KYUNGPOOK Math. J. 50(2010), 153-164

The Dynamics of Solutions to the Equation $x_{n+1}=\frac{p+x_{n-k}}{q+x_n}+\frac{x_{n-k}}{x_n}$

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Abstract. We study the global asymptotic stability, the character of the semicycles, the periodic nature and oscillation of the positive solutions of the difference equation

$$x_{n+1} = \frac{p + x_{n-k}}{q + x_n} + \frac{x_{n-k}}{x_n},\ n = 0, 1, 2, \cdots .$$

where $p, q \in (0, \infty), q \neq 2, k \in \{1, 2, \cdots\}$ and the initial values $x_{-k}, \cdots, x_0$ are arbitrary positive real numbers.

1. Introduction

During the last ten years there has been a fascination with discovering nonlinear difference equations of order greater than one. Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, ecology, physiology, physics, engineering and economics. Some nonlinear difference equations, especially the boundedness, global attractivity, oscillatory and some other properties of second order nonlinear difference equations have been investigated by many authors, see [1-6]. In particular, A. M. Amleh et al. [1] studied the global stability, the periodic character of solutions of the equation:

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n},\ n = 0, 1, 2, \cdots .$$

where $\alpha \in [0, \infty)$ and the initial values x_{-1}, x_0 are arbitrary positive real numbers.

H. M. E. Owaidy et al. [5] investigated the following equation:

$$x_{n+1} = \alpha + \frac{x_{n-k}}{x_n},\ n = 0, 1, 2, \cdots .$$

where $\alpha \in [1, \infty),\ k \in \{1, 2, \cdots\}$ and the initial values $x_{-k}, \cdots, x_0$ are arbitrary positive real numbers.

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* Corresponding Author.
Received July 6, 2009; accepted October 9, 2009.
2000 Mathematics Subject Classification: 39A10, 39A11.
Key words and phrases: Difference equations, Asymptotic stability, Periodicity, Semicycle, Oscillation.

Later, M. Saleh et al. [9] mainly studied the character of the semicycles, the asymptotic stability of solutions of the equation:

$$y_{n+1} = A + \frac{y_{n-k}}{y_n},\ n = 0, 1, 2, \cdots .$$

where $A \in (0, \infty),\ k \in \{2, 3, \cdots\}$ and the initial values x_{-1}, x_0 are arbitrary positive real numbers.

Recently, S. Ozen, et al. [8] studied the global stability, the periodic nature and the persistence of solutions of the equation:

$$x_{n+1} = \frac{\alpha + x_{n-1}}{\beta + x_n} + \frac{x_{n-1}}{x_n},\ n = 0, 1, 2, \cdots .$$

where $\alpha, \beta \in (0, \infty),\ \alpha \neq \beta,\ \beta \neq 2$ and the initial values x_{-1}, x_0 are arbitrary positive real numbers.

Motivated by the previous works, this paper addresses the difference equation:

$$x_{n+1} = \frac{p + x_{n-k}}{q + x_n} + \frac{x_{n-k}}{x_n},\ n = 0, 1, 2, \cdots . \tag{1}$$

where $p, q \in (0, \infty), q \neq 2,\ k \in \{1, 2, \cdots\}$ and the initial values $x_{-k}, \cdots, x_0$ are arbitrary positive real numbers.

We first recall some results which will be useful in the sequel.

Lemma 1.1([4]). *Assume that* $a, b \in R,\ k \in \{1, 2, \cdots\}$. *Then*

$$|a| + |b| < 1. \tag{2}$$

is a sufficient condition for asymptotic stability of the equation

$$x_{n+1} - ax_n + bx_{n-k} = 0,\ n = 0, 1, \cdots . \tag{3}$$

Suppose in addition that one of the following two cases hold.

(*a*) k *odd and* $b < 0$.

(*b*) k *even and* $ab < 0$.

Then (2) *is also a necessary condition for the asymptotic stability of Eq.*(3).

