On a Structure Defined by a Tensor Field $F$ of Type $(1,1)$ Satisfying \[
\prod_{j=1}^{k} [F^2 + a(j)F + \lambda^2(j)I] = 0
\]

**Abstract.** The differentiable manifold with $F$–structure were studied by many authors, for example: K. Yano [7], Ishihara [8], Das [4] among others but thus far we do not know the geometry of manifolds which are endowed with special polynomial $F_a(j)–$structure satisfying
\[
\prod_{j=1}^{K} [F^2 + a(j)F + \lambda^2(j)I] = 0
\]

However, special quadratic structure manifold have been defined and studied by Sinha and Sharma [8]. The purpose of this paper is to study the geometry of differentiable manifolds equipped with such structures and define special polynomial structures for all values of $j = 1, 2, \ldots, K \in N$, and obtain integrability conditions of the distributions $\pi_m^j$ and $\bar{\pi}_m^j$.

1. Introduction

Let $M^n$ be $n$–dimensional manifold of differentiability class $C^\infty$. Suppose there exist on $M^n$, a tensor field $F(\neq 0)$ of type $(1,1)$ satisfying
\[
\prod_{j=1}^{k} [F^2 + a(j)F + \lambda^2(j)I] = 0,
\]

where $\lambda(j)$ are scalars not equal to zero and $a(j)$ are real numbers for $j = 1, 2, \ldots, k \in N$, the set of natural numbers. For arbitrary vector field $X$ on $M^n$ the
above equation (1.1) can be put in the form

$$\prod_{j=1}^{k} [\bar{X} + a(j)\bar{X} + \lambda^2(j)X] = 0,$$

where

$$\bar{X} \overset{def}{=} F(X).$$

Let us call the manifold $M^n$ equipped with such a structure as the special $F_{a(j), \lambda(j)}$ - structure manifold.

**Theorem 1.1.** The rank of $F$ in the special polynomial $F_{a(j), \lambda(j)}$ - structure is equal to the dimension of the manifold.

**Proof.** Assuming $\bar{X} = 0 \Rightarrow \bar{X} = 0$. So from the equation (1.2) it follows that

$$\prod_{j=1}^{k} [\lambda^2(j)X] = 0 \Rightarrow X = 0 \text{ as } \lambda(j) \neq 0.$$

So the Kernel of $F$ is the trivial subspace $\{0\}$ of $TM^n$ where $TM^n$ denotes the tangent space of the manifold $M^n$. Hence if $\nu$ denotes the nullity of $F$, $\nu = 0$. If $\rho$ be the rank of $F$, then from a well-known theorem of linear algebra

$$\rho + \nu = n.$$

Since $\nu = 0$, hence $\rho = n$. This proves the theorem. $\square$

**Theorem 1.2.** The dimension of manifold $M^n$ equipped with the special polynomial $F_{a(j), \lambda(j)}$ - structure for $a^2(j) < 4\lambda^2(j)$ is even.

**Proof.** Let $\delta$ be the eigen value of $F$ and $V$ be the corresponding eigen vector. Then

$$\bar{V} = \delta V$$

which yields

$$\bar{V} = \delta^2 V.$$

Substituting these values of $\bar{V}$ and $\bar{V}$ in (1.2), we obtain

$$\prod_{j=1}^{k} [\delta^2 V + a(j)\delta V + \lambda^2(j)V] = 0$$

which gives

$$\prod_{j=1}^{k} [\delta^2 + a(j)\delta + \lambda^2(j)I] = 0.$$

The roots of the above equation are given by

$$\delta = -\frac{a(j) \pm \sqrt{a^2(j) - 4\lambda^2(j)}}{2}, \quad j = 1, 2, \ldots, k \in N.$$
If \( a^2(j) < 4\lambda^2(j) \), the eigen value of \( F \) are of the form \( \alpha(j) \pm \beta(j) \), where

\[
\alpha(j) = -\frac{a(j)}{2} \quad \text{and} \quad \beta(j) = \frac{\sqrt{4\lambda^2(j) - a^2(j)}}{2}.
\]

Since the complex eigen values occur in pairs, therefore the dimension \( n \) of the manifold must be even. \( \square \)

