Honam Mathematical J. **32** (2010), No. 4, pp. 675–687

TIGHT CLOSURE OF IDEALS RELATIVE TO MODULES

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Abstract. In this paper the dual notion of tight closure of ideals relative to modules is introduced and some related results are obtained.

1. Introduction

Throughout this paper R will denote a commutative Noetherian ring with identity and with a positive prime characteristic p. Further $\mathbf{N}$ will denote the set of natural integers.

The main idea of tight closure of an ideal in a commutative Noetherian ring (with a positive prime characteristic) was introduce by Hochster and Huneke in [5]. It is appropriate for us to begin by briefly summarizing some of main aspects.

Let I be an ideal of R. An element x of R is said to be in tight closure, I^*, of I, if there exists an element $c \in R^\circ$ (here R° denotes the subset of R consisting of all elements which are not contained in any minimal prime ideal of R) such that for all sufficiently large e, $cx^{p^e} \in (i^{p^e} : i \in I)$. The ideal $(i^{p^e} : i \in I)$ is denoted by $I^{[p^e]}$ and is called the eth Frobenius power of I. In particular if $I = (a_1, a_2, ..., a_n)$, then $I^{[p^e]} = (a_1^{p^e}, a_2^{p^e}, ..., a_n^{p^e})$.

In the remainder of this paper, to simplify notation, we will write q to stand for a power p^e of p. Then $I^{[p^e]} = I^{[q]}$.

For any ideals I and J, $I^{[q]} + J^{[q]} = (I + J)^{[q]}$, $I^{[q]}J^{[q]} = (IJ)^{[q]}$, in particular if n is any positive integer, $(I^n)^{[q]} = (I^{[q]})^n$.

Now let M be an $R-$module and let I be an ideal of R. In this paper we will introduce the notion of tight closure $I^{*[M]}$ of an ideal I of R relative to M (see 2.1) and establish some properties of this concept which reflect results of tight closure in the classical situation.

Received February 22, 2010. Accepted June 16, 2010.

AMS Subject Classification (2000): 13A35.

Keywords and phrases. F-reduction, Tight closure, Tight dependent, Associated prime ideal.

Let M be an $R-$module. A prime ideal P of R is said to be an associated prime of M if there exists an element $x \in M$ such that $Ann_R(x) = P$ (see [7]). The set of associated primes of M is denoted by $Ass_R(M)$.

We shall follow Macdonald's terminology (see [6]) concerning secondary representation. So whenever an $R-$module M has a secondary representation, then the set of attached primes of M, which is uniquely determined, is denoted by $Att_R(M)$.

Throughout the remainder of this paper R° will denote the subset of R consisting of all elements which are not contained in any minimal prime ideal of R.

The reader is referred to [10] for the tight closure of an ideal.

2. Tight closure of an ideal relative to module

Definition 2.1. Let M be an $R-$module and I and J be ideals of R. We say that I is an $F-$reduction of the ideal J relative to M, if $I \subseteq J$ and there exists $c \in R^\circ$ such that

$$(0 :_M I^{[q]}) \subseteq (0 :_M cJ^{[q]}) \; for \; all \; q \gg 0.$$

It is straightforward to see that if I is an $F-$reduction of an ideal J of R relative to M and also an $F-$reduction of an ideal J' of R relative to M, then I is an $F-$reduction of the ideal $J + J'$ relative to M. Thus, since R is a Noetherian ring, the set of ideals of R which have I as an $F-$reduction relative to M has a unique maximal member, denoted by $I^{*[M]}$ and called the tight closure of I relative to M. This is in fact the largest ideal which has I as $F-$reduction relative to M.

The proof of the next proposition is easy and is omitted.

Proposition 2.2. Let M be an $R-$module and I, J, I', J' and K be ideals of R.