Lemma 1.2([4]). *Consider the difference equation*

$$x_{n+1} = f(x_n, x_{n-k}), \quad n = 0, 1, 2, \cdots . \tag{4}$$

where $k \in \{1, 2, \cdots\}$. *Let* $I = [a, b]$ *be some interval of real numbers and assume that*

$$f :\ [a, b] \times [a, b] \to [a, b].$$

is a continuous function which satisfies the following properties:

(*a*) $f(x, y)$ *nonincreasing in* x *and nondecreasing in* y;

(b) If (m, M) is the solution of the system

$$f(m, M) = m, f(M, m) = M$$

then $m = M$.
Then the Eq.(4) has a unique positive equilibrium $\bar{x}$ and every solution converges to $\bar{x}$.

2. Local asymptotic stability

Let $\bar{x}$ be the positive equilibrium of Eq.(1). Consider the function $f(x, y) = (p + y)/(q + x) + y/x$. The equilibrium points are solutions of the equation

$$\bar{x}^2 + (q - 2)\bar{x} - (p + q) = 0.$$

So the unique positive equilibrium point of Eq.(1) is

$$\bar{x} = \frac{-(q-2) + \sqrt{(q-2)^2 + 4(p+q)}}{2}.$$

The linearized equation associated with Eq.(1) about equilibrium $\bar{x}$ is

$$(5) \qquad y_{n+1} + \frac{\bar{x}^2 + q}{\bar{x}(q + \bar{x})} y_n - \frac{2\bar{x} + q}{\bar{x}(q + \bar{x})} y_{n-k} = 0, \ n = 0, 1, 2, \cdots .$$

Hence its characteristic equation is

$$\lambda^{k+1} + \frac{\bar{x}^2 + q}{\bar{x}(q + \bar{x})} \lambda^k - \frac{2\bar{x} + q}{\bar{x}(q + \bar{x})} = 0, \ n = 0, 1, 2, \cdots .$$

By $\bar{x} = f(\bar{x}, \bar{x})$, we obtain

$$(q + \bar{x})(\bar{x} - 1) = p + \bar{x}.$$

So the positive equilibrium $\bar{x}$ satisfies $\bar{x} > 1$.

From here we obtain the following result.

Theorem 2.1. *Let $\bar{x}$ be the positive equilibrium point of Eq.(1), then the following statements are true:*
(a) Suppose that $p \geq q$. Then $2 \leq \bar{x} \leq p/q + 1$.
(b) Suppose that $p < q$. Than $p/q + 1 < \bar{x} < 2$.

Proof. Since $\bar{x}$ be the positive equilibrium point of Eq.(1), we have $\bar{x} = f(\bar{x}, \bar{x})$, thus

$$\bar{x} = \frac{p + \bar{x}}{q + \bar{x}} + 1.$$

Suppose that $p \geq q$, then

$$2 \leq \bar{x} \leq p/q + 1.$$

Suppose that $p < q$, then

$$\bar{x} = \frac{-(q-2) + \sqrt{(q-2)^2 + 4(p+q)}}{2} < \frac{-(q-2) + \sqrt{(q+2)^2}}{2} < 2.$$

Thus

$$\frac{p}{q} + 1 < \bar{x} < 2.$$

The proof is complete. □

Theorem 2.2. *Assume that $p, q \in R$ and $k \in \{0, 1, \cdots\}$. Then*

$$q > 2 \text{ and } p > 4q^2/(q-2)^2 + q \tag{6}$$

is a sufficient condition that the positive equilibrium $\bar{x}$ of eq.(1) is locally asymptotically stable. Suppose in addition that k is odd, then (6) is also a necessary condition for locally asymptotic stability of eq.(1).