**Theorem 1.3.** The special polynomial \( F_{a(j), \lambda(j)} \) - structure is not unique.

**Proof.** Let us put [5]

\[
\mu(F'(X)) = F(\mu(X)),
\]

where \( F' \) is a tensor field of type (1,1) and \( \mu \) is a non-singular vector valued function on \( M^n \). Thus

\[
\mu(F'^2(X)) = \mu F'(F'(X)) = F(\mu(F'(X))) = F(F(\mu(X))) = F^2(\mu(X)).
\]

Thus we get

\[
\prod_{j=1}^{k} \mu[F'^2(X)+a(j)F'(X)+\lambda^2(j)] = \prod_{j=1}^{k}[F^2(\mu(X))+a(j)F(\mu(X))+\lambda^2(j)] = 0.
\]

By virtue of the equation (1.1). Thus we obtain

\[
\prod_{j=1}^{k}[F'^2 + a(j)F' + \lambda^2(j)] = 0
\]

as \( \mu \) is non singular. Hence \( F' \) gives the special polynomial \( F_{a(j), \lambda(j)} \) - structure on the manifold \( M^n \). \( \square \)

2. Existence conditions

In this section, we shall prove the following:

**Theorem 2.1.** In order that the even dimensional manifold \( M^{2km} \) may admit the special polynomial \( F_{a(j), \lambda(j)} \) - structure for \( a^2(j) < 4\lambda^2(j) \), it is necessary and sufficient that it contains \( k \) distributions \( \pi^j_m \) of dimensions \( m \) and \( k \) distributions \( \tilde{\pi}^j_m \) conjugate to \( \pi^j_m \) such that they are mutually disjoint and span together a manifold of dimension \( 2km \).
Proof. Suppose first that the manifold $M^{2km}$ admits the special polynomial $F_{a(j), \lambda(j)}$ for $a^2(j) < 4\lambda^2(j)$. Hence the tensor $F$ has $k$ sets of $m$ eigenvalues each of the form $(\alpha(j) + i\beta(j))$ and other $k$ sets of eigenvalues of the form $(\alpha(j) - i\beta(j))$, $j = 1, 2, \ldots, k \in \mathbb{N}$. Let $P^j_x$, $x = 1, 2, \ldots, m$; $j = 1, 2, \ldots, k$ be $m$ eigen vectors for the $m$ eigen values $(\alpha(j) + i\beta(j))$ and $Q^j_x$, $x = 1, 2, \ldots, m$; $j = 1, 2, \ldots, k$ be $m$ eigen vectors for the $m$ eigen values $(\alpha(j) - i\beta(j))$ of $F$. Suppose

\[(2.1) \quad \prod_{j=1}^k [b^j_x P^j_x + c^j_x Q^j_x] = 0, \quad b^j_x, c^j_x \in \mathbb{R}, \quad x = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, k.\]

Operating the above equation (2.1) by $F$ and making use of the fact that $P^j_x$, $Q^j_x$ are eigen vectors for the eigen values $(\alpha(l) + i\beta(l))$ and $(\alpha(l) - i\beta(l))$ of $F$, $1 < l < k \in \mathbb{N}$, we get

\[(2.2) \quad |b^l_x P^l_x - c^l_x Q^l_x| \prod_{j=1, j \neq l}^k |b^j_x P^j_x + c^j_x Q^j_x| = 0.\]

Thus from equation (2.1) and (2.2), we get

\[(2.3) \quad b^j_x = 0 \quad \text{and} \quad c^j_x = 0, \quad x = 1, 2, \cdots, m; \quad j = l.\]

Hence the set $\{P^j_x, Q^j_x\}$ is linearly independent. Similarly, we get $b^j_x = 0$ and $c^j_x = 0$, for all values of $j = 1, 2, \cdots, k \in \mathbb{N}$: $x = 1, 2, \cdots, m$.

Hence the set $\{P^j_x, Q^j_x\}$ is linearly independent for all values of $x = 1, 2, \cdots, m$; $j = 1, 2, \cdots, k \in \mathbb{N}$.

