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BL-Algebras Based on Soft Set Theory

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Abstract. Molodtsov introduced the concept of soft sets, which can be seen as a new mathematical tool for dealing with uncertainty. In this paper, we initiate the study of soft BL-algebras by using the soft set theory. The notion of filteristic soft BL-algebras is introduced and some related properties are investigated.

1. Introduction

To solve complicated problems in economics, engineering, and environment, we cannot successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties cannot be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which have been pointed out in [12]. Maji et al. [10] and Molodtsov [12] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [12] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, research on the soft set theory is progressing rapidly. Maji et al. [11] described the application of soft set theory to a decision making problem. They also studied several operations on the theory of soft sets. The most appro-

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priate theory for dealing with uncertainties is the theory of fuzzy sets developed by Zadeh [15]. The present author [4] applied the notion of soft sets by Molodtsov to the theory of BCK/BCI-algebras, and introduced the notions of soft BCK/BCI-algebras, and then investigated their basic properties [5]. Aktas et al. [1] studied the basic concepts of soft set theory, and compared soft sets to fuzzy and rough sets, providing some examples to clarify their differences. They also discussed the notion of soft groups.

The concept of BL-algebra was introduced by Hájek's as the algebraic structures for his Basic Logic [3]. A well known example of a BL-algebra is the interval [0,1] endowed with the structure induced by a continuous t-norm. On the other hand, the MV-algebras, introduced by Chang in 1958 (see [2]), are one of the most well known classes of BL-algebras. In order to investigate the logic system whose semantic truth-value is given by a lattice, Xu [13] proposed the concept of lattice implication algebras and studied the properties of filters in such algebras [14]. In fact, the MV-algebras, G*ö*del algebras and product algebras are the most known classes of BL-algebras. BL-algebras are further discussed by many researchers, see [6, 7, 8, 9, 16, 17, 18, 19].

In this paper, we initiate the study of soft BL-algebras by using the soft set theory. The notion of filteristic soft BL-algebras is introduced and some related properties are investigated.

2. Basic Results on BL-algebras

Recall that an algebra $L = (L, \leq, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a BL-algebra [3] if it is a bounded lattice such that the following conditions are satisfied:

(i) $(L, \odot, 1)$ is a commutative monoid,

(ii) $\odot$ and $\rightarrow$ form an adjoin pair, i.e., $z \leq x \rightarrow y$ if and only if $x \odot z \leq y$ for all $x, y, z \in L$,

(iii) $x \wedge y = x \odot (x \rightarrow y)$,

(iv) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

In what follows, L is a BL-algebra unless otherwise specified.

In any BL-algebra L, the following statements are true:

(1) $x \leq y \Leftrightarrow x \rightarrow y = 1$,

(2) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z = y \rightarrow (x \rightarrow z)$,

(3) $x \odot y \leq x \wedge y$,

(4) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y), x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$.

(5) $x \rightarrow x' = x'' \rightarrow x$,

(6) $x \vee x' = 1 \Rightarrow x \wedge x' = 0$,

(7) $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$,

where $x' = x \rightarrow 0$.

A non-empty subset A of L is called a *filter* of L if it satisfies the following conditions: (i) $1 \in A$; (ii) $\forall x \in A, y \in L, x \rightarrow y \in A \Rightarrow y \in A$. It is easy to check that a non-empty subset A of L is a filter of L if and only if it satisfies: (i) $\forall x, y \in L, x \odot y \in A$; (ii) $\forall x \in A, x \leq y \Rightarrow y \in A$.

3. Basic Results on soft sets

Molodtsov [12] defined the soft set in the following way: Let U be an initial universe set and E be a set of parameters. Let $\mathscr{P}(U)$ denotes the non-empty power set of U and $A \subset E$.

Definition 3.1([12]). A pair $(\mathscr{F}, A)$ is called a *soft set* over U, where $\mathscr{F}$ is a mapping given by

$$\mathscr{F} : A \to \mathscr{P}(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $\mathscr{F}(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set $(\mathscr{F}, A)$. Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [12].

Definition 3.2([10]). Let $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ be two soft sets over a common universe U. The *intersection* of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ is defined to be the soft set $(\mathscr{H}, C)$ satisfying the following conditions:

(i) $C = A \cap B$,

(ii) $(\forall e \in C)$ ($\mathscr{H}(e) = \mathscr{F}(e)$ or $\mathscr{H}(e) = \mathscr{G}(e)$, (as both are same set)).

In this case, we write $(\mathscr{F}, A)\widetilde{\cap}(\mathscr{G}, B) = (\mathscr{H}, C)$.

Definition 3.3([10]). Let $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ be two soft sets over a common universe U. The *union* of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ is defined to be the soft set $(\mathscr{H}, C)$ satisfying the following conditions:

(i) $C = A \cup B$,

(ii) for all $e \in C$,

$$\mathscr{H}(e) = \begin{cases} \mathscr{F}(e) & \text{if } e \in A \setminus B, \\ \mathscr{G}(e) & \text{if } e \in B \setminus A, \\ \mathscr{F}(e) \cup \mathscr{G}(e) & \text{if } e \in A \cap B. \end{cases}$$

In this case, we write $(\mathscr{F}, A)\widetilde{\cup}(\mathscr{G}, B) = (\mathscr{H}, C)$.

