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Partial Sums of Starlike Harmonic Univalent Functions

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Abstract. Although, interesting properties on the partial sums of analytic univalent functions have been investigated extensively by several researchers, yet analogous results on partial sums of harmonic univalent functions have not been so far explored. The main purpose of the present paper is to establish some new and interesting results on the ratio of starlike harmonic univalent function to its sequences of partial sums.

1. Introduction

A continuous complex-valued function $f = u + iv$ is said to be harmonic in a simply connected domain D if both u and v are real harmonic in D. In any simply connected domain we can write$f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for$f$ to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|, z \in D$. See Clunie and Sheil-Small [2].

Denote by S_H the class of functions $f = h+\overline{g}$ which are harmonic univalent and sense-preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. For $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1.1}$$

For basic results on harmonic functions one may refer to the following standard introductory text book by Duren [4], see also Ahuja [1].

Note that S_H reduces to the class S of normalized analytic univalent functions if the co-analytic part of its member is zero.

A function$f$of the form (1.1) is harmonic starlike of order α, $0 \le \alpha < 1$, denoted by $S_H^*(\alpha)$, if it satisfies

$$Re\left\{\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right\} \ge \alpha, \quad z \in U.$$

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We further denote by $TS_H^*(\alpha)$ the subclass of $S_H^*(\alpha)$ such that functions h and g in $f = h + \overline{g}$ are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, g(z) = \sum_{k=1}^{\infty} |b_k| z^k, |b_1| < 1. \tag{1.2}$$

As shown recently by Jahangiri [7] a sufficient condition for a function of the form (1.1) to be in $S_H^*(\alpha)$ is that

$$\sum_{k=2}^{\infty} \frac{k-\alpha}{1-\alpha} |a_k| + \sum_{k=1}^{\infty} \frac{k+\alpha}{1-\alpha} |b_k| \leq 1. \tag{1.3}$$

For functions f of the form (1.2) above mentioned sufficient condition is also necessary. For detailed study see [7].

Several authors (e.g., see [3], [5], [6], [8], [9], [10], [12]) studied the partial sums of analytic univalent functions, yet analogous results on partial sums on harmonic univalent functions have not been so far explored. Motivated with the work of Silverman ([10], [11]) an attempt has been made to systematically study on the ratio of starlike harmonic univalent function to its sequences of partial sums.

We let the sequences of partial sums of functions of the form (1.1) with $b_1 = 0$ are

$$\begin{aligned} f_m(z) &= z + \sum_{k=2}^{m} a_k z^k + \sum_{k=2}^{\infty} \overline{b_k z^k}, f_n(z) \\ &= z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=2}^{n} \overline{b_k z^k}, f_{m,n}(z) \\ &= z + \sum_{k=2}^{m} a_k z^k + \sum_{k=2}^{n} \overline{b_k z^k}, \end{aligned}$$

when the coefficients of f are sufficiently small to satisfy the condition (1.3).

In the present paper, we determine sharp lower bounds for $Re\left\{\frac{f(z)}{f_m(z)}\right\}$, $Re\left\{\frac{f_m(z)}{f(z)}\right\}$, $Re\left\{\frac{f'(z)}{f'_m(z)}\right\}$, $Re\left\{\frac{f'_m(z)}{f'(z)}\right\}$, $Re\left\{\frac{f(z)}{f_n(z)}\right\}$, $Re\left\{\frac{f_n(z)}{f(z)}\right\}$, $Re\left\{\frac{f(z)}{f_{m,n}(z)}\right\}$, $Re\left\{\frac{f_{m,n}(z)}{f(z)}\right\}$, $Re\left\{\frac{f'(z)}{f'_{m,n}(z)}\right\}$ and $Re\left\{\frac{f'_{m,n}(z)}{f'(z)}\right\}$, where $f'(z) = \frac{\partial}{\partial\theta} f(re^{i\theta})$.

2. Main results

In our first theorem, we determine sharp lower bounds for $Re\left\{\frac{f(z)}{f_m(z)}\right\}$.