Let $L_j$ and $M_j$ be the linear transformation given by

\[(2.4) \quad L_j(X) = \bar{X} - (\alpha(j) - i\beta(j))X\]

and

\[(2.5) \quad M_j(X) = \bar{X} - (\alpha(j) + i\beta(j))X.\]

The results can be easily proved

\[(2.6) \quad L_j(P^j_x) = 2i\beta P^j_x, \quad L_j(Q^j_x) = 0, \quad M_j(P^j_x) = 0, \quad M_j(Q^j_x) = -2i\beta Q^j_x.\]

Thus there exist $k$ distributions $\pi^j_m$ and $k$ distributions $\tilde{\pi}^j_m$ each of dimension $m$ such that they are mutually disjoint and span together a manifold of dimension $2km$. The projections $L_j$ and $M_j$ are given by (2.4) and (2.5).
Suppose conversely that there exist \( k \) distributions \( \pi^j_m \) and \( k \) distributions \( \tilde{\pi}^j_m \) each of dimension \( m \) such that they have no common direction and span together a manifold of dimension \( 2km \).

Suppose in the \( k \) distributions \( \pi^j_m \) there are \( m \) linearly independent eigen vectors \( P^j_x \) and for the \( k \) distributions \( \tilde{\pi}^j_m \) the \( m \) linearly independent eigen vectors are \( \tilde{Q}^j_x \), \( x = 1, 2, \cdots, m; j = 1, 2, \cdots, k \in \mathbb{N} \). Then the set \( \{P^j_x, \tilde{Q}^j_x\} \) is linearly independent.

Let \( \{p^j_x, q^j_x\} \) be the set of 1-forms dual to the set \( \{P^j_x, \tilde{Q}^j_x\} \). Then

\[
\begin{align*}
p^j_x(P^j_y) &= \delta^y_x, \\
p^j_x(\tilde{Q}^j_y) &= 0, \\
q^j_x(P^j_y) &= 0, \\
q^j_x(\tilde{Q}^j_y) &= \delta^y_x,
\end{align*}
\]

also let

\[
\prod_{j=1}^k [p^j_x(X)P^j_x + q^j_x(X)\tilde{Q}^j_x] = X
\]

Barring the equation (2.8) both sides and using the fact that \( P^j_x, \tilde{Q}^j_x \) are eigen vectors for the eigen values \( \alpha(l) + i\beta(l) \) and \( \alpha(l) - i\beta(l) \) of \( F \) we get

\[
\begin{align*}
\left[ (\alpha(l) + i\beta(l)) p^j_x(X)P^j_x + (\alpha(l) - i\beta(l)) q^j_x(X)\tilde{Q}^j_x \right] \prod_{j=1, j \neq l}^k [p^j_x(X)P^j_x + q^j_x(X)\tilde{Q}^j_x] &= \bar{X}.
\end{align*}
\]

Thus from the equation (2.8) and (2.9), we get

\[
\begin{align*}
\bar{X} &= \alpha(l)X + [i\beta(l) (p^j_x(X)P^j_x - q^j_x(X)\tilde{Q}^j_x)] \prod_{j=1, j \neq l}^k [p^j_x(X)P^j_x + q^j_x(X)\tilde{Q}^j_x].
\end{align*}
\]

Barring (2.9) again and using the same fact that \( P^j_x, \tilde{Q}^j_x \) are eigen vectors for the eigen values \( \alpha(l) + i\beta(l) \) and \( \alpha(l) - i\beta(l) \) of \( F \), we get

\[
\bar{X} = [(\alpha(l) + i\beta(l))^2 (p^j_x(X)P^j_x + (\alpha(l) - i\beta(l))^2 q^j_x(X)\tilde{Q}^j_x)] \prod_{j=1, j \neq l}^k [p^j_x(X)P^j_x + q^j_x(X)\tilde{Q}^j_x].
\]

In view of the equation (2.8) and (2.10) and (2.11), we get

\[
\bar{X} - 2\alpha(l)X + (\alpha^2(l) + \beta^2(l))X = 0 .
\]
Since \( \alpha(l) = -\frac{a(l)}{2} \) and \( \beta(l) = \sqrt{4\lambda^2(l) - a^2(l)} \), where \( 1 \leq l \leq k \).