Proof. From (5), we have

$$|a| + |b| = |-\frac{\bar{x}^2 + q}{\bar{x}(q+\bar{x})}| + |-\frac{2\bar{x} + q}{\bar{x}(q+\bar{x})}|.$$

where $\bar{x}^2 + q > 0,\ \bar{x}(q+\bar{x}) > 0,\ 2\bar{x} + q > 0.$

By $|a| + |b| < 1$, we get

$$\frac{\bar{x}^2 + q}{\bar{x}(q+\bar{x})} + \frac{2\bar{x} + q}{\bar{x}(q+\bar{x})} < 1.$$

and easy computations give

$$2q < (q-2)\bar{x}.$$

Observe that $\bar{x} = \left(-(q-2) + \sqrt{(q-2)^2 + 4(p+q)}\right)/2$, we obtain

$$2q < (q-2)\frac{-(q-2) + \sqrt{(q-2)^2 + 4(p+q)}}{2}.$$

so

$$q(q-2)^2 + 4q^2 < p(q-2)^2.$$

which implies that

$$p > \frac{4q^2}{(q-2)^2} + q.$$

Since $b = -(2\bar{x} + q)/(\bar{x}(q+\bar{x})) < 0$, by Lemma 1, it is easy to see that $p > 4q^2/(q-2)^2 + q$ is necessary. □

3. Periodic nature

In this section, we will discuss the periodic nature of the positive solutions of Eq.(1).

Theorem 3.1. (1) *If k is even, then Eq.(1) has no period 2 solutions.*

(2) *If k is odd, and $q > 2, p \geq 4q^2/(q-2)^2+q$, then Eq.(1) has period 2 solutions.*

Proof. (1) If k is even, let

$$\cdots, \varphi, \psi, \varphi, \psi, \cdots, \varphi, \psi, \cdots$$

be a period 2 solution of Eq.(1), then

$$\varphi = \frac{p+\psi}{q+\psi} + 1 \tag{7}$$

$$\psi = \frac{p+\varphi}{q+\varphi} + 1. \tag{8}$$

Thus

$$q\varphi + \varphi\psi = p + q + 2\psi,$$
$$q\psi + \varphi\psi = p + q + 2\varphi.$$

Subtracting the relations above, we have

$$(\varphi - \psi)(q+2) = 0.$$

It is easy to see that $\varphi = \psi$ if $q+2 > 0$, which is a contradiction. So if k is even then Eq.(1) has no period 2 solutions.

(2) If k is odd, let

$$\cdots, \varphi, \psi, \varphi, \psi, \cdots, \varphi, \psi, \cdots$$

be a period 2 solution of Eq.(1), then

$$\varphi = \frac{p+\varphi}{q+\psi} + \frac{\varphi}{\psi} \tag{9}$$

$$\psi = \frac{p+\psi}{q+\varphi} + \frac{\psi}{\varphi}. \tag{10}$$

Thus

$$q\varphi\psi + \varphi\psi^2 = p\psi + q\varphi + 2\varphi\psi. \tag{11}$$

$$q\varphi\psi + \varphi^2\psi = p\varphi + q\psi + 2\varphi\psi.$$

Subtracting the relations above, we have

$$(\psi-\varphi)(\varphi\psi+q-p)=0.$$

so $\varphi\psi=p-q>0$, hence $\psi=(p-q)/\varphi$.

From (11), we get

$$q\varphi^2+A(2-q)\varphi+Aq=0.$$

where $A=p-q>0$.

And

$$\varphi=\frac{A(q-2)\pm A\sqrt{(q-2)^2-4q^2/A}}{2q}.$$

We proceed by treating three possible cases, respectively.

Case 1: $(q-2)^2-4q^2/A<0$. We have $q<p<4q^2/(q-2)^2+q$, it is impossible, hence Eq.(1) has no period 2 solutions.

Case 2: $(q-2)^2-4q^2/A\geq 0$. We have $p\geq 4q^2/(q-2)^2+q$, in addition that $q>2$, then Eq.(1) has period 2 solutions, which must be of the form

$$\cdots,\frac{A(q-2)+A\sqrt{(q-2)^2-4q^2/A}}{2q},\ \frac{A(q-2)-A\sqrt{(q-2)^2-4q^2/A}}{2q},\cdots$$

Case 3: $q<2$.

Since $A(q-2)<0$, we have $\varphi=\left(A(q-2)\pm A\sqrt{(q-2)^2-4q^2/A}\right)/(2q)<0$, which is a contradiction.