Theorem 2.1. *If f of the form* (1.1) *with $b_1 = 0$, satisfies condition* (1.3)*, then*

$$Re\left\{\frac{f(z)}{f_m(z)}\right\} \geq \frac{m}{m+1-\alpha}, \quad (z \in U) \tag{2.1}$$

The result (2.1) *is sharp with the function*

$$f(z) = z + \frac{1-\alpha}{m+1-\alpha} z^{m+1}. \tag{2.2}$$

Proof. We may write

$$\begin{aligned}
&\frac{1+\omega(z)}{1-\omega(z)} \\
&= \frac{m+1-\alpha}{1-\alpha}\left[\frac{f(re^{i\theta})}{f_m(re^{i\theta})} - \frac{m}{m+1-\alpha}\right] \\
&= \frac{1+\sum_{k=2}^{m} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{\infty} \overline{b_k} r^{k-1} e^{-i(k+1)\theta} + \frac{m+1-\alpha}{1-\alpha}\left[\sum_{k=m+1}^{\infty} a_k r^{k-1} e^{i(kv-1)\theta}\right]}{1+\sum_{k=2}^{m} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{\infty} \overline{b_k} r^{k-1} e^{-i(k+1)\theta}}.
\end{aligned}$$

So that

$$\begin{aligned}
&\omega(z) \\
&= \frac{\frac{m+1-\alpha}{1-\alpha}\left[\sum_{k=m+1}^{\infty} a_k r^{k-1} e^{i(k-1)\theta}\right]}{2+2\left(\sum_{k=2}^{m} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{\infty} \bar{b}_k r^{k-1} e^{-i(k+1)\theta}\right) + \frac{m+1-\alpha}{1-\alpha}\left(\sum_{k=m+1}^{\infty} a_k r^{k-1} e^{i(k-1)\theta}\right)}.
\end{aligned}$$

Then

$$|\omega(z)| \le \frac{\frac{m+1-\alpha}{1-\alpha}\left[\sum_{k=m+1}^{\infty} |a_k|\right]}{2-2\left(\sum_{k=2}^{m} |a_k| + \sum_{k=2}^{\infty} |b_k|\right) - \frac{m+1-\alpha}{1-\alpha}\left(\sum_{k=m+1}^{\infty} |a_k|\right)}.$$

This last expression is bounded above by 1, if and only if

$$\sum_{k=2}^{m} |a_k| + \sum_{k=2}^{\infty} |b_k| + \frac{m+1-\alpha}{1-\alpha}\left(\sum_{k=m+1}^{\infty} |a_k|\right) \le 1. \tag{2.3}$$

It suffices to show that L. H. S. of (2.3) is bounded above by $\sum_{k=2}^{\infty} \frac{k-\alpha}{1-\alpha}|a_k| + \sum_{k=2}^{\infty} \frac{k+\alpha}{1-\alpha}|b_k|$, which is equivalent to

$$\sum_{k=2}^{m} \frac{k-1}{1-\alpha}|a_k| + \sum_{k=2}^{\infty} \frac{k-1+2\alpha}{1-\alpha}|b_k| + \sum_{k=m+1}^{\infty} \frac{k-m-1}{1-\alpha}|a_k| \ge 0.$$

To see that $f(z)=z+\dfrac{1-\alpha}{m+1-\alpha}z^{m+1}$ gives the sharp result, we observe that for $z=re^{i\pi/m}$ that

$$\frac{f(z)}{f_m(z)}=1+\frac{1-\alpha}{m+1-\alpha}z^m\to 1-\frac{1-\alpha}{m+1-\alpha}=\frac{m}{m+1-\alpha},$$

when $r\to 1^-$. □

We next determine bounds for $Re\left\{\frac{f_m(z)}{f(z)}\right\}$.

Theorem 2.2. *If f of the form* (1.1) *with $b_1=0$, satisfies condition* (1.3)*, then*

$$Re\left\{\frac{f_m(z)}{f(z)}\right\}\geq\frac{m+1-\alpha}{m+2-2\alpha},\qquad (z\in U) \tag{2.4}$$

