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Meromorphic Function that Shares One Small Function with its Differential Polynomial

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Abstract. In this paper, we investigate the uniqueness problems of meromorphic functions that share a small function with its differential polynomials, and give a result which is related to a conjecture of R. Brück and improve the results of I. Lahiri and Q. C. Zhang.

1. Introduction and main result

In this paper, meromorphic functions mean meromorphic in the complex plane. We use the standard notations of Nevanlinna theory, which can be found in [10]. A meromorphic function $a(z)$ is called a small function with respect to $f(z)$ if $T(r,a) = S(r,f)$, i.e. $T(r,a) = o(T(r,f))$ as $r \to +\infty$ possibly outside a set of finite linear measure. We say that two meromorphic functions f and g share a small function a IM (ignoring multiplicities) when $f-a$ and $g-a$ have the same zeros. If $f-a$ and $g-a$ have the same zeros with the same multiplicities, then we say that f and g share a CM (counting multiplicities).

L. A. Rubel and C. C. Yang [7], G. Gundersen [3], L. Z. Yang [8], and many other authors have obtained elegant results on the uniqueness problems of entire functions that share values CM or IM with their first or k-th derivatives. In the respect of only one CM value, R. Bruck posed the following conjecture in 1996:

Brück Conjecture. *Let f be a non-constant entire function. suppose that $\sigma_2(f)$ is not a positive integer or infinite, if f and f' share a finite value a CM, then*

$$\frac{f'-a}{f-a} = c$$

for some non-zero constant c, where $\sigma_2(f)$ is the iterated order of f which is defined by

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{log\, log\, T(r,f)}{log\, r}.$$

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In 1998, Gundersen and Yang [4] verified that the Conjecture is true when f is of finite order. In 1999, Yang [9] confirmed that the Conjecture is also true when f' is replaced by $f^{(k)}(k \geq 2)$ and f is of finite order, in the recent years, many results have been published concerning the above conjecture, see [2], [5], [15], [6], [12], [16], [13], [14], etc., and Zhang [15] was the first author who consider the case when f is a meromorphic function. We need the following definition.

Definition 1. *Let l be a non-negative integer or infinite. Denote by $E_l(a, f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq l$ and $l+1$ times if $m > l$. If $E_l(a, f) = E_l(a, g)$, we say that f and g share (a, l). We also use $N_p(r, \frac{1}{f-a})$ to denote the counting function of the zeros of $f - a$ where a zero of multiplicity m is counted m times if $m \leq p$ and p times if $m > p$.*

Remark. *It is easy to see that f and g share (a, l) implies that f and g share (a, p) for $0 \leq p \leq l$. Also we note that f and g share the value a IM or CM if and only if f and g share $(a, 0)$ or (a, ∞), respectively.*

In 2004, Lahiri [5] improved the results of Zhang [15] by using the above definition and obtained the following two Theorems:

Theorem A. *Let f be a non-constant meromorphic function and k be a positive integer. If f and $f^{(k)}$ share (1,2) and*

$$2\overline{N}(r, f) + N_2(r, \frac{1}{f^{(k)}}) + N_2(r, \frac{1}{f}) < (\lambda + o(1))T(r, f^{(k)}),$$

for $r \in I$, where $0 < \lambda < 1$ and I is a set of infinite linear measure, then $\frac{f^{(k)}-a}{f-a} = c$ for $c \in \mathbf{C} \setminus \{0\}$.

Theorem B. *Let f be a non-constant meromorphic function and k be a positive integer. If f and $f^{(k)}$ share (1,1) and*

$$2\overline{N}(r, f) + N_2(r, \frac{1}{f^{(k)}}) + 2\overline{N}(r, \frac{1}{f}) < (\lambda + o(1))T(r, f^{(k)}),$$

for $r \in I$, where $0 < \lambda < 1$ and I is a set of infinite linear measure, then $\frac{f^{(k)}-a}{f-a} = c$ for $c \in \mathbf{C} \setminus \{0\}$.