(a) If I is an $F-$reduction of J relative to M and J is an $F-$reduction of K relative to M, then I is an $F-$reduction of K relative to M.
(b) If I is an $F-$reduction of J relative to M and I' is an $F-$reduction of J' relative to M, then II' is an $F-$reduction of JJ' relative to M.
(c) If $I \subseteq J \subseteq K$ and I is an $F-$reduction of K relative to M, then I is an $F-$reduction of J relative to M and J is an $F-$reduction of K relative to M.

(d) If I is an $F-$reduction of J relative to M and I' is an $F-$reduction of J' relative to M, then $I+I'$ is an $F-$reduction of $J+J'$ relative to M.

Definition 2.3. Let M be an $R-$module and let I be an ideal of R. An element x of R is said to be tight dependent on I relative to M, if there exists an element $c \in R^\circ$ such that

$$(0 :_M I^{[q]}) \subseteq (0 :_M cx^q) \ for\ all\ q \gg 0.$$

Lemma 2.4. Let M be an $R-$module and I be an ideal of R. An element x of R is tight dependent on I relative to M if and only if I is an $F-$reduction of $I + Rx$ relative to M.
Proof. The proof is straightforward.

Theorem 2.5. Let M be an $R-$module and I be an ideal of R. Then

$$I^{*[M]} = \{x \in R : x\ is\ tight\ dependent\ on\ I\ relative\ to\ M\}.$$

Proof. Let $x \in R$ be tight dependent on I relative to M. Then I is an $F-$reduction of $I + Rx$ relative to M by Lemma 2.4. Hence $I + Rx \subseteq I^{*[M]}$ so that $x \in I^{*[M]}$. Now let $x' \in I^{*[M]}$. Then $I \subseteq (I + Rx') \subseteq I^{*[M]}$. Since $I^{*[M]}$ is an $F-$reduction of I relative to M, $I + Rx'$ is an $F-$reduction of I relative to M by Proposition 2.2(c). Now the claim follows from Lemma 2.4.

Lemma 2.6. Let M be an $R-$module. We have the following.

(a) If $dimR = 0$ then $\sqrt{Ann_R(M)} = 0^{*[M]}$.
(b) If I be an ideal of R with $ht(I) > 0$, then there exists $d \in R^\circ$ such that $(0 :_M I^{[q]}) \subseteq (0 :_M dx^q)$ for every $q = p^e$.

Proof. (a) Clearly, $\sqrt{Ann_R(M)} \subseteq 0^{*[M]}$. To see the reverse inclusion, let $t \in 0^{*[M]}$. Then there exists $c \in R^\circ$, such that

$$(0 :_M 0^{[q]}) \subseteq (0 :_M ct^q)$$

for all $q \gg 0$. Since $dimR = 0$, $R^\circ = R \setminus \bigcup_{P \in Spec(R)} P$. Thus $c \notin z(M)$. This implies that $(0 :_M t^q) = M$ for all $q \gg 0$. Hence $\sqrt{Ann_R(M)} = 0^{*[M]}$.

(b) Since $I \subseteq I^{*[M]}$, we have $ht(I^{*[M]}) > 0$. Hence $I^{*[M]} =< x_1, ..., x_n >$, where $x_1, ..., x_n \in I^{*[M]} \cap R^\circ$. For each $x_i \in I^{*[M]}$ $(1 \leq$

$i \leq n$), there exists $c_i \in R^\circ$ and $q_i = p^{e_i}$ such that

$$(0 :_M I^{[q]}) \subseteq (0 :_M c_i x_i^q) \textit{ for every } q \geq q_i.$$

Set $d_i = c_i x^{q_i}$. Then it is easy to see that

$$(0 :_M I^{[q]}) \subseteq (0 :_M d_i x_i^q) \textit{ for every } q = p^e.$$

Let $d = d_1 d_2 ... d_n$. Then we have

$$(0 :_M I^{[q]}) \subseteq (0 :_M dx^q) \textit{ for every } q = p^e$$

where $d \in R^\circ$.

