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When Some Complement of an EC-Submodule is a Direct Summand

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Abstract. A module M is said to satisfy the EC_{11} condition if every ec-submodule of M has a complement which is a direct summand. We show that for a multiplication module over a commutative ring the EC_{11} and P-extending conditions are equivalent. It is shown that the EC_{11} property is not inherited by direct summands. Moreover, we prove that if M is an EC_{11}-module where $SocM$ is an ec-submodule, then it is a direct sum of a module with essential socle and a module with zero socle. An example is given to show that the reverse of the last result does not hold.

1. Introduction

Throughout this article, all rings are associative with unity and R denotes such a ring. All modules are unital right R-modules. Recall that a module is said to be *extending* or *CS* or said to satisfy the C_1 condition if every submodule is essential in a direct summand. Following [9], we call a (closed) submodule as *ec-(closed) submodule* if it contains essentially a cyclic submodule. A module M is said to be *principally extending (or P-extending)* if every cyclic submodule of M is essential in a direct summand. Recall that, an R-module M is said to be a *multiplication module* if for each $X \leq M$ there exists $A_R \leq R_R$ such that $X = MA$ (see, for example [1], [8]). Following [6], a module is said to be *ECS* if every ec-closed submodule is a direct summand. In [11], the authors investigated a weakened form of the C_1 condition: Every submodule has a complement which is a direct summand. This weakened C_1 property is called the C_{11} condition. For recent results on C_{11}-

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modules and rings, refer to [3] and [12].

In this article, we study modules whose every ec-submodule has a complement which is a direct summand. We call this property as EC_{11} condition. It is easy to check that for a module M EC_{11} condition is equivalent to the property if every cyclic submodule of M has a complement which is a direct summand of M. Clearly, the C_{11} condition implies the EC_{11} property.

In section 1, we consider connections between the EC_{11} condition, and various other generalizations of the C_1 condition. As an application we show that the EC_{11} condition is equivalent to the P-extending for the class of multiplication modules. In section 2, we show that the EC_{11} property is not inherited by direct summands. However, we obtain conditions which make direct summands of an EC_{11}-module have EC_{11} condition. We also show that if M is an EC_{11}-module and $r(M)$ is an ec-submodule of M where r is any left exact preradical, then M has a decomposition $M_1 \oplus M_2$ such that $r(M_1)$ is essential in M_1 and $r(M_2) = 0$. Finally, we provide a counter example which shows that the converse of the latter decomposition result does not hold, in general.

Let R be a ring and M a right R-module. If $X \subseteq M$, then $X \leq M$ denotes X is a submodule of M. Moreover, $SocM$, $End(M)$ and $J(R)$ symbolize the socle of M, the ring of endomorphisms of M and the Jacobson radical of R, respectively. We use $S(R, M)$ to denote the split-null extension of M by R. A ring is called *Abelian* if every idempotent is central. Other terminology and notation can be found in [2], [7] and [10].

2. Preliminary results

In this section, we study relationships between the EC_{11} condition and various generalizations of the C_1 condition. Recall from [4], a module is *FI-extending* if every fully invariant submodule is essential in a direct summand.

Lemma 2.1. *Let N, K be submodules of M such that $N \cap K = 0$. Then K is a complement of N in M if and only if K is closed in M and $N \oplus K$ is essential in M.*

Proof. Simple to check. □

Proposition 2.2. *Let M be a module. Then the following statements are equivalent.*
(i) *M has EC_{11}.*
(ii) *For any ec-closed submodule L in M, there exists a direct summand K of M such that K is a complement of L in M.*
(iii) *For any ec-submodule N in M, there exists a direct summand K of M such that $K \cap N = 0$ and $K \oplus N$ is essential submodule of M.*
(iv) *For any ec-closed submodule L in M, there exists a direct summand K of M such that $K \cap L = 0$ and $K \oplus L$ is essential submodule of M.*

Proof. (i) ⇔ (ii) and (iii) ⇔ (iv) Obvious.

