

A STRONG LAW OF LARGE NUMBERS FOR AANA RANDOM VARIABLES IN A HILBERT SPACE AND ITS APPLICATION

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Abstract. In this paper we introduce the concept of asymptotically almost negatively associated random variables in a Hilbert space and obtain the strong law of large numbers for a strictly stationary asymptotically almost negatively associated sequence of H -valued random variables with zero means and finite second moments. As an application we prove a strong law of large numbers for a linear process generated by asymptotically almost negatively random variables in a Hilbert space with this result.

1. Introduction

A finite family $\{Y_i, 1 \leq i \leq n\}$ of real-valued random variables is said to be associated if for any coordinatewise increasing functions $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$

$$\text{Cov}(f(Y_1, \dots, Y_n), g(Y_1, \dots, Y_n)) \geq 0$$

whenever this covariance exists. A finite family $\{Y_i, 1 \leq i \leq n\}$ is said to be negatively associated (NA) if for any disjoint subsets $A, B \subset \{1, \dots, n\}$ and any real coordinatewise nondecreasing functions f on \mathbb{R}^A , g on \mathbb{R}^B ,

$$\text{Cov}(f(Y_i, i \in A), g(Y_j, j \in B)) \leq 0$$

whenever the covariance exists. An infinite family of random variables is associated (negatively associated) if every finite subfamily is associated (negatively associated). These concepts of dependence were

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introduced by Esary, Proschan and Walkup(1967) and Joag-Dev and Proschan(1983), respectively.

From notion of negative association we can consider the following dependence notion.

A sequence $\{Y_n, n \geq 1\}$ of random variables is called asymptotically almost negatively associated(AANA) if there is a nonnegative sequence $q(m) \rightarrow 0$ such that

$$(1.1) \quad \begin{aligned} &Cov(f(Y_m), g(Y_{m+1}, \dots, Y_{m+k})) \\ &\leq q(m)(Var(f(Y_m))Var(g(Y_{m+1}, \dots, Y_{m+k})))^{\frac{1}{2}} \end{aligned}$$

for all $m, k \geq 1$ and for all coordinatewise increasing continuous functions f and g whenever the right hand side of (1.1) is finite. This definition was introduced by Chandra and Ghosal(1996a, 1996b).

Note that the concept of AANA is weaker than negative association, that is, the family of AANA sequences contains NA(in particular, independent) sequence and some more sequences of random variables which are not much deviated from being negatively associated.

Chandra and Ghosal(1996a) proved the strong law of large numbers of Marcinkiewicz and Zygmund for AANA random variables and they(1996b) also obtained the strong laws of large numbers for weighted averages of AANA random variables. Recently, Kim et al.(2004) obtained the Hájek – Renyi type inequality for AANA random variables and proved the strong law of large number by applying this inequality.

Let $\{\xi_1, \dots, \xi_m\}$ be a sequence of \mathbb{R}^d -valued random vectors. $\{\xi_1, \dots, \xi_m\}$ is said to be associated if $Cov(f(\xi_1, \dots, \xi_m), g(\xi_1, \dots, \xi_m)) \geq 0$ for any nondecreasing functions f and g on \mathbb{R}^{md} , such that the covariance exists. This definition was introduced by Burton, Dabrowski and Dehling(1986).

As Burton, Dabrowski and Dehling(1986) introduced the definition of association for random vectors with values in \mathbb{R}^d and random variables with values in a Hilbert space, Ko(2009) and Ko et al.(2009) gave the definition of negative association for random vectors with values in \mathbb{R}^d and random variables with values in a Hilbert space as follows. $\{\xi_1, \dots, \xi_m\}$ is said to be negatively associated if $Cov(f(\xi_i, i \in A), g(\xi_j, j \in B)) \leq 0$ for any disjoint subsets $A, B \subset \{1, \dots, n\}$ and for any nondecreasing functions f on $\mathbb{R}^{|A|}$, g on $\mathbb{R}^{|B|}$ such that the covariance exists where $|A|$ is the cardinality of A .

Now by the similar idea we can define the concept of AANA in \mathbb{R}^d as follows: Let $\{\xi_n, n \geq 1\}$ be a sequence of \mathbb{R}^d -valued random vectors.

$\{\xi_n, n \geq 1\}$ is said to be AANA if there exists a nonnegative sequence $q(m) \rightarrow 0$ such that for all $m, k \geq 1$,

$$(1.2) \quad \begin{aligned} &Cov(f(\xi_m), g(\xi_{m+1}, \dots, \xi_{m+k})) \\ &\leq q(m)(Var(f(\xi_m))Var(g(\xi_{m+1}, \dots, \xi_{m+k})))^{\frac{1}{2}} \end{aligned}$$

for every coordinatewise increasing continuous functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ and $g : \mathbb{R}^{kd} \rightarrow \mathbb{R}$ so that the right-hand side of (1.2) is finite.