Lemma 2.7. Let M be an $R-$module. Then the operation $I \to I^{*[M]}$ is semiprime on the set of ideals of R in the sense of [9]. More precisely for all ideals I and J of R the following conditions hold.

(a) $I \subseteq I^* \subseteq I^{*[M]}$.
(b) If $I \subseteq J$, then $I^{*[M]} \subseteq J^{*[M]}$.
(c) $(I^{*[M]})^{*[M]} = I^{*[M]}$.
(d) $I^{*[M]} J^{*[M]} \subseteq (IJ)^{*[M]}$.

Proof. (a), (b), and (c) are clear.
(d) Use Lemma 2.4 and Proposition 2.2 (b).

Corollary 2.8. Let M be an R-module and let Λ be an index set. Then for every ideals I and J of R, we have

(a) $(I^{*[M]} J^{*[M]})^{*[M]} = (IJ)^{*[M]}$,
(b) $(\sum_{i \in \Lambda} (I_i)^{*[M]})^{*[M]} = (\sum_{i \in \Lambda} I_i)^{*[M]}$,
(c) $(\bigcap_{i \in \Lambda} (I_i)^{*[M]})^{*[M]} = \bigcap_{i \in \Lambda} (I_i)^{*[M]}$.

Proof. By Lemma 2.7, the operation $I \to I^{*[M]}$ is semiprime on the set of ideals of R. It is easy see that if Λ is an index set and $I \to I_x$ is any semiprime operation on the set of ideals of R, then we have

$$(I_x J_x)_x = (IJ)_x, \quad (\sum_{i \in \Lambda} (I_i)_x)_x = (\sum_{i \in \Lambda} I_i)_x, \quad (\bigcap_{i \in \Lambda} (I_i)_x)_x = \bigcap_{i \in \Lambda} (I_i)_x.$$

Definition 2.9. Let M be an $R-$module. The ideal I of R is tightly closed relative to M, if $I^{*[M]} = I$.

Lemma 2.10. Let M be an $R-$module. Then we have the following.

(a) The intersection of ideals tightly closed relative to M is tightly closed relative to M.
(b) If I and J are ideals of R and I is tightly closed relative to M, then so is $(I :_R J)$.

Proof. (a) This follows from Corollary 2.8 (c).

(b) Set $J = \sum_{i=1}^{n} Ru_i$. Thus $(I :_R J) = \bigcap_{i=1}^{n} (I :_R Ru_i)$. So by part (a), it is enough to prove the assertion for the case that J is a principal ideal. So let $J = Ru$ and let $x \in (I : J)^{*[M]}$. Then there exists $c \in R^{\circ}$ such that

$$(0 :_M (I : u)^{[q]}) \subseteq (0 :_M cx^q) \; for \; all \; q \gg 0.$$

Since $(I : u)^{[q]} \subseteq (I^{[q]} : u^q)$, it follows that

$$(0 :_M (I^{[q]} : u^q)) \subseteq (0 :_M cx^q) \; for \; all \; q \gg 0.$$

This in turn implies that

$$(0 :_M I^{[q]}) \subseteq (0 :_M c(ux)^q) \; for \; all \; q \gg 0.$$

Thus $ux \in I^{*[M]} = I$ by Theorem 2.5. This yields that

$$(I : Ru)^{*[M]} \subseteq (I : Ru).$$

The reverse inclusion follows from Lemma 2.7 (a). Hence $(I : Ru)^{*[M]} = (I : Ru)$ as desired.

Remark 2.11. Let M be an $R-$module. An element $x \in R$ is said to be $M-$coregular if $xM = M$. Further an ideal I of R is said to be $M-$coregular if there exists an element $x \in I$ such that $xM = M$.