The result (2.4) *is sharp with the function given by* (2.2)*.*

Proof. To prove Theorem 2.2, we may write

$$\begin{aligned}&\frac{1+\omega(z)}{1-\omega(z)}\\&=\frac{m+2(1-\alpha)}{1-\alpha}\left[\frac{f_m(z)}{f(z)}-\frac{m+1-\alpha}{m+2(1-\alpha)}\right]\\&=\frac{1+\sum_{k=2}^{m}a_kr^{k-1}e^{i(k-1)\theta}+\sum_{k=2}^{\infty}\overline{b_k}r^{k-1}e^{-i(k+1)\theta}-\frac{m+1-\alpha}{1-\alpha}\left[\sum_{k=m+1}^{\infty}a_kr^{k-1}e^{i(k-1)\theta}\right]}{1+\sum_{k=2}^{\infty}a_kr^{k-1}e^{i(k-1)\theta}+\sum_{k=2}^{\infty}\overline{b_k}r^{k-1}e^{-i(k+1)\theta}},\end{aligned}$$

where

$$|\omega(z)|\leq\frac{\dfrac{m+2(1-\alpha)}{1-\alpha}\left[\sum_{k=m+1}^{\infty}|a_k|\right]}{2-2\left(\sum_{k=2}^{m}|a_k|+\sum_{k=2}^{\infty}|b_k|\right)-\dfrac{m}{1-\alpha}\left(\sum_{k=m+1}^{\infty}|a_k|\right)}\leq 1.$$

This last inequality is equivalent to

$$\sum_{k=2}^{m}|a_k|+\sum_{k=2}^{\infty}|b_k|+\frac{m+1-\alpha}{1-\alpha}\left(\sum_{k=m+1}^{\infty}|a_k|\right)\leq 1. \tag{2.5}$$

Since the L. H. S. of (2.5) is bounded above by $\sum_{k=2}^{\infty}\dfrac{k-\alpha}{1-\alpha}|a_k|+\sum_{k=2}^{\infty}\dfrac{k+\alpha}{1-\alpha}|b_k|$, the proof is evidently complete. □

We next turn to ratios for $Re\left\{\dfrac{f'(z)}{f'_m(z)}\right\}$ and $Re\left\{\dfrac{f'_m(z)}{f'(z)}\right\}$.

Theorem 2.3. *If f of the form* (1.1) *with $b_1 = 0$, satisfies condition* (1.3), *then*

$$Re\left\{\frac{f'(z)}{f'_m(z)}\right\} \geq \frac{\alpha m}{m+1-\alpha}, \qquad (z \in U). \tag{2.6}$$

The result (2.6) *is sharp with the function given by* (2.2).

Proof. We may write

$$\begin{aligned}
&\frac{1+\omega(z)}{1-\omega(z)}\\
&= \frac{(m+1-\alpha)}{(m+1)(1-\alpha)}\left[\frac{f'(z)}{f'_m(z)} - \frac{\alpha m}{m+1-\alpha}\right]\\
&= \frac{1+\sum_{k=2}^{m} k a_k r^{k-1} e^{i(k-1)\theta} - \sum_{k=2}^{\infty} k\overline{b_k} r^{k-1} e^{-i(k+1)\theta} + \frac{(m+1-\alpha)}{(m+1)(1-\alpha)}\left[\sum_{k=m+1}^{\infty} k a_k r^{k-1} e^{i(k-1)\theta}\right]}{1+\sum_{k=2}^{m} k a_k r^{k-1} e^{i(k-1)\theta} - \sum_{k=2}^{\infty} k\overline{b_k} r^{k-1} e^{-i(k+1)\theta}}.
\end{aligned}$$

The required result follows by using the techniques as used in Theorem 2.1. □

Theorem 2.4. *If f of the form* (1.1) *with $b_1 = 0$, satisfies condition* (1.3), *then*

$$Re\left\{\frac{f'_m(z)}{f'(z)}\right\} \geq \frac{(m+1-\alpha)}{(m+1-\alpha)+(m+1)(1-\alpha)}, \qquad (z \in U). \tag{2.7}$$

The result (2.7) *is sharp with the function given by* (2.2).

Proof. Proceeding exactly as in the proof of Theorem 2.3, we evidently have the required result. □

We next turn to ratios for $Re\left\{\frac{f(z)}{f_n(z)}\right\}$ and $Re\left\{\frac{f_n(z)}{f(z)}\right\}$.