(i) $\Leftrightarrow$ (iii) Follows from Lemma 2.1. □

Lemma 2.3. *Let M_R be a module. Consider the following statements:*
(i) *M_R is ECS*
(ii) *M_R is P-extending*
(iii) *M_R is EC_{11}-module*
(iv) *M_R is C_{11}-module*
Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) *and* (iv) $\Rightarrow$ (iii). *In general, the converses to these implications do not hold.*

Proof. (i) $\Rightarrow$ (ii). Clear by [6, Proposition 1.1].
(ii) $\Rightarrow$ (iii). Let K be an ec-closed submodule of M. Then there exists $x \in K$ such that xR is essential in K. Since M is P-extending, there exists a direct summand D of M such that xR is essential in D. Now $M = D \oplus D'$ for some submodule D' of M. Then $K \cap D' = 0$ and $K \oplus D'$ is essential in M. By Lemma 2.1, M is an EC_{11}-module.
(iv) $\Rightarrow$ (iii) Clear.

Let R be the ring as in [5, Example 3.2] i.e., $R = \left[\begin{smallmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z} \end{smallmatrix}\right]$. Then R is right P-extending. However, R_R is not ECS-module. Thus (ii) $\not\Rightarrow$ (i). Now, let M be the $\mathbb{Z}[x]$-module $\mathbb{Z}[x] \oplus \mathbb{Z}[x]$. So M is an EC_{11}-module. But M is not P-extending, by [6, Proposition 1.2]. Thus (iii) $\not\Rightarrow$ (ii).

Finally, let R be the ring as in [10, Example 7.54]. Then R is a commutative, regular ring which is not Baer. Now by [4, Theorem 4.7 (iii)], R_R is not FI-extending. Hence, [3, Proposition 1.2] yields that R_R is not C_{11}-module. Thus (iii) $\not\Rightarrow$ (iv). □

Corollary 2.4. *Let M_R be an indecomposable module. Then the following statements are equivalent.*
(i) *M_R is ECS*
(ii) *M_R is P-extending*
(iii) *M_R is EC_{11}-module*
(iv) *M_R is uniform*

Proof. (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) follow from Lemma 2.3.
(iii) $\Rightarrow$ (iv) Let $0 \neq X \leq M$. Then there exists $0 \neq x \in X$. Let L be any closure of xR in M. Thus L is an ec-closed submodule of M. By hypothesis there exists a direct summand D of M such that $L \cap D = 0$ and $L \oplus D$ is essential in M. It follows that L is essential in M. Since L is complement of M, then $L = M$. Hence X is essential in M. Thus M_R is uniform.
(iv) $\Rightarrow$ (i) Obvious. □

Theorem 2.5. *Let M be an R-module such that $End(M_R)$ is Abelian and $X \leq M$ implies $X = \sum_{i \in I} h_i(M)$, where $h_i \in End(M_R)$. Then M is EC_{11}-module if and only if M is P-extending.*

Proof. Assume M is EC_{11}-module and X is a cyclic submodule of M. Let Y be a closure of X in M. Then X is essential in Y. So Y is an ec-closure submodule of M. Now $Y = \sum_{i\in I} h_i(M)$, where each $h_i \in End(M_R)$. By hypothesis, eM is a complement of Y where $e^2 = e \in End(M_R)$. Let $0 \neq y \in Y$. Then $y = ey+(1-e)y$. But $y = \sum_{i\in I} h_i(m_i)$ where $m_i \in M$. Thus $ey = e\sum_{i\in I} h_i(m_i) = \sum_{i\in I} h_i((em_i) \in Y \cap eM = 0$ i.e., $y = (1-e)y$. Hence Y is essential in $(1-e)M$. Then $Y = (1-e)M$ is direct summand of M. Hence M_R is P-extending. The converse follows from Lemma 2.3. □

Corollary 2.6. *If M is an R-module satisfying any of the following conditions, then M is EC_{11}-module if and only if M is P-extending.* (i) *$M_R = R_R$ and R is Abelian.* (ii) *M is cyclic and R is commutative.* (iii) *M is a multiplication module and R is commutative.*