Theorem 2.12. Let M be an $R-$module and let I, J, and K be ideals of R. If K consists of $M-$regular elements or K is an $M-$coregular principal ideal, then we have

$$(IK)^{*[M]} \subseteq (JK)^{*[M]} \Rightarrow I^{*[M]} \subseteq J^{*[M]}.$$

Proof. Let $x \in I^{*[M]}$. By Lemma 2.7 (d), we have

$$xK \subseteq (IK)^{*[M]} \subseteq (JK)^{*[M]}.$$

Now we can find $c \in R^{\circ}$ such that

$$(0 :_M J^{[q]}K^{[q]}) \subseteq (0 :_M cx^q K^{[q]})$$

for all $q \gg 0$. If K consists of $M-$regular elements or K is an $M-$coregular principal ideal, then $K^{[q]}$ consists of $M-$regular elements or $K^{[q]}$ is an

$M-$coregular principal ideal. This follows that

$$(0:_M J^{[q]}) \subseteq (0:_M cx^q)$$

for all $q \gg 0$. Hence $x \in J^{*[M]}$ and the proof is completed.

Corollary 2.13 (Cancelation law). Let M be an $R-$module and let I, J, and K be ideals of R. If K consists of $M-$regular elements or K is an $M-$coregular principal ideal, then

$$(IK)^{*[M]} = (JK)^{*[M]} \Rightarrow I^{*[M]} = J^{*[M]}.$$

Proof. This follows from Theorem 2.12.

Theorem 2.14. Let M be an $R-$module. Let I and K be ideals of R. If K consists of $M-$regular elements or K is an $M-$coregular principal ideal, then

$$(I^{*[M]}K^{*[M]} :_R K^{*[M]}) = ((IK)^{*[M]} :_R K^{*[M]}) = ((IK)^{*[M]} :_R K)$$
$$= (I^{*[M]}K :_R K) = I^{*[M]}$$

Proof. It is clear that

$$I^{*[M]} \subseteq (I^{*[M]}K^{*[M]} :_R K^{*[M]}) \subseteq ((IK)^{*[M]} :_R K^{*[M]}) \subseteq ((IK)^{*[M]} :_R K)$$

and

$$(I^{*[M]}K :_R K) \subseteq ((IK)^{*[M]} :_R K).$$

Hence it is enough to prove that $((IK)^{*[M]} :_R K) \subseteq I^{*[M]}$. Since IK is an $F-$reduction of $(IK)^{*[M]}$ relative to M, there exists $c \in R^\circ$ such that

$$(0:_M (IK)^{[q]}) \subseteq (0:_M c((IK)^{*[M]})^{[q]}) \text{ for all } q \gg 0.$$

Since $((IK)^{*[M]} :_R K)K \subseteq (IK)^{*[M]}$,

$$(0:_M I^{[q]}K^{[q]}) \subseteq (0:_M c((IK)^{*[M]} :_R K)^{[q]}K^{[q]}) \text{ for all } q \gg 0.$$

If K consists of $M-$regular elements or K is an $M-$coregular principal ideal, then $K^{[q]}$ consists of $M-$regular elements or $K^{[q]}$ is an $M-$coregular principal ideal. This implies that

$$(0:_M I^{[q]}) \subseteq (0:_M c((IK)^{*[M]} :_R K)^{[q]}) \text{ for all } q \gg 0.$$

Thus I is an $F-$reduction of $((IK)^{*[M]} :_R K)$ relative to M. So

$$((IK)^{*[M]} :_R K) \subseteq I^{*[M]}$$

and the proof is completed.

Corollary 2.15. Let I be an ideal of R and let M be an $R-$module.

If I consists of $M-$regular elements or I is an $M-$coregular principal ideal, then for $0 < m < n$, we have

$$\begin{aligned}((I^n)^{*[M]} :_R (I^m)^{*[M]}) &= ((I^n)^{*[M]} :_R I^m)\\ &= ((I^{n-m})^{*[M]}(I^m)^{*[M]} :_R (I^m)^{*[M]})\\ &= ((I^{n-m})^{*[M]} I^m :_R I^m) = (I^{n-m})^{*[M]}\end{aligned}$$

Proof. This follows from Theorem 2.14.