In 2005, Zhang [16] further improved the above two results of Lahiri [5] and got the following Theorem:

Theorem C. *Let f be a non-constant meromorphic function and $k(\geq 1)$, $l(\geq 0)$ be integers. Also, let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic function such that $T(r, a) = S(r, f)$. Suppose that $f - a$ and $f^{(k)} - a$ share $(0, l)$.*
If $l \geq 2$ and

$$2\overline{N}(r,f)+N_2(r,\frac{1}{f^{(k)}})+N_2(r,\frac{1}{(\frac{f}{a})'})<(\lambda+o(1))T(r,f^{(k)}),$$

or $l=1$ *and*

$$2\overline{N}(r,f)+N_2(r,\frac{1}{f^{(k)}})+2\overline{N}(r,\frac{1}{(\frac{f}{a})'})<(\lambda+o(1))T(r,f^{(k)}),$$

or $l=0$, *i.e.* $f-a$ *and* $f^{(k)}-a$ *share the value 0 IM and*

$$4\overline{N}(r,f)+3N_2(r,\frac{1}{f^{(k)}})+2\overline{N}(r,\frac{1}{(\frac{f}{a})'})<(\lambda+o(1))T(r,f^{(k)}),$$

for $r\in I$, *where* $0<\lambda<1$ *and* I *is a set of infinite linear measure, then* $\frac{f^{(k)}-a}{f-a}=c$ *for* $c\in\mathbf{C}\setminus\{0\}$.

Definition 2. *Let* $p_0,p_1,\ldots,p_k$ *be non-negative integers. We call*

$$M[f]=f^{p_0}(f')^{p_1}\cdots(f^{(k)})^{p_k}$$

a differential monomial in f *with degree* $d_M=p_0+p_1+\cdots+p_k$ *and weight* $\Gamma_M=p_0+2p_1+\cdots+(k+1)p_k$, *and*

$$Q[f]=\sum_{j=1}^{n}a_jM_j[f],$$

where a_j *are small functions of* f, *is called a differential polynomial in* f *of degree* $d=\max\{d_{M_j},1\le j\le n\}$ *and weight* $\Gamma=\max\{\Gamma_{M_j},1\le j\le n\}$.

In this paper, we will study the problem of a meromorphic function sharing one small function with its differential polynomials and obtain the following result which is an improvement and complement of the above Theorem of Zhang [16].

Theorem 1. *Let* f *be a non-constant meromorphic function and* $Q[f]$ *be a non-constant differential polynomial of degree* d *and weight* Γ. *Let* $a(z)$ *be a small meromorphic function of* f *such that* $a(z)\not\equiv 0,\infty$. *Suppose that* $f-a$ *and* $Q[f]-a$ *share* $(0,l)$, *and* $(n-1)d\le\sum_{j=1}^{n}d_{M_j}$. *Then* $\frac{Q[f]-a}{f-a}=C$ *for some non-zero constant* C *if one of the following assumptions holds,*

(i) $l\ge 2$ *and*

$$2\overline{N}(r,f)+N_2(r,\frac{1}{Q})+N_2(r,\frac{1}{(\frac{f}{a})'})<(\lambda+o(1))T(r,Q),\tag{1.1}$$

(ii) $l=1$ *and*

$$2\overline{N}(r,f)+N_2(r,\frac{1}{Q})+2\overline{N}(r,\frac{1}{(\frac{f}{a})'})<(\lambda+o(1))T(r,Q),\tag{1.2}$$

(iii) $l = 0$ *and*

$$4\overline{N}(r,f) + 3N_2(r,\frac{1}{Q}) + 2\overline{N}(r,\frac{1}{(\frac{f}{a})'}) < (\lambda + o(1))T(r,Q). \tag{1.3}$$

for $r \in I$, *where* $0 < \lambda < 1$ *and* I *is a set of infinite linear measure.*

2. Some lemmas

Lemma 2.1([5]). *Let* f *be a nonconstant meromorphic function,* k *be a positive integer. Then*

$$N_p(r,\frac{1}{f^{(k)}}) \le N_{p+k}(r,\frac{1}{f}) + k\overline{N}(r,f) + S(r,f). \tag{2.1}$$

Suppose that F and G are two non-constant meromorphic functions such that $F$and G share the value 1 IM. Let z_0 be a 1-point of F of order p, a 1-point of G of order q. We denote by $N_L(r,\frac{1}{F-1})$ the counting function of those 1-points of F where $p > q$, by $N_E^{1)}(r,\frac{1}{F-1})$ the counting function of those 1-points of F where $p = q = 1$, by $N_E^{(2}$ the counting function of those 1 -points of F where $p = q \ge 2$; each point in these counting functions is counted only one time. Similarly, we can define $N_L(r,\frac{1}{G-1})$, $N_E^{1)}(r,\frac{1}{G-1})$ and $N_E^{(2}(r,\frac{1}{G-1})$.