Proof. By Theorem 2.5 the result is true for condition (i). Now assume that M is cyclic and R is commutative. There exists $B_R \leq R_R$ such that M_R is isomorphic to R/B. Let Y/B be an R-submodule of R/B. So $Y/B = (\sum_{i\in I} y_iR) + B = (\sum_{i\in I} y_iR+B)R$, where each $y_i \in Y$. Define $h_i : R/B \to R/B$ by $h_i(r+B) = y_i+B$. Then $h_i \in End((R/B)_R)$. Hence $Y/B = \sum_{i\in I} h_i(R/B)$. Since R is commutative, $End((R/B)_R)$ is commutative. Thus Theorem 2.5 yields the result for condition (ii).

Finally, assume that M is a multiplication module and R is commutative. Let $X = MA$, where $A_R \leq R_R$. For each $a \in A$ define $h_a : M \to M$ by $h_a(m) = ma$ for $m \in M$. Then $X = MA = \sum_{a\in A} h_a(M)$. Observe that every submodule of a multiplication module is fully invariant. By [4, Lemma 1.9], if $e^2 = e \in End(M_R)$, then e and $1-e \in S_l(End(M_R))$ where $S_l(End(M_R))$ is the set of all left semicentral idempotent elements of $End(M_R)$. Hence e is central. So $End(M_R)$ is Abelian. Again, Theorem 2.5 yields the result. □

3. Direct summands of an EC_{11}-module

In contrast to CS-modules, direct summands of a C_{11}-module need not satisfy the C_{11} condition, in general (see [12]). Our next result shows that EC_{11} property does not inherited by direct summands of a module which satisfies the EC_{11} condition.

Proposition 3.1. *Let $n \geq 3$ be any odd integer. Let $\mathbb{R}$ be the real field and S the polynomial ring $\mathbb{R}[x_1, x_2, \cdots, x_n]$. Then the ring $R = S/Ss$, where $s = \sum_{i=1}^{n} x_i^2 - 1$, is a commutative Noetherian domain and the free R-module $M = \bigoplus_{i=1}^{n} R$ contains a direct summand which does not satisfy EC_{11}.*

Proof. It is clear that M_R satisfies EC_{11}. By the proof of [12, Example 4], $M = K \oplus K'$ for some submodules K, K' of M such that $K' \cong R$ and K is indecomposable. Since K has uniform dimension 2, Corollary 2.4 yields that K_R does not satisfy EC_{11} condition. □

Observe that the submodule K_R in the proof of Proposition 3.1 is a complement which is not an ec-closed submodule of M. In the rest of this note we deal with direct summands of an EC_{11}-module.

Lemma 3.2. *Let M be an EC_{11}-module and X a submodule. If the intersection of X with any direct summand of M is a direct summand of X, then X is an EC_{11}-module.*

Proof. Clear. □

Recall that a module M has *SIP* if the intersection of two direct summands of M is also a direct summand (see [15]).

Corollary 3.3. *Let M be an EC_{11}-module.*
(i) *If X is a submodule of M such that $eX \subseteq X$ for all $e^2 = e \in End(M_R)$, then X is an EC_{11}-module. In particular, every fully invariant submodule of M is an EC_{11}-module.*
(ii) *If M has SIP, then every direct summand of M has EC_{11}.*

Proof. (i) Let D be a direct summand of M and $e : M \to D$ be the canonical projection. By Lemma 3.2, X is an EC_{11}-module.
(ii) This part is an immediate consequence of Lemma 3.2. □

Lemma 3.4. *Let $M = M_1 \oplus M_2$. Then M_1 satisfies EC_{11} if and only if for every ec-submodule N of M_1, there exists a direct summand K of M such that $M_2 \subseteq K$, $K \cap N = 0$ and $K \oplus N$ is an essential submodule of M.*