Corollary 2.16. Let M be an $R-$module. Let I and K be ideals of R. If K consists of $M-$regular elements or K is an $M-$coregular principal ideal, then

$$Ass_R(R/I^{*[M]}) \subseteq Ass_R(R/(I^{*[M]}K)) \cap Ass_R(R/(IK)^{*[M]})$$

Proof. Let $P \in Ass_R(R/I^{*[M]})$. Then there exists an element $x \in R$ such that

$$P = (I^{*[M]} :_R x).$$

Then by Theorem 2.14,

$$P = ((IK)^{*[M]} :_R Kx) = (I^{*[M]}K :_R Kx).$$

So there exist $a, b \in R$ such that $P = ((IK)^{*[M]} :_R a)$ and $P = (I^{*[M]}K :_R b)$. Hence $P \in Ass_R(R/(I^{*[M]}K)) \cap Ass_R(R/(IK)^{*[M]})$.

Theorem 2.17. Let M be an $R-$module. Let I be an ideal of R such that I consists of $M-$regular elements or I is an $M-$coregular principal ideal.

(a) The sequence of sets $(Ass_R(R/(I^n)^{*[M]}))_{n\in\mathbf{N}}$ is an increasing sequence.

(b) If $A(n) = Ass_R(R/(I^n)^{*[M]})$ and $B(n) = Ass_R((I^{n-1})^{*[M]})/(I^n)^{*[M]})$, then $A(n) = B(n)$ for every $n \in \mathbf{N}$

Proof. (a) Let $n \in \mathbf{N}$ and let $P \in Ass_R(R/(I^n)^{*[M]})$. Then there exists $c \in R$ such that $P = ((I^n)^{*[M]} :_R c)$. Now by Corollary 2.15, we have

$$P = ((I^{n+1})^{*[M]} :_R cI).$$

So there exists $y \in R$ such that $P = ((I^{n+1})^{*[M]} :_R y)$. This implies that $P \in Ass_R(R/(I^{n+1})^{*[M]})$.

(b) Let $n \in \mathbf{N}$. It is clear that $B(n) \subseteq A(n)$. Let $P \in A(n)$. Then there exists $c \in R$ such that $P = ((I^n)^{*[M]} :_R c)$. By Lemma 2.7(d), $(I^{*[M]})^n \subseteq (I^n)^{*[M]}$. So $I^{*[M]} \subseteq P$. Thus $c \in ((I^n)^{*[M]} :_R I^{*[M]})$. But $((I^n)^{*[M]} :_R I^{*[M]}) = (I^{n-1})^{*[M]}$ by Corollary 2.15. Hence $c \in (I^{n-1})^{*[M]}$ so that $P \in B(n)$ as required.

We recall that (see [4]) the sequence of sets $(Ass_R(R/I^n))_{n\in\mathbf{N}}$ is ultimately constant. we will denote the ultimate constant value of this sequence by $As^*(I,R)$.

Theorem 2.18. Let M be an $R-$module. Then for every ideal I of R which consists of a regular element, the sequence of sets $(Ass_R((I^n)^{*[M]}/I^n))_{n\in\mathbf{N}}$ is increasing and ultimately constant.

Proof. By [8, 8.1], there exists a positive integer m such that for $n \geq m$, we have

$$(I^{n+1} :_R I) = I^n.$$

Let $n \geq m$ and let $P \in Ass_R((I^n)^{*[M]}/I^n)$. Then there exists $x \in (I^n)^{*[M]}$ such that $P = (I^n :_R x)$. It follows that $P = (I^{n+1} :_R xI)$. Now by using Lemma 2.7(d), we have

$$xI \subseteq (I^n)^{*[M]}I \subseteq (I^n)^{*[M]}I^{*[M]} \subseteq (I^{n+1})^{*[M]}.$$