Theorem 2.5. *If f of the form* (1.1) *with $b_1 = 0$ satisfies condition* (1.3), *then*

$$Re\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{n+2\alpha}{n+1+\alpha} \qquad (z \in U). \tag{2.8}$$

The result (2.8) *is sharp with the function*

$$f(z) = z + \frac{1-\alpha}{n+1+\alpha}\overline{z}^{n+1}. \tag{2.9}$$

Proof. We may write

$$\frac{1+\omega(z)}{1-\omega(z)}$$
$$=\frac{n+1+\alpha}{1-\alpha}\left[\frac{f(re^{i\theta})}{f_n(re^{i\theta})}-\frac{n+2\alpha}{n+1+\alpha}\right]$$
$$=\frac{1+\sum\limits_{k=2}^{\infty}a_kr^{k-1}e^{i(k-1)\theta}+\sum\limits_{k=2}^{n}\overline{b_k}r^{k-1}e^{-i(k+1)\theta}+\frac{n+1+\alpha}{1-\alpha}\left[\sum\limits_{k=n+1}^{\infty}\overline{b_k}r^{k-1}e^{-i(k+1)\theta}\right]}{1+\sum\limits_{k=2}^{\infty}a_kr^{k-1}e^{i(k-1)\theta}+\sum\limits_{k=2}^{n}\overline{b_k}r^{k-1}e^{-i(k+1)\theta}}.$$

The details involved are fairly straightforward and may be omitted.

To see that $f(z)=z+\frac{1-\alpha}{n+1+\alpha}\bar{z}^{n+1}$gives the sharp result, we observe that for $z=re^{\frac{i\pi}{n+2}}$ that

$$\frac{f(z)}{f_n(z)}=1+\frac{1-\alpha}{n+1+\alpha}r^ne^{-i(n+2)\frac{\pi}{n+2}}\to 1-\frac{1-\alpha}{n+1+\alpha}=\frac{n+2\alpha}{n+1+\alpha}$$

when $r\to 1^-$. □

Theorem 2.6. *If f of the form* (1.1) *with $b_1=0$, satisfies condition* (1.3)*, then*

$$Re\left\{\frac{f_n(z)}{f(z)}\right\}\geq\frac{n+1+\alpha}{n+2},\quad (z\in U) \tag{2.10}$$

The result (2.10) *is sharp with the function given by* (2.9).

Proof. We may write

$$\frac{1+\omega(z)}{1-\omega(z)}$$
$$=\frac{n+2}{1-\alpha}\left[\frac{f_n(z)}{f(z)}-\frac{n+1+\alpha}{n+2}\right]$$
$$=\frac{1+\sum\limits_{k=2}^{\infty}a_kr^{k-1}e^{i(k-1)\theta}+\sum\limits_{k=2}^{n}\overline{b_k}r^{k-1}e^{-i(k+1)\theta}-\frac{n+1+\alpha}{1-\alpha}\left[\sum\limits_{k=n+1}^{\infty}\overline{b_k}r^{k-1}e^{-i(k+1)\theta}\right]}{1+\sum\limits_{k=2}^{\infty}a_kr^{k-1}e^{i(k-1)\theta}+\sum\limits_{k=2}^{\infty}\overline{b_k}r^{k-1}e^{-i(k+1)\theta}}.$$

We omit the details of the proof, because it runs parallel to that from Theorem 2.2. □

We next determine bounds for $Re\left\{\frac{f(z)}{f_{m,n}(z)}\right\}$.

Theorem 2.7. *If f of the form* (1.1) *with $b_1 = 0$, satisfies condition* (1.3), *then*

(2.11) (i) $Re\left\{\dfrac{f(z)}{f_{m,n}(z)}\right\} \geq \dfrac{m}{m+1-\alpha}$ $(z \in U)$ *if* $m \leq n+2\alpha$ *or* $b_k = 0, \forall k \geq 2,$

(2.12) (ii) $Re\left\{\dfrac{f(z)}{f_{m,n}(z)}\right\} \geq \dfrac{n+2\alpha}{n+1+\alpha}$ $(z \in U)$ *if* $m \geq n+2\alpha$ *or* $a_k = 0, \forall k \geq 2.$

The results (2.11) *and* (2.12) *are sharp with the function given by* (2.2) *and* (2.9), *respectively.*