Lemma 2.2([11]). *Let* F *and* G *are two nonconstant meromorphic functions,*

$$\Delta = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{2.2}$$

If F *and* G *share* 1 *IM and* $\Delta \not\equiv 0$, *then*

$$N_E^{1)}(r,\frac{1}{F-1}) \le N(r,\Delta) + S(r,F) + S(r,G). \tag{2.3}$$

Lemma 2.3. *Let* $Q[f]$ *be a non-constant differential polynomial. Let* z_0 *be a pole of* f *of order* p *and neither a zero nor a pole of coefficients of* $Q[f]$. *Then* z_0 *is a pole of* $Q[f]$ *with order at most* $pd + (\Gamma - d)$.

Proof. Let

$$Q[f] = \sum_{j=1}^{n} a_j M_j[f], \qquad M_j[f] = f^{p_0}(f')^{p_1}\cdots(f^{(k)})^{p_k},$$

$$d_{M_j} = p_0 + p_1 + \cdots + p_k, \qquad \Gamma_{M_j} = p_0 + 2p_1 + \cdots + (k+1)p_k.$$

Let z_0 be a pole of f of order p, then z_0 be a pole of $M_j[f]$ of order $pd_{M_j} + (\Gamma_{M_j} - d_{M_j})$.

Because $d = \max\{d_{M_j}, 1 \le j \le n\}$, $\Gamma = \max\{\Gamma_{M_j}, 1 \le j \le n\}$ and z_0 neither be a zero nor be a pole of a_j, then z_0 is a pole of $Q[f]$ with order at most $pd + (\Gamma - d)$. □

Lemma 2.4. *Let f be a transcendental meromorphic function, $Q[f]$ is a differential polynomial in f of degree d and weight Γ. Then $T(r,Q) = O(T(r,f))$, $S(r,Q) = S(r,f)$.*

Proof. From Lemma 2.3, we have

$$\begin{aligned} T(r,Q) &= m(r,Q) + N(r,Q) \le m(r,\frac{Q}{f^d})) + m(r,f^d) + N(r,Q) \\ &\le (nd - \sum_{j=1}^{n} d_{M_j}) m(r,\frac{1}{f}) + dm(r,f) + dN(r,f) + (\Gamma - d)\overline{N}(r,f) + S(r,f) \\ &= [(n+1)d - \sum_{j=1}^{n} d_{M_j}]T(r,f) + (\Gamma - d)\overline{N}(r,f) - (nd - \sum_{j=1}^{n} d_{M_j})N(r,\frac{1}{f}) + S(r,f). \end{aligned}$$

So we obtain $T(r,Q) = O(T(r,f))$.
Since

$$\frac{S(r,Q)}{T(r,f)} = \frac{S(r,Q)}{T(r,Q)} \times \frac{T(r,Q)}{T(r,f)} = \frac{S(r,Q)}{T(r,Q)} \times \frac{O(T(r,f))}{T(r,f)} \longrightarrow 0,$$

we get $S(r,Q) = S(r,f)$. □

3. Proof of Theorem 1

Let $F = \frac{Q}{a}$, $G = \frac{f}{a}$, then $F - 1 = \frac{Q-a}{a}$, $G - 1 = \frac{f-a}{a}$. Since $f - a$ and $Q - a$ share $(0,l)$, F and G share $(1,l)$ except the zero and poles of a(z). From Lemma 2.4, we have

$$T(r,F) = O(T(r,f)) + S(r,f), \quad T(r,G) = T(r,f) + S(r,f), \tag{3.1}$$

and

$$S(r,F) = S(r,G) = S(r,f). \tag{3.2}$$

It is obvious that f is a transcendental meromorphic function. Let Δ be defined by (2.2). We distinguish two cases.