The proof is complete. □

4. Semicycle analysis and global asymptotic stability

The method of semicycles analysis is very useful in consideration of positive solutions. So in this section, we will analyze the character of the semicycles.

Theorem 4.1. (1) *Suppose k is odd, either*

$$x_{-k},\ x_{2-k},\cdots,x_{-1}<\bar{x},\quad x_{1-k},\ x_{3-k},\cdots,x_0>\bar{x} \tag{12}$$

or

$$x_{-k},\ x_{2-k},\cdots,x_{-1}>\bar{x},\quad x_{1-k},\ x_{3-k},\cdots,x_0<\bar{x}. \tag{13}$$

Then every solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1) strictly oscillates about $\bar{x}$, and has semicycles of length one.

(2) *Let $p\neq q$, if the solutions of Eq.(1) have only one semicycle, then every solution of Eq.(1) converges to the positive equilibrium $\bar{x}$, which means $\bar{x}$ is a global attractor.*

Proof. (1) We consider only the case (12), the other case is similar and will be omitted.

We complete the proof by induction. Firstly, we notice that

$$x_1 = f(x_0, x_{-k}) < f(\bar{x}, \bar{x}) = \bar{x},$$
$$x_2 = f(x_1, x_{1-k}) > f(\bar{x}, \bar{x}) = \bar{x}.$$

where the function $f(x, y)$ decreases in x and increases in y.

If there exists some integer $N > 0$, such that $x_{2n-1} < \bar{x},\ x_{2n} > \bar{x}$ if $n < N$, then

$$x_{2N-1} = f(x_{2N-2}, x_{2N-2-k}) < f(\bar{x}, \bar{x}) = \bar{x},$$
$$x_{2N} = f(x_{2N-1}, x_{2N-1-k}) > f(\bar{x}, \bar{x}) = \bar{x}.$$

By induction, we have completed the proof.

(2) We consider only the case of positive semicycle, the one of the negative semicycle is similar and is omitted.

Suppose the solution of Eq.(1) has only one positive semicycle, i.e. for any $n \geq -k,\ x_n \geq \bar{x}$. We will complete the proof in two cases:

Case 1: $p < q$.

By Theorem 2.1, if $p < q$, then $p/q + 1 < \bar{x} < 2$. Set $I = [1 + p/q, \infty)$, then $x_n \in I$ if $n \geq -k$. Consider the function $f(x, y) = (p+y)/(q+x)+y/x$, it decreases in x and increases in y.

By Theorem 3.1, if $p < q$, then for any k, Eq.(1) has no period 2 solutions, which implies that if (m, M) is a solution of the system $f(m, M) = m, f(M, m) = M$, we have $m = M$.

By Lemma 1.2, Eq.(1) has a unique positive equilibrium $\bar{x}$, and every solution converges to $\bar{x}$.

This completes the proof.

Case 2: $p > q$.

Firstly, we will prove $x_{n-k} \geq x_n$. For the sake of contradiction, consider

$$x_{n+1} = \frac{p + x_{n-k}}{q + x_n} + \frac{x_{n-k}}{x_n} \geq \bar{x}.$$

Because of the monotonicity of the function $f(x, y)$, we have

$$f(\bar{x}, \bar{x}) = \bar{x} \leq x_{n+1} = f(x_n, x_{n-k}) < f(x_n, x_n).$$

When $p > q$, the function $f(x, x)$ is decreasing. From the inequality $f(\bar{x}, \bar{x}) < f(x_n, x_n)$, we have $\bar{x} > x_n$, which is a contradiction.