Similarly it follows that
\[
\prod_{j=1}^{k} (\bar{X} + a(j)\bar{X} + \lambda^2(j)X) = 0 \quad \text{for all} \quad j = 1, 2, \cdots, k \in N.
\]

Thus the manifold \( M^{2km} \) admits the special polynomial \( F_{a(j), \lambda(j)} \) - structure for \( j = 1, 2, \cdots, k \in N \).

**Theorem 2.2.** We have
\[
\begin{align*}
L^2j &= 2i\beta(j)Lj, \\
M^2j &= -2i\beta(j)Mj, \\
LjMj &= MjLj = 0.
\end{align*}
\]

**Proof.** We have in view of the equation (2.4)
\[
L_j = F - (\alpha(j) - i\beta(j))I.
\]

Thus
\[
L^2j = F^2 - 2[\alpha(j) - i\beta(j)]F + (\alpha(j) - i\beta(j))^2I.
\]

Since \( \alpha(j) \pm \beta(j) \) is the root \( \prod_{j=1}^{k} [F^2 + a(j)F + \lambda^2(j)I] = 0 \), so
\[
L^2j = -a(j)F - \lambda^2(j)I - 2[\alpha(j) - i\beta(j)]F + (\alpha(j) - i\beta(j))^2I
\]
\[
L^2j = 2i\beta(j)[F - (\alpha(j) - i\beta(j))I]
\]
\[
L^2j = 2i\beta(j)L(j).
\]

Similarly, it can be shown that
\[
M^2j = -2i\beta(j)M(j).
\]

Also,
\[
LjMj = MjLj = [F - (\alpha(j) - i\beta(j))][F - (\alpha(j) + i\beta(j))]
\]
\[
\text{or}
\]
\[
(2.13) \quad LjMj = MjLj = F^2 + [\alpha^2(j) + \beta^2(j)]I - 2\alpha(j)F.
\]

Since \( \alpha(j) = -\frac{a(j)}{2} \) and \( \alpha^2(j) + \beta^2(j) = \lambda^2(j) \).
Hence
\begin{equation}
L_j M_j = M_j L_j = F^2 + a(j)F + \lambda^2(j)I = 0,
\end{equation}

Thus
\begin{equation}
L_j M_j = M_j L_j = 0.
\end{equation}

Thus the theorem is proved. \(\square\)

3. Nijenhuis Tensor \(F_{a(j), \lambda(j)} - \text{structure}\)

The Nijenhuis Tensor \(F_{a(j), \lambda(j)} - \text{structure}\) is the skew-symmetric tensor of type \((1,2)\) given by
\begin{equation}
N(X, Y) = [X, Y] + [X, \overline{Y}] - [\overline{X}, Y] - [X, \overline{Y}]
\end{equation}
for arbitrary vector fields \(X, Y\) in \(M^n\).

**Theorem 3.1.** We have
\begin{align}
(3.2) & \quad N(X, \overline{Y}) = N(\overline{X}, Y), \\
(3.3) & \quad N(\overline{X}, Y) = -\lambda^2(j)N(X, Y) - a(j)N(X, \overline{Y}), \\
(3.4) & \quad N(\overline{X}, \overline{Y}) = -\lambda^2(j)N(X, Y) - a(j)N(\overline{X}, Y).
\end{align}

**Proof.** Barring \(X\) in (3.1), we have
\begin{align}
N(\overline{X}, Y) & = [\overline{X}, Y] + [\overline{X}, \overline{Y}] - [\overline{X}, Y] - [\overline{X}, \overline{Y}]
\end{align}
which in view of (1.2) reduces to
\begin{align}
(3.5) & \quad N(\overline{X}, Y) = -\lambda^2(j)[X, Y] - a(j)[\overline{X}, Y] - \lambda^2(j)[\overline{X}, Y] + \lambda^2(j)[\overline{X}, \overline{Y}] - [\overline{X}, \overline{Y}].
\end{align}

Barring \(Y\) in (3.1) and using (1.2), we have
\begin{align}
(3.6) & \quad N(X, \overline{Y}) = -\lambda^2(j)[X, \overline{Y}] - a(j)[X, \overline{Y}] - \lambda^2(j)[X, \overline{Y}] + \lambda^2(j)[X, \overline{Y}] - [\overline{X}, \overline{Y}].
\end{align}