Case 1. $\Delta \equiv 0$. Integrating (2.2), yields

$$\frac{1}{G-1} = \frac{C}{F-1} + D, \tag{3.3}$$

where C and D are constants and $C \neq 0$. If there exists a pole z_0 of f with multiplicity p which is not zero or pole of a, then z_0 is a pole of F with multiplicity

$pd+(\Gamma-d)$, a pole of G with multiplicity p. This contradicts with (3.3) as Q contains at least one derivative. Therefore, we have

$$\overline{N}(r,f)\leq\overline{N}(r,a)+\overline{N}(r,\frac{1}{a})=S(r,f), \tag{3.4}$$

$$\overline{N}(r,F)=\overline{N}(r,G)=\overline{N}(r,f)=S(r,f). \tag{3.5}$$

From(3.3), we also get that F and G share the value 1 CM .
Next, we will prove $D=0$.
Suppose $D\neq 0$, then we have

$$\frac{1}{G-1}=\frac{D(F-1+\frac{C}{D})}{F-1}. \tag{3.6}$$

Since F and G share the value 1 CM, we have

$$\overline{N}(r,\frac{1}{D(F-1+\frac{C}{D})})=S(r,f). \tag{3.7}$$

If $\frac{C}{D}\neq 1$, then by using (3.2), (3.5), (3.7) and the second fundamental theorem, we have

$$\begin{aligned}T(r,F) &\leq \overline{N}(r,F)+\overline{N}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{F-1+\frac{C}{D}})+S(r,F)\\ &\leq \overline{N}(r,\frac{1}{F})+S(r,f)\leq N_2(r,\frac{1}{F})+S(r,f)\\ &\leq T(r,F)+S(r,f).\end{aligned}$$

This gives that

$$N_2(r,\frac{1}{F})=T(r,F)+S(r,f).$$

So we have

$$N_2(r,\frac{1}{Q})=T(r,Q)+S(r,f).$$

This contradicts with conditions (1.1), (1.2), (1.3).
If $\frac{C}{D}=1$, from (3.6) we know

$$\frac{1}{G-1}\equiv C\frac{F}{F-1}.$$

This gives us that

$$(G-1-\frac{1}{C})F\equiv-\frac{1}{C}.$$

Using that $F=\frac{Q}{a}$ and $G=\frac{f}{a}$, we get

$$f-a(1+\frac{1}{C})\equiv-\frac{a^2}{C}\cdot\frac{1}{Q}. \tag{3.8}$$

Using (3.4) (3.8), Lemma 2.3 and the first fundamental theorem, we get

$$\begin{aligned}(d+1)T(r,f) &= T(r,\frac{1}{f^d(f-(1+\frac{1}{C})a)})+O(1)\\ &= T(r,-\frac{CQ}{f^da^2})+O(1)\\ &= N(r,\frac{Q}{f^d})+m(r,\frac{Q}{f^d})+S(r,f)\\ &\le dN(r,\frac{1}{f})+m(r,\frac{M_1}{f^d})+\ldots+m(r,\frac{M_n}{f^d})+S(r,f)\\ &\le dN(r,\frac{1}{f})+m(r,\frac{1}{f^{d-d_{M_1}}})+\ldots+m(r,\frac{1}{f^{d-d_{M_n}}})+S(r,f)\\ &\le dN(r,\frac{1}{f})+(nd-\sum_{j=1}^{n}d_{M_j})m(r,\frac{1}{f})+S(r,f)\\ &\le dN(r,\frac{1}{f})+dm(r,\frac{1}{f})+S(r,f)\\ &\le (d+o(1))T(r,f)+S(r,f),\end{aligned}$$

which is a contradiction, hence D=0. This gives from (3.3) that

$$\frac{F-1}{G-1}\equiv C,$$

which implies

$$\frac{Q[f]-a}{f-a}\equiv C.$$

Case 2. $\Delta\not\equiv 0$. By the similar method that used in the proof of Theorem C [16], we get a contradiction. The proof is complete.

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