Definition 3.4([10]). If $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ are two soft sets over a common universe U, then "$(\mathscr{F}, A)$ *AND* $(\mathscr{G}, B)$" denoted by $(\mathscr{F}, A)\widetilde{\wedge}(\mathscr{G}, B)$ is defined by $(\mathscr{F}, A)\widetilde{\wedge}(\mathscr{G}, B) = (\mathscr{H}, A \times B)$, where $\mathscr{H}(\alpha, \beta) = \mathscr{F}(\alpha) \cap \mathscr{G}(\beta)$ for all $(\alpha, \beta) \in A \times B$.

Definition 3.5([10]). If $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ are two soft sets over a common universe U, then "$(\mathscr{F}, A)$ *OR* $(\mathscr{G}, B)$" denoted by $(\mathscr{F}, A)\widetilde{\vee}(\mathscr{G}, B)$ is defined by $(\mathscr{F}, A)\widetilde{\vee}(\mathscr{G}, B) = (\mathscr{H}, A \times B)$, where $\mathscr{H}(\alpha, \beta) = \mathscr{F}(\alpha) \cup \mathscr{G}(\beta)$ for all $(\alpha, \beta) \in A \times B$.

Definition 3.6([10]). For two soft sets $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ over a common universe

U, we say that $(\mathscr{F}, A)$ is a *soft subset* of $(\mathscr{G}, B)$, denoted by $(\mathscr{F}, A) \widetilde{\subset} (\mathscr{G}, B)$, if it satisfies:

(i) $A \subset B$,

(ii) For every $\varepsilon \in A$, $\mathscr{F}(\varepsilon)$ and $\mathscr{G}(\varepsilon)$ are identical approximations.

The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets developed by Zadeh [15].

4. Filteristic soft BL-algebras

In what follows let L and A be a BL-algebra and a non-empty set, respectively, and R will refer to an arbitrary binary relation between an element of A and an element of L, that is, R is a subset of $A \times L$ without otherwise specified. A set-valued function $\mathscr{F} : A \to \mathscr{P}(L)$ can be defined as $\mathscr{F}(x) = \{y \in L \mid (x, y) \in R\}$ for all $x \in A$. The pair $(\mathscr{F}, A)$ is then a soft set over L.

For a soft set $(\mathscr{F}, A)$, the set

$$\operatorname{supp}(\mathscr{F}, A) := \{x \in A \mid \mathscr{F}(x) \neq \emptyset\}$$

is called the *support* of $(\mathscr{F}, A)$. Thus a null soft set is indeed a soft set with an empty support, and we say that a soft set $(\mathscr{F}, A)$ is *non-null* if $\operatorname{supp}(\mathscr{F}, A) \neq \emptyset$.

Definition 4.1. Let $(\mathscr{F}, A)$ be a non-null soft set over L. Then $(\mathscr{F}, A)$ is called a *filteristic soft BL-algebra* over L if $\mathscr{F}(x)$ is a filter of L for all $x \in \operatorname{supp}(\mathscr{F}, A)$.

Example 4.2. Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$. Then we define $x \wedge y = \min\{x, y\}, x \vee y = \max\{x, y\}$, and $\odot$ and $\to$ as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

x It is clear that $(L, \wedge, \vee, \odot, \to, 1)$ is now a BL-algebra. Let $(\mathscr{F}, A)$ be a soft set over L, where $A = (0, 1]$ and $\mathscr{F} : A \to \mathcal{P}(L)$ is a set-valued function defined by

$$\mathscr{F}(x) = \begin{cases} \{0, a, b, 1\} & \text{if } 0 < x \leq 0.3, \\ \{1, b\} & \text{if } 0.3 < t \leq 0.6, \\ \{1\} & \text{if } 0.6 < t \leq 0.8, \\ \emptyset & \text{if } 0.8 < t \leq 1. \end{cases}$$

Thus, $\mathscr{F}(x)$ is a filter of L for all $x \in A$, and so $(\mathscr{F}, A)$ is a filteristic soft BL-algebra over L.

Theorem 4.3. *Let $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ be filteristic soft BL-algebras over L. Then* *the set* $(\mathscr{F},A)\widetilde{\wedge}(\mathscr{G},B)$ *is a filteristic soft BL-algebra over* L *when it is non-null.*

Proof. Note from Definition 3.4 that $(\mathscr{F},A)\widetilde{\wedge}(\mathscr{G},B) = (\mathscr{H},A\times B)$, where $\mathscr{H}(x,y) = \mathscr{F}(x)\cap\mathscr{G}(y)$ for all $(x,y)\in A\times B$. Let $(x,y)\in \text{supp}(\mathscr{H},A\times B)$. Then $\mathscr{H}(x,y) = \mathscr{F}(x)\cap\mathscr{G}(y)\neq\emptyset$, and is a filter of L for all $(x,y)\in(\text{supp}\mathscr{H},A\times B)$. Hence $(\mathscr{F},A)\widetilde{\wedge}(\mathscr{G},B)$ is a filteristic soft BL-algebra over L. □