Consider the subsequences of $\{x_{i+kt}\}_{t=-1}^{\infty}$, where $i \in \{0, 1, \cdots, k-1\}$. Then every subsequence is decreasing and bounded from below, so there exists some constant L_i, such that

$$\lim_{t\to\infty} x_{i+kt} = L_i \geq \bar{x}, \text{ for any } i \in \{0, 1, \cdots, k-1\}.$$

Next, we will prove $L_i = L_j$ for $i \neq j$. Taking limits on both sides of Eq.(1), we obtain

$$\begin{aligned} L_0 &= \frac{p + L_{k-1}}{q + L_{k-1}} + 1, \\ L_1 &= \frac{p + L_0}{q + L_0} + 1, \\ \vdots & \quad \vdots \\ L_{k-1} &= \frac{p + L_{k-2}}{q + L_{k-2}} + 1. \end{aligned}$$

so

$$L_0 = L_1 = \cdots = L_{k-1} = \bar{x}.$$

Thus every subsequence $\{x_{i+kt}\}_{t=-1}^{\infty}$ converges to $\bar{x}$, hence the solution $\{x_n\}_{n=-k}^{\infty}$ converges to $\bar{x}$, which means $\bar{x}$ is a global attractor.

This completes the proof. □

Especially, if $k = 1$, Eq.(1) is changed into the one in Ozen [8].

$$x_{n+1} = \frac{p + x_{n-1}}{q + x_n} + \frac{x_{n-1}}{x_n}, \ n = 0, 1, 2, \cdots . \tag{14}$$

Corollary 4.2. (1) *If Eq.*(14) *has only one semicycle, then the positive equilibrium* $\bar{x}$ *is a global attractor; moreover, when* $p > 4q^2/(q-2)^2 + q, \bar{x}$ *is globally asymptotically stable;*

(2) *If the initial values of Eq.*(14) *satisfy the condition* $x_{-1} < \bar{x} < x_0$*, then* $x_{2k-1} < \bar{x} < x_{2k}$*, for any* $k \in \{0, 1, 2, \cdots\}$*;*

(3) *If the initial values of Eq.*(14) *satisfy the condition* $x_{-1} > \bar{x} > x_0$*, then* $x_{2k-1} > \bar{x} > x_{2k}$*, for any* $k \in \{0, 1, 2, \cdots\}$*;*

(4) *If Eq.*(14) *has at least two semicycles, then the solution* $\{x_n\}_{n=-1}^{\infty}$ *oscillates, moreover, except the first semicycle, every semicycle has length of one.*

For the global asymptotic stability, we have the following theorem.

Theorem 4.3. *When* $p > 4q^2/(q-2)^2 + q$*, if Eq.(1) has only one semicycle, then the unique positive equilibrium* $\bar{x}$ *is globally asymptotically stable.*

Proof. By Theorem 2.2, when $p > 4q^2/(q-2)^2 + q$, the positive equilibrium $\bar{x}$ is locally asymptotically stable. By Theorem 4.1, when $p \neq q$, if Eq.(1) has only one semicycle, the positive equilibrium $\bar{x}$ is a global attractor. This means when $p > 4q^2/(q-2)^2 + q$, if Eq.(1) has only one semicycle, then the unique positive equilibrium $\bar{x}$ is globally asymptotically stable.

This completes the proof. □

5. Question and Discussions

Question: If Eq.(1) has more than one semicycle, under what condition is the positive equilibrium a global attractor? Moreover, under what condition is the positive equilibrium globally asymptotically stable?

Discussion 1: Firstly, let $k = 1$ in Eq.(1), then we the following result by computation.

If the initial values strictly oscillate about the positive equilibrium, then the solutions strictly oscillate, which shows the positive equilibrium is not a global attractor. So we can present a conjecture as follows:

Conjecture: If the initial values of Eq.(1) strictly oscillate, then the solutions strictly oscillate, and have two different cases: either, the odd subsequence converges to 0, and the even subsequence converges to ∞; or, the odd subsequence converges to ∞, and the even subsequence converges to 0.

Here is some date when $k = 1$, which is concerned with Eq.(14):

Case 1: $p = 3,\ q = 1,\ 2.5 < \bar{x} < 3$, set $x_{-1} = 2,\ x_0 = 4$, which satisfy $4q^2/(q-2)^2 + q > p > q,\ x_{-1} < \bar{x} < x_0$.

The following date shows that the odd subsequence converges to 0, and the even subsequence converges to ∞.