We also extend the concept of AANA for random vectors with values in \mathbb{R}^d to random vectors with values in a separable Hilbert space as follows : Let $\{\xi_n, n \geq 1\}$ be a sequence of random variables taking values in a Hilbert space $(H, \langle \cdot, \cdot \rangle)$. $\{\xi_n, n \geq 1\}$ is said to be AANA if for some orthonormal basis $\{e_j, j \geq 1\}$ of H and for any $d \geq 1$ the d -dimensional sequence $(\langle \xi_i, e_1 \rangle, \dots, \langle \xi_i, e_d \rangle), i \geq 1$ is AANA.

Let H be a separable real Hilbert space with the norm $\|\cdot\|_H$ generated by an inner product, $\langle \cdot, \cdot \rangle_H$ and let $\{e_k, k \geq 1\}$ be an orthonormal basis in H . Let $L(H)$ be the class of bounded linear operators from H to H and denote by $\|\cdot\|_{L(H)}$ its usual norm. Let $\{\xi_k, k \in \mathbb{Z}\}$ be a strictly stationary sequence of H -valued random variables and $\{a_k, k \in \mathbb{Z}\}$ be a sequence of bounded operator, that is $a_k \in L(H)$. We define the stationary linear process in a Hilbert space by

$$(1.3) \quad X_k = \sum_{j=0}^{\infty} a_j \xi_{k-j}, k \in \mathbb{Z}.$$

(See Melève et al.(1997), and Bosq(2003) for more details.) The sequence $\{X_k, k \in \mathbb{Z}\}$ is a natural extension of the multivariate linear processes(Brockwell and Davis(1987), Chap. 11). We define

$$(1.4) \quad S_n = \sum_{k=1}^n X_k.$$

Notice that if $\sum_{j=0}^{\infty} \|a_j\|_{L(H)} < \infty$ and $\{\xi_k, k \in \mathbb{Z}\}$ is a sequence of H -valued i.i.d. random variables centered in $L_2(H)$, then it is well known that the series in (1.3) converges almost surely(Araujo and Gine(1980), Chap. 3.2).

In Section 2 we will study the strong law of large numbers for AANA random variables in a Hilbert space and in Section 3 we show the strong law of large numbers for a strictly stationary linear process generated by AANA random variables in a Hilbert space as its application.

2. Results

Lemma 2.1(Chandra, Ghosal (1996 a,b)) Let $\{Y_n, n \geq 1\}$ be a sequence of mean zero, square integrable random variables such that (1.1) holds for $1 \leq m < k + m \leq n$ and for all coordinatewise increasing continuous functions f and g whenever the right-hand side of (1.1) is finite. Let $A^2 = \sum_{m=1}^{n-1} q^2(m)$ and $\sigma_k^2 = EY_k^2, k \geq 1$. Then

$$(2.1) \quad E\left(\max_{1 \leq j \leq n} \sum_{i=1}^j Y_i\right)^2 \leq 2(A + (1 + A^2)^{\frac{1}{2}})^2 \sum_{k=1}^n \sigma_k^2,$$

From Lemma 2.1 we obtain the following maximal inequality for H-valued AANA random variables:

Lemma 2.2 Let $\{\xi_i, 1 \leq i \leq n\}$ be a sequence of mean zero, square integrable H-valued random variables such that (1.2) holds for $1 \leq m \leq k + m \leq n$ and for all coordinatewise increasing continuous functions f and g whenever the right-hand side of (1.2) is finite. Let $A^2 = \sum_{m=1}^{n-1} q^2(m)$. Then

$$(2.2) \quad P\{\max_{1 \leq j \leq n} \|\sum_{i=1}^j \xi_i\| \geq \epsilon\} \leq 2\epsilon^{-2}(A + (1 + A^2)^{\frac{1}{2}})^2 \sum_{i=1}^n E\|\xi_i\|^2.$$

Proof It follows from Lemma 2.1 that

$$\begin{aligned} (2.3) \quad E \max_{1 \leq j \leq n} \left\| \sum_{i=1}^j \xi_i \right\|^2 &= E \max_{1 \leq j \leq n} \sum_{k=1}^{\infty} \left(\sum_{i=1}^j \langle \xi_i, e_k \rangle \right)^2 \\ &\leq \sum_{k=1}^{\infty} E \left(\max_{1 \leq j \leq n} \sum_{i=1}^j \langle \xi_i, e_k \rangle \right)^2 \\ &\leq 2 \sum_{k=1}^{\infty} (A + (1 + A^2)^{\frac{1}{2}})^2 \sum_{i=1}^n E \langle \xi_i, e_k \rangle^2 \\ &\leq 2(A + (1 + A^2)^{\frac{1}{2}})^2 \sum_{i=1}^n E \sum_{k=1}^{\infty} \langle \xi_i, e_k \rangle^2 \\ &= 2(A + (1 + A^2)^{\frac{1}{2}})^2 \sum_{i=1}^n E\|\xi_i\|^2 \end{aligned}$$