Theorem 4.4. *Let* $(\mathscr{F},A)$ *be a filteristic soft BL-algebra over* L*. If* B *is a subset of* A*, then* $(\mathscr{F}|_B,B)$ *is a filteristic soft BL-algebra over* L*.*

Proof. Straightforward. □

Theorem 4.5. *Let* $(\mathscr{F},A)$ *and* $(\mathscr{G},B)$ *be two filteristic soft BL-algebras over* L*. If* $A\cap B\neq\emptyset$*, then the intersection* $(\mathscr{F},A)\widetilde{\cap}(\mathscr{G},B)$ *is a filteristic soft BL-algebra over* L*.*

Proof. Using Definition 3.2, we can write $(\mathscr{F},A)\widetilde{\cap}(\mathscr{G},B) = (\mathscr{H},C)$, where $C = A\cap B$ and $\mathscr{H}(x) = \mathscr{F}(x)$ or $\mathscr{G}(x)$ for all $x\in C$. Note that $\mathscr{H}: C\to\mathscr{P}(L)$ is a mapping, and therefore $(\mathscr{H},C)$ is a soft set over X. Since $(\mathscr{F},A)$ and $(\mathscr{G},B)$ are filteristic soft BL-algebras over L, it follows that $\mathscr{H}(x) = \mathscr{F}(x)$ is a filter of L, or $\mathscr{H}(x) = \mathscr{G}(x)$ is a filter of L for all $x\in C$. Hence $(\mathscr{H},C) = (\mathscr{F},A)\widetilde{\cap}(\mathscr{G},B)$ is a filteristic soft BL-algebra over L. □

Theorem 4.6. *Let* $(\mathscr{F},A)$ *and* $(\mathscr{G},A)$ *be two filteristic soft BL-algebras over* L*. Then their intersection* $(\mathscr{F},A)\widetilde{\cap}(\mathscr{G},A)$ *is a filteristic soft BL-algebra over* L*.*

Proof. Straightforward. □

Theorem 4.7. *Let* $(\mathscr{F},A)$ *and* $(\mathscr{G},A)$ *be two filteristic soft BL-algebras over* L*. If* A *and* B *are disjoint, then the union* $(\mathscr{F},A)\widetilde{\cup}(\mathscr{G},A)$ *is a filteristic soft BL-algebra over* L*.*

Proof. Using Definition 3.3, we can write $(\mathscr{F},A)\widetilde{\cup}(\mathscr{G},B) = (\mathscr{H},C)$, where $C = A\cup B$ and for every $e\in C$,

$$\mathscr{H}(e) = \begin{cases} \mathscr{F}(e) & \text{if } e\in A\setminus B,\\ \mathscr{G}(e) & \text{if } e\in B\setminus A,\\ \mathscr{F}(e)\cup\mathscr{G}(e) & \text{if } e\in A\cap B.\end{cases}$$

Since $A\cap B=\emptyset$, either $x\in A\setminus B$ or $x\in B\setminus A$ for all $x\in C$. If $x\in A\setminus B$, then $\mathscr{H}(x) = \mathscr{F}(x)$ is a filter of L since $(\mathscr{F},A)$ is a filteristic soft BL-algebra over L. If $x\in B\setminus A$, then $\mathscr{H}(x) = \mathscr{G}(x)$ is a filter of L since $(\mathscr{G},B)$ is a filteristic soft BL-algebra over L. Hence $(\mathscr{H},C) = (\mathscr{F},A)\widetilde{\cup}(\mathscr{G},A)$ is a filteristic soft BL-algebra over L. □

Theorem 4.8. *If* $(\mathscr{F},A)$ *and* $(\mathscr{G},B)$ *are filteristic soft BL-algebras over* L*, then* $(\mathscr{F},A)\widetilde{\wedge}(\mathscr{G},B)$ *is a filteristic soft BL-algebra over* L*.*

Proof. By means of Definition 3.4, we know that

$$(\mathscr{F},A)\widetilde{\wedge}(\mathscr{G},B) = (\mathscr{H},A\times B),$$

where $\mathscr{H}(x, y) = \mathscr{F}(x) \cap \mathscr{G}(y)$ for all $(x, y) \in A \times B$. Since $\mathscr{F}(x)$ and $\mathscr{G}(y)$ are filters of L, the intersection $\mathscr{F}(x) \cap \mathscr{G}(y)$ is also a filter of L. Hence $\mathscr{H}(x, y)$ is a filter of L for all $(x, y) \in A \times B$, and therefore $(\mathscr{F}, A)\widetilde{\wedge}(\mathscr{G}, B) = (\mathscr{H}, A \times B)$ is a filteristic soft BL-algebra over L. □

5. Conclusions

In this paper, we apply soft set theory to BL-algebras. We hope that the research along this direction can be continued, and in fact, some results in this paper have already constituted a platform for further discussion concerning the future development of soft BL-algebras and other algebraic structure. In our future, we will consider soft BL-algebras based on fuzzy set theory.

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