Hence there exists $c \in (I^{n+1})^{*[M]}$ such that $P = (I^{n+1} :_R c)$. Thus for $n \geq m$, the sequence of sets $(Ass_R((I^n)^{*[M]}/I^n))_{n\in\mathbf{N}}$ becomes an increasing sequence. Now the result follows from the fact that for large n,

$$Ass_R((I^n)^{*[M]}/I^n) \subseteq Ass_R(R/I^n) \subseteq As^*(I,R).$$

Corollary 2.19. Let E be an injective $R-$module. Then for every ideal I of R which consists of a regular element, the sequence of sets

$$(Att_R((0 :_E I^n)/(0 :_E (I^n)^{*[E]})))_{n\in\mathbf{N}},$$

is increasing and ultimately constant.

Proof. This follows from Theorem 2.18 and the fact that for every $n \in \mathbf{N}$, we have

$$Att_R(Hom_R((I^n)^{*[E]}/I^n, E)) =$$
$$\{P \in Ass_R((I^n)^{*[E]}/I^n) : P \subseteq Q \ for\ some\ Q \in Ass_R(E)\}$$

by [2, 3.2].

Lemma 2.20. Let I be an ideal of R. Further let M be a finitely generated $R-$module such that $\sqrt{Ann_R(M)} = Ann_R(M)$. If for all minimal primes P of R, the image of x modulo P is in the $(\frac{I+P}{P})^{*[M/PM]}$, then $x \in I^{*[M]}$.

Proof. Let $Min(R) = \{P_1, ..., P_n\}$ and let $\overline{x} \in (\frac{I+P_i}{P_i})^{*[M/P_iM]}$ for every $i = 1, ..., n$. Then for each i $(1 \leq i \leq n)$, there exists $\overline{c_i} = c_i + P_i \in (R/P_i)^{\circ}$ and $q_i = p^{e_i}$ such that

$$(0 :_{M/P_iM} (\frac{I+P_i}{P_i})^{[q]}) \subseteq (0 :_{M/P_iM} \overline{c_i}\ \overline{x}^q)\ for\ all\ q \geq q_i.$$

Since for each i $(1 \leq i \leq n)$, $Rc_i + P_i$ is not contained in $P_1 \cup ... \cup P_n$, we can find $c_i' \in R^{\circ}$ such that for all $q \geq q_i$,

$$(0 :_{M/P_iM} (\frac{I+P_i}{P_i})^{[q]}) \subseteq (0 :_{M/p_iM} \overline{c_i'}\ \overline{x}^q).$$

Set $q' = Max\{q_1, q_2, ..., q_n\}$. Let $q \geq q'$ and let $m \in (0 :_M I^{[q]})$. Further for each i $(1 \leq i \leq n)$, choose $0 \neq \lambda_i \in \bigcap_{\substack{j=1 \\ j\neq i}}^{n} P_j \setminus P_i$. Then for every $i = 1, ..., n$,

$$c_i' \lambda_i x^q m \in \sqrt{Ann(M)}\ M.$$

Since $\sqrt{Ann(M)} = Ann(M)$, $c_i'\lambda_i x^q m = 0$ for every $i = 1, ..., n$. Set $c'' = \sum_{i=1}^{n} c_i'\lambda_i$. It follows that $c'' x^q m = 0$, where $c'' \in R^{\circ}$. Therefore

$$(0 :_M I^{[q]}) \subseteq (0 :_M c'' x^q)\ for\ all\ q \geq q'.$$

This completes the proof.

Definition 2.21 (see [1, 1.1, 2.5]). Let I be an ideal of R. Let T be a subset of $Spec(R)$. The notation $I(T)$ will denote (I if I=R and) , if I is a proper, the intersection of those primary terms in a minimal primary decomposition of I which are contained in at least one member of T (the intersection of an empty family of ideals of R is assumed to be R itself). This definition is unambiguous and $I(\{P\})$ is denoted by $I(P)$. It is clear that $I(P) = (IR_P)^c$ is just the contraction back to R of the extension of I to R_P under the natural ring homomorphism. Also we have $I(T) = \bigcap_{P\in T} I(P)$ and $(J \cap K)(T) = (J(T) \cap K(T))$ for every ideal J and K of R.