which yields (2.2). □

Theorem 2.3 Let $\{\xi_n, n \geq 1\}$ be an AANA sequence of H-valued random variables with $E\xi_n = 0$ and $E\|\xi_n\|^2 < \infty$. Assume

$$(2.4) \quad \sum_{i=1}^{\infty} E\|\xi_i\|^2 < \infty.$$

Then S_n converges almost surely, where $S_n = \xi_1 + \cdots + \xi_n$.

Proof Let $\epsilon > 0$. Then

$$\begin{aligned} P\left\{\sup_{k,m \geq n} \|S_k - S_m\| > \epsilon\right\} &\leq P\left\{\sup_{k \geq n} \|S_k - S_n\| > \frac{1}{2}\epsilon\right\} \\ &\quad + P\left\{\sup_{m \geq n} \|S_m - S_n\| > \frac{1}{2}\epsilon\right\} \\ &\leq 2 \lim_{N \rightarrow \infty} P\left\{\max_{n \leq k \leq N} \|S_k - S_n\| > \frac{1}{2}\epsilon\right\}. \end{aligned}$$

Hence, by Lemma 2.2 and (2.4) we have

$$\begin{aligned} P\left\{\sup_{k,m \geq n} \|S_k - S_m\| > \epsilon\right\} &\leq 8(A + (1 + A^2)^{\frac{1}{2}})^2 \epsilon^{-2} \sum_{i=n}^{\infty} E\|\xi_i\|^2 \rightarrow 0, \\ &\text{as } n \rightarrow \infty, \end{aligned}$$

where A is defined in Lemma 2.2.

Thus we may conclude that the sequence $\{S_n, n \geq 1\}$ is Cauchy almost surely and therefore convergent almost surely. \square

The above theorem and the Kronecker lemma(cf Stout(1995)) imply the following result.

Theorem 2.4 Let $\{\xi_n, n \geq 1\}$ be an AANA sequence of H-valued random variables with $E\xi_n = 0$ and $E\|\xi_n\|^2 < \infty$, $n \geq 1$. Assume

$$(2.5) \quad \sum_{j=1}^{\infty} j^{-2} E\|\xi_j\|^2 < \infty.$$

Then, as $n \rightarrow \infty$ we have $n^{-1}S_n \rightarrow 0$ almost surely, where $S_n = \xi_1 + \cdots + \xi_n$.

Corollary 2.5 Let $\{\xi_n, n \geq 1\}$ be an AANA sequence of R^d -valued random vectors, centered at expectations and finite second moments. If (2.4) holds, then S_n converges almost surely.

Corollary 2.6 Let $\{\xi_n, n \geq 1\}$ be an AANA sequence of R^d -valued random vectors, centered at expectations and finite second moments. If (2.5) holds, then, as $n \rightarrow \infty$, $n^{-1}S_n \rightarrow 0$ a.s..

3. Applications

Now we apply the strong law of large numbers for AANA random variables in a Hilbert space to the linear processes.

Lemma 3.1 Let $\{a_k, k \in \mathbb{Z}\}$ be a sequence of bounded linear operators and let $\tilde{a}_i = \sum_{j=i+1}^{\infty} a_j$. If $\sum_{j=0}^{\infty} j\|a_j\| < \infty$ then $\sum_{j=0}^{\infty} \|\tilde{a}_j\| < \infty$.

Proof. Clearly we have $\sum_{j=0}^{\infty} \|\tilde{a}_j\| \leq \sum_{j=0}^{\infty} \sum_{i=j+1}^{\infty} \|a_i\| = \sum_{j=0}^{\infty} j\|a_j\|$ by induction. \square

Theorem 3.2 Let $\{\xi_k, k \in \mathbb{Z}\}$ be a strictly stationary asymptotically almost negatively associated sequence of H -valued random variables with $E\xi_n = 0$ and $E\|\xi_n\|^2 < \infty$. Let $\{a_k, k \in \mathbb{Z}\}$ be a sequence of bounded linear operators on H satisfying

$$(3.1) \quad \sum_{j=0}^{\infty} j\|a_j\|_{L(H)} < \infty.$$

Define the stationary Hilbert space process by $X_k = \sum_{j=0}^{\infty} a_j \xi_{k-j}$, $k \in \mathbb{Z}$. Assume $\sum_{j=1}^{\infty} j^{-2} E\|\xi_j\|^2 < \infty$. Then