Proof. Suppose M_1 satisfies EC_{11}. Let N be any ec-submodule of M_1. By Proposition 2.2, there exists a direct summand L of M_1 such that $N \cap L = 0$ and $N \oplus L$ is essential in M_1. Clearly, $(L \oplus M_2) \cap N = 0$ and $(L \oplus M_2 \oplus N)$ is essential in M. Conversely, suppose M_1 has the stated property. Let H be an ec-submodule of M_1. By hypothesis, there exists a direct summand K of M such that $M_2 \subseteq K$, $K \cap H = 0$ and $K \oplus H$ is an essential submodule of M. Now $K = K \cap (M_1 \oplus M_2) = (K \cap M_1) \oplus M_2$ so that $K \cap M_1$ is a direct summand of M, and hence also of M_1, $H \cap (K \cap M_1) = 0$, and $H \oplus (K \cap M_1) = M_1 \cap (H \oplus K)$ which is an essential submodule of M_1. By Proposition 2.2, M_1 satisfies EC_{11}. □

Theorem 3.5. *Let $M = M_1 \oplus M_2$ be an EC_{11}-module such that for every ec-submodule K of M with $K \cap M_2 = 0$, $K \oplus M_2$ is a direct summand of M. Then M_1 is an EC_{11}-module. In this case M_1 is a P-extending module.*

Proof. By Lemma 3.4, M_1 is an EC_{11}-module. For the second part, let K be an ec-submodule of M_1. Hence K is an ec-submodule of M with $K \cap M_2 = 0$. By hypothesis, $K \oplus M_2$ is a direct summand of M. Therefore K is a direct summand of M and hence also of M_1. It follows that M_1 is a P-extending module. □

Theorem 3.6. *Let M be an EC_{11}-module. If $SocM$ is cyclic then $M = M_1 \oplus M_2$ where M_1 is a submodule of M with essential socle and M_2 a submodule of M with zero socle.*

Proof. Let S denote the socle of M. By hypothesis, there exist submodules M_1 and M_2 of M such that $M = M_1 \oplus M_2$, $S \cap M_2 = 0$ and $S \oplus M_2$ is an essential submodule of M. So $S = SocM = SocM_1 \oplus SocM_2$. Clearly $SocM_2 = 0$ so that

$S \leq M_1$. Now $S \oplus M_2$ essential in M implies S essential in M_1. Thus we have the required decomposition. □

It is clear that for a module M $SocM$ is cyclic submodule if and only if it is an ec-submodule. Note that Theorem 3.6 holds true if we replace socle with any left exact preradical in the category of right R-modules. For the definition and basic properties of left exact preradicals, consult [13]. However, the converse of the Theorem 3.6 is not true, in general. We conclude with such a counterexample.

Exmaple 3.7. Let S be a commutative domain, which is not a field, and whose Jacobson radical $J(S) = 0$. Let V be a faithful semisimple S-module. Note that, since $J(S) = 0$, such a module exists and it has infinite Goldie dimension, because it should contain an infinite direct sum of pairwise non-isomorphic simple S-modules. Let $R = S(S,V) = \{\left[\begin{smallmatrix} s & v \\ 0 & s \end{smallmatrix}\right] : s \in S, v \in V\}$. Let I be the ideal of R, $I = S(0,V) = \{\left[\begin{smallmatrix} 0 & v \\ 0 & 0 \end{smallmatrix}\right] : v \in V\}$. Since V is faithful, I is an essential ideal of R. Thus R is a commutative ring with essential socle I. Let $M_1 = R$, $M_2 = R/I$ and $M = M_1 \oplus M_2$. Note that $SocM = I \oplus 0$ and $SocM_2 = 0$. Now, let N be any simple submodule of M. It is clear that N is an ec-submodule of M. By [14, Lemma 3.1], there is no direct summand L of M such that $L \cap N = 0$ and $L \oplus N$ is essential in M. Because this would imply that $L \oplus N$ contains $SocM$, and [14, Lemma 3.2], combined with the fact that $SocM$ is not simple, shows that this is impossible. It follows that M is not an EC_{11}-module.

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