Lemma 2.22. Let I be an ideal of R and M be an $R-$module. Then

$$I^*(Ass_R(M)) \subseteq I^{*[M]}.$$

Proof. There exists $c \in R^{\circ}$ such that

$$c(I^*)^{[q]} \subseteq I^{[q]}\ for\ all\ q \gg 0.$$

By [3, 2.7], we have $(0:_M (I^*)^{[q]}) = (0:_M (I^*)^{[q]}(Ass_R(M)))$. Then

$$(0:_M I^{[q]}) \subseteq (0:_M c(I^*)^{[q]}(Ass_R(M))).$$

It follows that

$$(0:_M I^{[q]}) \subseteq (0:_M c(I^*(Ass_R(M))^{[q]}) \ for\ all\ q \gg 0.$$

Hence $I^*(Ass_R(M))$ is an $F-$reduction of I relative to M so that $I^*(Ass_R(M)) \subseteq I^{*[M]}$. This completes the proof.

3. Tight closure of an ideal relative to injective modules

Definition 3.1 (see [1]). Let I and J be ideals of R and let E be an injective $R-$module. Then I is said to be a reduction of J relative to E if $I \subseteq J$ and there exists $n \in \mathbf{N}$ such that $(0:_E IJ^n) = (0:_E J^{n+1})$. An element x of R is said to be integrally dependent on I relative to E if there exists $n \in \mathbf{N}$ such that

$$(0:_E \sum_{i=1}^{n} x^{n-i} I^i) \subseteq (0:_E x^n).$$

The set of ideals of R which have I as a reduction relative to E has a unique maximal member, which denoted by $I^{*(E)}$ and called the integral closure of I relative to E.

Lemma 3.2. Let I be ideals of R and let E be an injective $R-$module such that $\bigcup_{P \in Ass_R(E)} P \subseteq \bigcup_{P \in Min(R)} P$. Then $I^{*[E]} \subseteq I^{*(E)}$.

Proof. Let $x \in I^{*[E]}$. Then there exists a positive integer q and $c \in R \setminus \bigcup_{P \in Min(R)} P$ such that

$$(0:_E I^{[q]}) \subseteq (0:_E cx^q).$$

Since $I^q \subseteq \sum_{i=1}^{q} x^{q-i} I^i$,

$$(0:_E \sum_{i=1}^{q} x^{q-i} I^i) \subseteq (0:_E I^q) \subseteq (0:_E I^{[q]}).$$

Hence $(0 :_E \sum_{i=1}^{q} x^{q-i} I^i) \subseteq (0 :_E cx^q)$. Now since $c \in R \setminus \bigcup_{P \in Ass_R(E)} P$,

$$(0 :_E \sum_{i=1}^{q} x^{q-i} I^i) \subseteq (0 :_E x^q).$$

Hence x is integrally dependent on I relative to E and the proof is completed by [1 , 2.7].

Proposition 3.3. Let $P \in Spec(R)$ and $E = E(R/P)$ (where for an $R-$module L, we will use $E(L)$ to denote the injective envelope of L). Suppose that I is an ideal of R. We have the following.

(a) If $x \in I^{*[E]}$, then $\frac{x}{1} \in (IR_P)^*$.
(b) If $P \in V(\bigcup_q Ass \frac{R}{I^{[q]}})$, then $x \in I^{*[E]}$ if and only if $\frac{x}{1} \in (IR_P)^*$.

Proof. (a) Let $x \in I^{*[E]}$. Then there exists $c \in R^\circ$, such that

$$(0 :_E I^{[q]}) \subseteq (0 :_E cx^q).$$

Then $\frac{c}{1}\frac{x^q}{1} R_P \subseteq I^{[q]} R_P = (IR_P)^{[q]}$ by [1, 1.6]. Since $\frac{c}{1} \in (R_P)^\circ$, $\frac{x}{1} \in (IR_P)^*$.