$$(3.2) \quad n^{-1} \sum_{k=1}^n X_k \rightarrow 0 \text{ a.s.}$$

Proof Letting $\tilde{a}_j = \sum_{i=j+1}^{\infty} a_i$ and $Y_k = \sum_{j=0}^{\infty} \tilde{a}_j \xi_{k-j}$, which is well define since $\sum_{j=0}^{\infty} \|\tilde{a}_j\|_{L(H)} < \infty$ according to Lemma 3.1, we have

$$\begin{aligned}
X_k &= \sum_{j=0}^{\infty} a_j \xi_{k-j} \\
&= a_0 \xi_k + \sum_{j=1}^{\infty} a_j \xi_{k-j} \\
&= \left(\sum_{j=0}^{\infty} a_j \right) \xi_k - \tilde{a}_0 \xi_k - \sum_{j=1}^{\infty} (\tilde{a}_j - \tilde{a}_{j-1}) \xi_{k-j} \\
&= \left(\sum_{j=0}^{\infty} a_j \right) \xi_k + Y_{k-1} - Y_k,
\end{aligned}$$

which implies that

$$\sum_{k=1}^n X_k = \left(\sum_{j=0}^{\infty} a_j \right) \sum_{k=1}^n \xi_k + Y_0 - Y_n.$$

Using Theorem 2.4 on $(\sum_{j=0}^{\infty} a_j) \sum_{k=1}^n \xi_k$, the theorem is proved if

$$(3.3) \quad n^{-1} Y_0 \rightarrow 0 \text{ a.s. and } n^{-1} Y_n \rightarrow 0 \text{ a.s., as } n \rightarrow \infty.$$

To prove (3.3) we note that

$$E\|Y_k\| \leq \sum_{j=0}^{\infty} \|\tilde{a}_j\| E\|\xi_{k-j}\| = E\|\xi_{k-j}\| \sum_{j=0}^{\infty} \|\tilde{a}_j\|_{L(H)} < \infty$$

and that $\sum_{n=1}^{\infty} P(\|X\| > n) < \infty \Leftrightarrow E\|X\| < \infty$.

Hence

$$\sum_{n=1}^{\infty} P\{n^{-1}\|Y_n\| > \epsilon\} = \sum_{n=1}^{\infty} P\{n^{-1}\|Y_0\| > \epsilon\} < \infty$$

for any $\epsilon > 0$, which yields (3.3). \square

Corollary 3.3 Let $\{X_k, k \in \mathbb{Z}\}$ be a d -dimensional linear process of the form $X_k = \sum_{j=0}^{\infty} a_j \xi_{k-j}$ where $\{\xi_k, k \in \mathbb{Z}\}$ is a strictly stationary sequence of d -dimensional asymptotically almost negatively associated random vectors with $E\xi_n = 0$, $E\|\xi_n\| < \infty$ and $E\|\xi_n\|^2 < \infty$ and $\{a_k, k \in \mathbb{Z}\}$ is a sequence of $d \times d$ matrix with $\sum_{j=0}^{\infty} j\|a_j\| < \infty$.

Assume

$$\sum_{j=1}^{\infty} j^{-2} E \|\xi_j\|^2 < \infty.$$

Then, as $n \rightarrow \infty$

$$n^{-1} \sum_{k=1}^n X_k \rightarrow 0 \text{ a.s.}$$

Theorem 3.4 Let $\{\xi_k, k \in \mathbb{Z}\}$ be a strictly stationary sequence of H -valued asymptotically almost negatively associated random variables with $E\xi_0 = 0$, $E\|\xi_0\| < \infty$ and let $\{a_k, k \in \mathbb{Z}\}$ be a sequence of bounded linear operators on H satisfying (3.1). Then

$$n^{-1} \sum_{k=1}^n \xi_k \rightarrow 0 \text{ a.s. implies } n^{-1} \sum_{k=1}^n X_k \rightarrow 0 \text{ a.s.}$$

Proof The proof of Theorem 3.3 is similar to that of Theorem 3.2. \square

Corollary 3.5 Let $\{X_k, k \in \mathbb{Z}\}$ be a d -dimensional linear process of the form $X_k = \sum_{j=0}^{\infty} a_j \xi_{k-j}$, where $\{\xi_k, k \in \mathbb{Z}\}$ is a strictly stationary sequence of d -dimensional asymptotically almost negatively associated random vectors with $E\xi_0 = 0$, $E\|\xi_0\| < \infty$ and $\{a_k, k \in \mathbb{Z}\}$ is a sequence of $d \times d$ matrix with $\sum_{j=0}^{\infty} j \|a_j\| < \infty$. Then

$$n^{-1} \sum_{k=1}^n \xi_k \rightarrow 0 \text{ a.s. implies } n^{-1} \sum_{k=1}^n X_k \rightarrow 0 \text{ a.s.}$$

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