Proof. To prove (i) part, we may write

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{m+1-\alpha}{1-\alpha}\left[\frac{f(re^{i\theta})}{f_{m,n}(re^{i\theta})} - \frac{m}{m+1-\alpha}\right]$$

$$= \frac{1+\displaystyle\sum_{k=2}^{m} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{n} \overline{b_k} r^{k-1} e^{-i(k+1)\theta} + \frac{m+1-\alpha}{1-\alpha}\left[\sum_{k=m+1}^{\infty} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=n+1}^{\infty} \overline{b_k} r^{k-1} e^{-i(k+1)\theta}\right]}{1+\displaystyle\sum_{k=2}^{m} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{n} \overline{b_k} r^{k-1} e^{-i(k+1)\theta}}.$$

So that

$$\omega(z) = \frac{\dfrac{m+1-\alpha}{1-\alpha}\left[\displaystyle\sum_{k=m+1}^{\infty} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=n+1}^{\infty} \bar{b}_k r^{k-1} e^{-i(k+1)\theta}\right]}{2+2\left(\displaystyle\sum_{k=2}^{m} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{n} \bar{b}_k r^{k-1} e^{-i(k+1)\theta}\right) + \dfrac{m+1-\alpha}{1-\alpha}\left(\displaystyle\sum_{k=m+1}^{\infty} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=n+1}^{\infty} \bar{b}_k r^{k-1} e^{-i(k+1)\theta}\right)}.$$

Then

$$|\omega(z)| \leq \frac{\dfrac{m+1-\alpha}{1-\alpha}\left[\displaystyle\sum_{k=m+1}^{\infty} |a_k| + \sum_{k=n+1}^{\infty} |b_k|\right]}{2-2\left(\displaystyle\sum_{k=2}^{m} |a_k| + \sum_{k=2}^{n} |b_k|\right) - \dfrac{m+1-\alpha}{1-\alpha}\left(\displaystyle\sum_{k=m+1}^{\infty} |a_k| + \sum_{k=n+1}^{\infty} |b_k|\right)}.$$

This last expression is bounded above by 1 if and only if

$$\sum_{k=2}^{m} |a_k| + \sum_{k=2}^{n} |b_k| + \frac{m+1-\alpha}{1-\alpha}\left(\sum_{k=m+1}^{\infty} |a_k| + \sum_{k=n+1}^{\infty} |b_k|\right) \leq 1. \tag{2.13}$$

It suffices to show that L. H. S. of (2.13) is bounded above by $\sum_{k=2}^{\infty}\frac{k-\alpha}{1-\alpha}|a_k|+\sum_{k=2}^{\infty}\frac{k+\alpha}{1-\alpha}|b_k|$, which is equivalent to

$$\sum_{k=2}^{m}\frac{k-1}{1-\alpha}|a_k|+\sum_{k=2}^{n}\frac{k-1+2\alpha}{1-\alpha}|b_k|$$
$$+\sum_{k=m+1}^{\infty}\frac{k-m-1}{1-\alpha}|a_k|+\sum_{k=n+1}^{\infty}\frac{k-m-1+2\alpha}{1-\alpha}|b_k|\geq 0.$$

To see that $f(z)=z+\frac{1-\alpha}{m+1-\alpha}z^{m+1}$ gives the sharp result, we observe that for $z=re^{i\pi/m}$ that

$$\frac{f(z)}{f_{m,n}(z)}=1+\frac{1-\alpha}{m+1-\alpha}z^m\to 1-\frac{1-\alpha}{m+1-\alpha}=\frac{m}{m+1-\alpha},$$

when $r\to 1^-$.

