l \geq 2$), by Lemma 3 and the equality (2), we have

$$LG(M) = 2(-1)^{r-l+1} (T_0 + T_1)^{l-2} T_0^{r-l+1} T_1^{r-l+1} + \sum_{s+t > 2r-l} a_{s,t}(M) T_0^s T_1^t,$$

which implies

$$a_{i,j}(M) = \begin{cases} -2 \binom{r-2}{i-1} & \text{if } i + j = r = l, \\ 0 & \text{if } i + j < 2r - l. \end{cases}$$

□

Proof of Theorem 1. By the definition of the R-matrix, we have

$$R - R^{-1} \Big|_{\substack{t_0=1+\alpha_0 h \\ t_1=1+\alpha_1 h}} = hX,$$

for some 16×16 matrix X such that $X|_{h=0} \in M_{16}(\mathbb{C}[\alpha_0^{\pm 1/2}, \alpha_1^{\pm 1/2}])$. Then, for a singular link L with more than d singular points ($d \geq 2$), we have

$$LG(L; 1 + \alpha_0 h, 1 + \alpha_1 h) = \sum_{i+j > d} a_{i,j}(L) \alpha_0^i \alpha_1^j h^{i+j},$$

which implies that $\sum_{i+j=d} a_{i,j}(L) \alpha_0^i \alpha_1^j$ is a finite type invariant of type d . This argument is essentially the same with [2]. Hence $a_{i,j}(L)$ is a finite type invariant of type $i + j$.

For $M \in \widetilde{\mathcal{M}}^{d,r}$ and an r -component singular link L , there exists an integer $m^{d,r}(M; L)$ such that

$$v(L) = \sum_{M \in \widetilde{\mathcal{M}}^{d,r}} m^{d,r}(M; L) v(M),$$

for any finite type invariant v of type d (cf. [2], [10], [12]). We suppose $i + j \leq r - 2$. Putting $v = a_{i,j}$, by Lemma 4, we have

$$a_{i,j}(L) = \sum_{M \in \widetilde{\mathcal{M}}_\infty^{i+j,r}} m^{i+j,r}(M; L) a_{i,j}(M) = 0.$$

We suppose $i + j = r - 1$. Putting $v = a_{i,j}$, by Lemma 4, we have

$$a_{i,j}(L) = \sum_{M \in \widetilde{\mathcal{M}}_\infty^{r-1,r} \cup \mathcal{M}_0^{r-1,r}} m^{i+j,r}(M; L) a_{i,j}(M) = 0.$$

We suppose $i + j = r = 2$. By [12], we have

$$m^{2,2}(\theta_1; L) = \lambda_{12}(L), \quad m^{2,2}(\theta_2(2); L) = \frac{\lambda_{12}^2(L) - \lambda_{12}(L)}{2}.$$

Putting $v = a_{1,1}$, by Lemma 4, we have

$$\begin{aligned} a_{1,1}(L) &= \sum_{M \in \widetilde{\mathcal{M}}_\infty^{2,2} \cup \mathcal{M}_0^{1,2} \cup \mathcal{M}_2^{2,2}} m^{i+j,r}(M; L) a_{1,1}(M) \\ &= m^{2,2}(\theta_1; L) a_{1,1}(\theta_1) + m^{2,2}(\theta_2(2); L) a_{1,1}(\theta_2(2)) \\ &= -\lambda_{12}^2(L) \\ &= -\sum_{C \in \mathcal{C}_2} \Lambda_C(L). \end{aligned}$$

We suppose $i + j = r \geq 3$. Let $N \in \mathcal{M}_r^{r,r}$. Putting $v = \Lambda_{\Phi(N)}$, by Lemma 2, we have

$$\Lambda_{\Phi(N)}(L) = \sum_{M \in \widetilde{\mathcal{M}}^{r,r}} m^{r,r}(M; L) \Lambda_{\Phi(N)}(M) = m^{r,r}(N; L).$$

Putting $v = a_{i,j}$, by Lemma 4, we have

$$\begin{aligned} a_{i,j}(L) &= \sum_{M \in \mathcal{M}_0^{r-1,r}} m^{r,r}(M; L) a_{i,j}(M) + \sum_{l=2}^r \sum_{M \in \mathcal{M}_l^{r,r}} m^{r,r}(M; L) a_{i,j}(M) \\ &= \sum_{M \in \mathcal{M}_r^{r,r}} -2 \binom{r-2}{i-1} \Lambda_{\Phi(M)}(L) \\ &= -2 \binom{r-2}{i-1} \sum_{C \in \mathcal{C}_r} \Lambda_C(L). \quad \square \end{aligned}$$

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