$x_1 : 1.5$	$x_2 : 5.46667$	$x_3 : 0.970267$	$x_4 : 9.93141$
$x_5 : 0.460895$	$x_6 : 30.3998$	$x_7 : 0.125381$	$x_8 : 272.137$
$x_9 : 0.0119033$	$x_{10} : 23134.3$	$x_{11} : 0.000130701$	$x_{12} : 1.77026e+008$
$x_{13} : 1.69482e-008$	$x_{14} : 1.04451e+016$	$x_{15} : 2.87216e-016$	$x_{16} : 3.63668e+031$
$x_{17} : 8.24929e-032$	$x_{18} : 4.40847e+062$	$x_{19} : 6.80508e-063$	$x_{20} : 6.4782e+124$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x_{33} : 0$	$x_{34} : \infty$	$x_{35} : 0$	$x_{36} : \infty$
$x_{37} : 0$	$x_{38} : \infty$	$x_{39} : 0$	$x_{40} : \infty$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Case 2: $p = 2,\ q = 3,\ 1.5 < \bar{x} < 2$, set $x_{-1} = 1,\ x_0 = 3$, which satisfy $q > p,\ x_{-1} < \bar{x} < x_0$.

The following date shows that the odd subsequence converges to 0, and the even subsequence converges to ∞.

$x_1: 0.833333$	$x_2: 4.90435$	$x_3: 0.52837$	$x_4: 11.2388$
$x_5: 0.224581$	$x_6: 54.1492$	$x_7: 0.0430733$	$x_8: 1275.59$
$x_9: 0.00163168$	$x_{10}: 782192$	$x_{11}: 2.56108e-006$	$x_{12}: 3.05415e+011$
$x_{13}: 6.54848e-012$	$x_{14}: 4.66391e+022$	$x_{15}: 4.28825e-023$	$x_{16}: 1.0876e+045$
$x_{17}: 1.83891e-045$	$x_{18}: 5.9144e+089$	$x_{19}: 3.38158e-090$	$x_{20}: 1.74901e+179$
$x_{21}: 1.14351e-179$	$x_{22}: \infty$	$x_{23}: 0$	$x_{24}: \infty$
$x_{25}: 0$	$x_{26}: \infty$	$x_{27}: 0$	$x_{28}: \infty$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Case 3: $p = 40,\ q = 3,\ 6 < \bar{x} < 6.5$, set $x_{-1} = 3,\ x_0 = 8$, which satisfy $p \geq 4q^2/(q-2)^2 + q,\ x_{-1} < \bar{x} < x_0$.

The following date shows that the odd subsequence converges to 0, and the even subsequence converges to ∞.

x_1 : 4.28409	x_2 : 8.45708	x_3 : 4.37179	x_4 : 8.50778
x_5 : 4.36966	x_6 : 8.5291	x_7 : 4.36082	x_8 : 8.54874
x_9 : 4.35129	x_{10} : 8.56875	x_{11} : 4.34152	x_{12} : 8.5893
x_{13} : 4.33153	x_{14} : 8.61041	x_{15} : 4.32131	x_{16} : 8.63212
x_{17} : 4.31086	x_{18} : 8.65445	x_{19} : 4.30016	x_{20} : 8.67743
x_{21} : 4.28922	x_{22} : 8.70109	x_{23} : 4.278	x_{24} : 8.72546
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{61} : 3.99676	x_{62} : 9.38889	x_{63} : 3.977	x_{64} : 9.43962
x_{65} : 3.95655	x_{66} : 9.49274	x_{67} : 3.93536	x_{68} : 9.54845
x_{69} : 3.9134	x_{70} : 9.60695	x_{71} : 3.89062	x_{72} : 9.66846
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{181} : 0	x_{182} : ∞	x_{183} : 0	x_{184} : ∞
x_{185} : 0	x_{186} : ∞	x_{187} : 0	x_{188} : ∞
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

In the three cases above, if $x_{-1} > \bar{x} > x_0$, we also have similar results, that is : either the odd subsequence converges to 0, and the even subsequence converges to ∞; or the odd subsequence converges to ∞, and the even subsequence converges to 0. The related date is omitted in here.