To prove second part, we write

$$\frac{1+\omega(z)}{1-\omega(z)}=\frac{n+1+\alpha}{1-\alpha}\left[\frac{f(re^{i\theta})}{f_{m,n}(re^{i\theta})}-\frac{n+2\alpha}{n+1+\alpha}\right]$$

$$=\frac{1+\sum_{k=2}^{m}a_kr^{k-1}e^{i(k-1)\theta}+\sum_{k=2}^{n}\overline{b_k}r^{k-1}e^{-i(k+1)\theta}+\frac{n+1+\alpha}{1-\alpha}\left[\sum_{k=m+1}^{\infty}a_kr^{k-1}e^{i(k-1)\theta}+\sum_{k=n+1}^{\infty}\overline{b_k}r^{k-1}e^{-i(k+1)\theta}\right]}{1+\sum_{k=2}^{m}a_kr^{k-1}e^{i(k-1)\theta}+\sum_{k=2}^{n}\overline{b_k}r^{k-1}e^{-i(k+1)\theta}}$$

where

$$\omega(z)=\frac{\frac{n+1+\alpha}{1-\alpha}\left[\sum_{k=m+1}^{\infty}a_kr^{k-1}e^{i(k-1)\theta}+\sum_{k=n+1}^{\infty}\bar{b}_kr^{k-1}e^{-i(k+1)\theta}\right]}{2+2\left(\sum_{k=2}^{m}a_kr^{k-1}e^{i(k-1)\theta}+\sum_{k=2}^{n}\bar{b}_kr^{k-1}e^{-i(k+1)\theta}\right)+\frac{n+1+\alpha}{1-\alpha}\left(\sum_{k=m+1}^{\infty}a_kr^{k-1}e^{i(k-1)\theta}+\sum_{k=n+1}^{\infty}\bar{b}_kr^{k-1}e^{-i(k+1)\theta}\right)}.$$

So that

$$|\omega(z)| \le \frac{\frac{n+1+\alpha}{1-\alpha}\left[\sum_{k=m+1}^{\infty}|a_k| + \sum_{k=n+1}^{\infty}|b_k|\right]}{2-2\left(\sum_{k=2}^{m}|a_k| + \sum_{k=2}^{n}|b_k|\right) - \frac{n+1+\alpha}{1-\alpha}\left(\sum_{k=m+1}^{\infty}|a_k| + \sum_{k=n+1}^{\infty}|b_k|\right)}$$
$$\le 1.$$

If

$$\sum_{k=2}^{m}|a_k| + \sum_{k=2}^{n}|b_k| + \frac{n+1+\alpha}{1-\alpha}\left(\sum_{k=m+1}^{\infty}|a_k| + \sum_{k=n+1}^{\infty}|b_k|\right) \le 1. \tag{2.14}$$

The L. H. S. of (2.14) is bounded above by $\sum_{k=2}^{\infty}\frac{k-\alpha}{1-\alpha}|a_k| + \sum_{k=2}^{\infty}\frac{k+\alpha}{1-\alpha}|b_k|$ if

$$\sum_{k=2}^{m}\frac{k-1}{1-\alpha}|a_k| + \sum_{k=2}^{n}\frac{k-1+2\alpha}{1-\alpha}|b_k| + \sum_{k=m+1}^{\infty}\frac{k-n-1-2\alpha}{1-\alpha}|a_k| + \sum_{k=n+1}^{\infty}\frac{k-n-1}{1-\alpha}|b_k| \ge 0,$$

and the proof is complete.

To see that $f(z) = z + \frac{1-\alpha}{n+1+\alpha}\bar{z}^{n+1}$gives the sharp result, we observe that for $z = re^{i\frac{\pi}{n+2}}$ that

$$\frac{f(z)}{f_{m,n}(z)} = 1 + \frac{1-\alpha}{n+1+\alpha}r^n e^{-i(n+2)\frac{\pi}{n+2}} \to 1 - \frac{1-\alpha}{n+1+\alpha} = \frac{n+2\alpha}{n+1+\alpha}$$

when $r \to 1^-$.

We next determine bounds for $Re\left\{\frac{f_{m,n}(z)}{f(z)}\right\}$. □

Theorem 2.8. *If f of the form* (1.1) *with $b_1 = 0$, satisfies condition* (1.3)*, then*

$$\text{(i) } Re\left\{\frac{f_{m,n}(z)}{f(z)}\right\} \ge \frac{m+1-\alpha}{m+2-2\alpha},\ (z \in U),\ \textit{if } m \le n+2\alpha \textit{ or } b_k = 0, \forall k \ge 2, \tag{2.15}$$

$$\text{(ii) } Re\left\{\frac{f_{m,n}(z)}{f(z)}\right\} \ge \frac{n+1+\alpha}{n+2},\ (z \in U),\ \textit{if } m \ge n+2\alpha \textit{ or } a_k = 0, \forall k \ge 2. \tag{2.16}$$