(b) ($\Rightarrow$) It follows from (a). Conversely let $\frac{x}{1} \in (IR_P)^*$. Then there exists $\frac{c}{1} \in (R_P)^\circ$ such that $\frac{c}{1}\frac{x^q}{1} R_P \subseteq (IR_P)^{[q]} = I^{[q]} R_P$. Then

$$(0 :_E I^{[q]}) \subseteq (0 :_E cx^q).$$

by [1, 1.6]. By choice of P, we have $c \in R^\circ$ so that $x \in I^{*[E]}$. This completes the proof.

Remark 3.4. Let I be an ideal of R and let E be an injective $R-$module. Then $I^{*(E)} = I^{-}(Ass_R(E))$, where I^{-} is integral closure of ideal I [1, 2.6].

Theorem 3.5. Let I be an ideal of R and let E be an injective $R-$module.

(a) If I is generated by at most n elements, then for all $m \geq 0$ we have

$$(I^{m+n})^{*(E)} \subseteq (I^{m+1})^{*[E]}.$$

(b) If I is generated by a regular sequence, then

$$I^{*[E]} = I^*(Ass_R(E)).$$

Proof. (a) By Briançon-Skoda theorem, for all $m \geq 0$,

$$(I^{m+n})^{-} \subseteq (I^{m+1})^{*}.$$

Now by using Remark 3.4 and Lemma 2.22, we have

$$(I^{m+n})^{*(E)} = (I^{m+n})^{-}(Ass_R(E)) \subseteq (I^{m+1})^{*}(Ass_R(E)) \subseteq (I^{m+1})^{*[E]}.$$

(b) We have $E \cong \bigoplus_{P \in Ass_R(E)} E(R/P)$. Then $I^{*[E]} \subseteq \bigcap_{P \in Ass_R(E)} I^{*[E(R/P)]}$. But by using Proposition 3.3 (a), for every $P \in Ass_R(E)$, we have $I^{*[E(R/P)]} \subseteq I^{*}(P)$. Therefore

$$I^{*[E]} \subseteq \bigcap_{P \in Ass_R(E)} I^{*}(P) = I^{*}(Ass_R(E)).$$

Now the assertion follows from Lemma 2.22 and the proof is completed.

References

[1] H. Ansari-Toroghy and R.Y. Sharp, *Integral closure of ideals relative to injective modules over commutative Noetherian rings*, Quart. J. Math. Oxford, (2) **42** (1991), 393-402.

[2] H. Ansari-Toroghy, *Secondary representation of some modules over a commutative ring*, Acta Math. Hungar., (3) **100** (2003), 257-262.

[3] H. Ansari-Toroghy and F. Dorostkar, *On the integral closure of ideals*, Honam Math. J., (4) **29** (2007), 653-666.

[4] M. Brodmann, *Asymptotic stability of* $Ass(M/I^nM)$, Proc. Amer. Math. Soc., **74** (1979), 16-18.

[5] M. Hochster and C. Huneke, *Tight closure, invariant theory, and Briançon-Skoda theorem*, J. Amer. Math. Soc., (1) **3** (1990), 31-116.

[6] I. G. Macdonald, *Secondary representation of modules over a commutative ring*, Symp. Math. XI(1973), 23-43.

[7] H. Matsumura, *Commutative Algebra*, Benjamin, New York, 1970.

[8] S. McAdam, *Asymptotic prime divisors*, Lecture Notes in Mathematics 1023, Springer, Berlin, 1983.

[9] J. W. Petro, *Some results on the asymptotic completion of an ideal* , Proc. Amer. Math. Soc., **15** (1964), 519-524.

[10] I. Swanson and C. Huneke, *Integral closure of ideals, rings, and modules*, Cambridge Univ. Press, New York, 2006.

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