Discussion 2: We will discuss Eq.(1) in the case that the initial values lie in the same way about the positive equilibrium, at the same time, Eq.(1) has more than one semicycle.

Firstly, we take $k = 1$ in Eq.(1) to investigate Eq.(14), and have a result by computation.

When $p > 4q^2/(q-2)^2+q,\ x_{-1} > x_0 > \bar{x}$, or $p > 4q^2/(q-2)^2+q,\ x_{-1} < x_0 < \bar{x}$, then the positive equilibrium $\bar{x}$ is a global attractor.

So, we have the conjecture as follows:

Conjecture: When $p > 4q^2/(q-2)^2 + q,\ x_{-k} > \cdots > x_0 > \bar{x}$, or $p > 4q^2/(q-2)^2 + q,\ x_{-k} < \cdots < x_0 < \bar{x}$, then the positive equilibrium $\bar{x}$ is a global attractor.

Here is some date when $k = 1$, which is concerned with Eq.(14):

Case 1: $p = 40,\ q = 3,\ \bar{x} \approx 6.07647$, set $x_{-1} = 2,\ x_0 = 4$, which satisfy $p > 4q^2/(q-2)^2 + q,\ x_{-1} < x_0 < \bar{x}$.

$x_1 : 6.5$	$x_2 : 5.24696$	$x_3 : 6.87725$	$x_4 : 5.34387$
$x_5 : 6.90511$	$x_6 : 5.35173$	$x_7 : 6.90647$	$x_8 : 5.35288$
$x_9 : 6.90584$	$x_{10} : 5.35352$	$x_{11} : 6.90506$	$x_{12} : 5.35413$
$x_{13} : 6.90427$	$x_{14} : 5.35473$	$x_{15} : 6.90348$	$x_{16} : 5.35533$
$x_{17} : 6.90268$	$x_{18} : 5.35594$	$x_{19} : 6.90188$	$x_{20} : 5.35655$
$x_{21} : 6.90108$	$x_{22} : 5.35716$	$x_{23} : 6.90029$	$x_{24} : 5.35777$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x_{99901} : 6.07647$	$x_{99902} : 6.07647$	$x_{99903} : 6.07647$	$x_{99904} : 6.07647$
$x_{99905} : 6.07647$	$x_{99906} : 6.07647$	$x_{99907} : 6.07647$	$x_{99908} : 6.07647$
$x_{99909} : 6.07647$	$x_{99910} : 6.07647$	$x_{99911} : 6.07647$	$x_{99912} : 6.07647$
$x_{99913} : 6.07647$	$x_{99914} : 6.07647$	$x_{99915} : 6.07647$	$x_{99916} : 6.07647$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

However, if the initial values don't satisfy $x_{-1} < x_0 < \bar{x}$, the positive equilibrium $\bar{x}$ is not a global attractor in spite of $p > 4q^2/(q-2)^2+q$. The related date is omitted in here.

Case 2: $p = 40,\ q = 3,\ \bar{x} \approx 6.07647$, set $x_{-1} = 10,\ x_0 = 8$, which satisfy $p > 4q^2/(q-2)^2 + q,\ x_{-1} > x_0 > \bar{x}$.

This means, in the case $p > 4q^2/(q-2)^2 + q,\ x_{-1} > x_0 > \bar{x}$, the positive equilibrium $\bar{x}$ is a global attractor. The related date is omitted in here.

However, if the initial values don't satisfy $x_{-1} > x_0 > \bar{x}$, the positive equilibrium $\bar{x}$ is not a global attractor in spite of $p > 4q^2/(q-2)^2+q$. The related date is omitted in here.

Case 3: when $p < q$, or $q < p < 4q^2/(q-2)^2 + q$, the positive equilibrium $\bar{x}$ is not a global attractor, no matter how the initial values lie about $\bar{x}$. The related date is omitted in here.

Acknowledgements The work of this paper was supported by National Natural Science Foundation of China (10871213).

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