The results (2.15) *and* (2.16) *are sharp with the function given by* (2.2) *and* (2.9) *respectively.*

Proof. To prove (i) part we may write

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{m+2(1-\alpha)}{1-\alpha}\left[\frac{f_{m,n}(z)}{f(z)} - \frac{m+1-\alpha}{m+2(1-\alpha)}\right]$$

$$= \frac{1+\sum_{k=2}^{m} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{n} \overline{b_k} r^{k-1} e^{-i(k+1)\theta} - \frac{m+1-\alpha}{1-\alpha}\left[\sum_{k=m+1}^{\infty} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=n+1}^{\infty} \overline{b_k} r^{k-1} e^{-i(k+1)\theta}\right]}{1+\sum_{k=2}^{\infty} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{\infty} \overline{b_k} r^{k-1} e^{-i(k+1)\theta}},$$

where

$$|\omega(z)| \le \frac{\frac{m+2(1-\alpha)}{1-\alpha}\left[\sum_{k=m+1}^{\infty}|a_k| + \sum_{k=n+1}^{\infty}|b_k|\right]}{2-2\left(\sum_{k=2}^{m}|a_k| + \sum_{k=2}^{n}|b_k|\right) - \frac{m}{1-\alpha}\left(\sum_{k=m+1}^{\infty}|a_k| + \sum_{k=n+1}^{\infty}|b_k|\right)} \le 1.$$

This last inequality is equivalent to

$$\sum_{k=2}^{m}|a_k| + \sum_{k=2}^{n}|b_k| + \frac{m+1-\alpha}{1-\alpha}\left(\sum_{k=m+1}^{\infty}|a_k| + \sum_{k=n+1}^{\infty}|b_k|\right) \le 1. \tag{2.17}$$

Since the L. H. S. of (2.17) is bounded above by $\sum_{k=2}^{\infty}\frac{k-\alpha}{1-\alpha}|a_k| + \sum_{k=2}^{\infty}\frac{k+\alpha}{1-\alpha}|b_k|$, the proof is complete.

To prove (ii) part, we write

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{n+2}{1-\alpha}\left[\frac{f_{m,n}(z)}{f(z)} - \frac{n+1+\alpha}{n+2}\right]$$

$$= \frac{1+\sum_{k=2}^{m} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{n} \overline{b_k} r^{k-1} e^{-i(k+1)\theta} - \frac{n+1+\alpha}{1-\alpha}\left[\sum_{k=m+1}^{\infty} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=n+1}^{\infty} \overline{b_k} r^{k-1} e^{-i(k+1)\theta}\right]}{1+\sum_{k=2}^{\infty} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{\infty} \overline{b_k} r^{k-1} e^{-i(k+1)\theta}},$$

where

$$|\omega(z)| \le \frac{\frac{n+2}{1-\alpha}\left[\sum_{k=m+1}^{\infty}|a_k| + \sum_{k=n+1}^{\infty}|b_k|\right]}{2-2\left(\sum_{k=2}^{m}|a_k| + \sum_{k=2}^{n}|b_k|\right) - \frac{n+2\alpha}{1-\alpha}\left(\sum_{k=m+1}^{\infty}|a_k| + \sum_{k=n+1}^{\infty}|b_k|\right)}$$

$$\le 1.$$

This last inequality is equivalent to

$$\sum_{k=2}^{m}|a_k|+\sum_{k=2}^{n}|b_k|+\frac{n+1+\alpha}{1-\alpha}\left(\sum_{k=m+1}^{\infty}|a_k|+\sum_{k=n+1}^{\infty}|b_k|\right)\leq 1. \tag{2.18}$$

Since the L. H. S. of (2.18) is bounded above by $\sum_{k=2}^{\infty}\frac{k-\alpha}{1-\alpha}|a_k|+\sum_{k=2}^{\infty}\frac{k+\alpha}{1-\alpha}|b_k|$, the proof is complete. □