From (3.5) and (3.6), we obtain (3.2). Barring \(X\) and \(Y\) in (3.1) and using (1.2), we have
\begin{align}
(3.7) & \quad N(\overline{X}, \overline{Y}) = -\lambda^4(j)[X, Y] + a(j)\lambda^2(j)[X, \overline{Y}] + a(j)\lambda^2(j)[X, Y] + a^2(j)[\overline{X}, \overline{Y}] - \lambda^2(j)[\overline{X}, Y] + \lambda^2(j)[\overline{X}, \overline{Y}] + a(j)[\overline{X}, \overline{Y}] + \lambda^2(j)[\overline{X}, Y].
\end{align}
(3.8) \[ \lambda^2(j)N(X, Y) = \lambda^2(j)[X, Y] - \lambda^4(j)[X, Y] - a(j)\lambda^2(j)[X, Y] - \lambda^2(j)[X, Y] - \lambda^2(j)[X, Y] \]

and

(3.9) \[ a(j)N(X, Y) = -a(j)\lambda^2[X, Y] - a^2(j)[X, Y] - a(j)\lambda^2(j)[X, Y] \]

\[ - a(j)[X, Y] + a(j)\lambda^2(j)[X, Y] \]

from (3.1), (3.7), (3.8) and (3.9), we get (3.3).

Equation (3.4) follows from (3.2) and (3.3).

\[ \square \]

4. Integrability conditions

In this section, we shall establish some results on the integrability of the \( k \)
distributions \( \tilde{\pi}_m^l \) and \( \tilde{\pi}_m^l \).

**Theorem 4.1.** The necessary and sufficient condition that the \( k \) distributions \( \pi_m^l \) integrable is that

(4.1) \[ (dMj)(X, Y) = 0 \quad \text{for all} \quad j = 1, 2, \ldots, k \in N. \]

**Proof.** Suppose for particular value \( j = l \), distribution \( \pi_m^l \) is integrable. Now

\[ X, Y \in \pi_m^l \Rightarrow [X, Y] \in \pi_m^l. \]

Hence

(4.2) \[ Ml(X) = 0, \quad Ml(Y) = 0 \quad \text{and} \quad Ml([X, Y]) = 0, \]

we have [3]

(4.3) \[ (dMl)(X, Y) = X.Ml(Y) - Y.Ml(X) - Ml([X, Y]). \]

Thus in view of equation (4.2), we have

(4.4) \[ (dMl)(X, Y) = 0. \]

Similarly it follows that \( (dMj)(X, Y) = 0 \) for all \( j = 1, 2, \ldots, k. \)

Hence the condition is necessary.

Suppose conversely that

\[ (dMj)(X, Y) = 0 \quad \text{for all} \quad X, Y \in k \text{ distributions} \pi_m^l \]

\[ (dMj)(X, Y) = 0 \quad \text{for all} \quad j = 1, 2, \ldots, k. \]

Thus
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\[ M_j([X, Y]) = 0 \text{ as } M_j(X) = 0 = M_j(Y) \text{ for all } j = 1, 2, \ldots, k. \]

Also
\[ L_j([X, Y]) = [X, Y] - (\alpha(j) - i\beta(j))[X, Y] \text{ for all } j = 1, 2, \ldots, k \]
\[ = (\alpha(j) + i\beta(j))[X, Y] - (\alpha(j) - i\beta(j))[X, Y] \text{ for all } j = 1, 2, \ldots, k \]
or
\[ L_j([X, Y]) = 2i\beta(j)[X, Y] \text{ for all } j = 1, 2, \ldots, k. \]

Thus it follows that if $X, Y \in k$ distributions $\pi^j_m$ then $[X, Y]$ also belongs to $k$ distributions $\pi^j_m$. Thus the $k$ distributions $\pi^j_m$ is integrable. \(\square\)

**Theorem 4.2.** The necessary and sufficient condition for the $k$ distributions $\pi^j_m$ to be integrable is that

\[ (dL_j)(X, Y) = 0 \text{ for all } j = 1, 2, \ldots, k. \]

**Proof.** Proof follows easily in a way similar to that of the Theorem 4.1. \(\square\)

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**References**