Theorem 2.9. *If f of the form* (1.1) *with $b_1=0$, satisfies condition* (1.3)*, then*

$$Re\left\{\frac{f'(z)}{f'_{m,n}(z)}\right\}\geq\frac{\alpha m}{m+1-\alpha},\quad (z\in U),\ \text{if } m<n+2 \tag{2.19}$$

The result (2.19) *is sharp with the function $f(z)=z+\frac{1-\alpha}{m+1-\alpha}z^{m+1}$.*

Proof. We may write

$$\frac{1+\omega(z)}{1-\omega(z)}=\frac{m+1-\alpha}{(m+1)(1-\alpha)}\left[\frac{f'(z)}{f'_{m,n}(z)}-\frac{\alpha m}{m+1-\alpha}\right]$$

$$=\frac{1+\sum_{k=2}^{m}ka_kr^{k-1}e^{i(k-1)\theta}-\sum_{k=2}^{n}k\bar{b}_kr^{k-1}e^{-i(k+1)\theta}+\frac{(m+1-\alpha)}{(m+1)(1-\alpha)}\left[\sum_{k=m+1}^{\infty}ka_kr^{k-1}e^{i(k-1)\theta}-\sum_{k=n+1}^{\infty}k\overline{b_k}r^{k-1}e^{-i(k+1)\theta}\right]}{1+\sum_{k=2}^{m}ka_kr^{k-1}e^{i(k-1)\theta}-\sum_{k=2}^{n}k\overline{b_k}r^{k-1}e^{-i(k+1)\theta}},$$

where

$$\omega(z)=\frac{\frac{(m+1-\alpha)}{(m+1)(1-\alpha)}\left[\sum_{k=m+1}^{\infty}ka_kr^{k-1}e^{i(k-1)\theta}-\sum_{k=n+1}^{\infty}k\bar{b}_kr^{k-1}e^{-i(k+1)\theta}\right]}{2+2\left(\sum_{k=2}^{m}ka_kr^{k-1}e^{i(k-1)\theta}-\sum_{k=2}^{n}k\bar{b}_kr^{k-1}e^{-i(k+1)\theta}\right)+\frac{(m+1-\alpha)}{(m+1)(1-\alpha)}\left(\sum_{k=m+1}^{\infty}ka_kr^{k-1}e^{i(k-1)\theta}-\sum_{k=n+1}^{\infty}k\bar{b}_kr^{k-1}e^{-i(k+1)\theta}\right)}.$$

So that

$$|\omega(z)|\leq\frac{\frac{(m+1-\alpha)}{(m+1)(1-\alpha)}\left[\sum_{k=m+1}^{\infty}k|a_k|+\sum_{k=n+1}^{\infty}k|b_k|\right]}{2-2\left(\sum_{k=2}^{m}k|a_k|+\sum_{k=2}^{n}k|b_k|\right)-\frac{(m+1-\alpha)}{(m+1)(1-\alpha)}\left(\sum_{k=m+1}^{\infty}k|a_k|+\sum_{k=n+1}^{\infty}k|b_k|\right)}$$

$$\leq 1.$$

This last inequality is equivalent to

$$\sum_{k=2}^{m} k|a_k| + \sum_{k=2}^{n} k|b_k| + \frac{(m+1-\alpha)}{(m+1)(1-\alpha)}\left(\sum_{k=m+1}^{\infty} k|a_k| + \sum_{k=n+1}^{\infty} k|b_k|\right) \le 1. \tag{2.20}$$

Since the L. H. S. of (2.20) is bounded above by $\sum_{k=2}^{\infty} \frac{k-\alpha}{1-\alpha}|a_k| + \sum_{k=1}^{\infty} \frac{k+\alpha}{1-\alpha}|b_k|$, the proof is complete. □

Theorem 2.10. *If f of the form* (1.1) *with $b_1 = 0$, satisfies condition* (1.3)*, then*

$$Re\left\{\frac{f'_{m,n}(z)}{f'(z)}\right\} \ge \frac{(m+1-\alpha)}{(m+1-\alpha)+(m+1)(1-\alpha)}(z \in U) \tag{2.21}$$

The result (2.21) *is sharp with the function* $f(z) = z + \frac{1-\alpha}{m+1-\alpha} z^{m+1}$.

Proof. The proof of the above theorem is similar to that of Theorem 2.9 so we omit the details involved. □

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