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A HYBRID ITERATIVE METHOD OF SOLUTION FOR MIXED EQUILIBRIUM AND OPTIMIZATION PROBLEMS

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Abstract. In this paper, we introduce a hybrid iterative method for finding a common element of the set of solutions of a mixed equilibrium problem, the set of common fixed points of finitely many nonexpansive mappings and the set of solutions of the variational inequality for an inverse strongly monotone mapping in a Hilbert space. We show that the iterative sequences converge strongly to a common element of the three sets. The results extended and improved the corresponding results of L.-C.Ceng and J.-C.Yao.

1. Introduction

Let H be a Hilbert space and let C be a nonempty closed convex subset of H and let P_C be the metric projection of H onto C. Let $\varphi : C \to R$ be a real-valued function and $\Theta : C \times C \to R$ be an equilibrium bifunction, i.e., $\Theta(u,u) = 0$ for each $u \in C$. We consider the mixed equilibrium problem MEP which is to find $x^* \in C$ such that

$$MEP : \Theta(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \quad \forall\, y \in C.$$

In particular, if $\varphi \equiv 0$, this problem reduces to the equilibrium problem EP, which is to find $x^* \in C$ such that

$$EP : \Theta(x^*, y) \geq 0, \quad \forall\, y \in C.$$

Denote the set of solution of MEP by Ω, some methods have been proposed to solve the MEP.

A mapping $T : C \to H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C$. Denote the set of fixed points of T by $F(T)$. Recall that if C is a nonempty bounded closed convex subset of H and $T : C \to C$ is nonexpansive,

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then $F(T) \neq \emptyset$. Also, recall that a mapping $f : H \to H$ is contractive if there exists a constant $\alpha \in [0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|, \forall x, y \in H$.

A mapping A of C into H is called monotone if $\langle Au - Av, u - v\rangle \geq 0$, for all $u, v \in C$. The variational inequality problem is to find $u \in C$ such that $\langle Au, v - u\rangle \geq 0$ for all $v \in C$. The set of solutions of the variational inequality is denoted by $VI(C, A)$. A mapping A of C into H is called inverse-strongly monotone if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2$$

for all $x, y \in C$[1]. For such a case, A is called α-inverse-strongly monotone. If A is α-inverse-strongly monotone mapping of C into H, then A is $\frac{1}{\alpha}$-Lipschitz continuous.

Let $\lambda_{n1}, \lambda_{n2}, ..., \lambda_{nN} \in (0, 1], n \in \mathbf{N}$. Given the mappings $T_1, T_2, ..., T_N$ of C into itself, as in Ref [4] one can define, for each $n \in \mathbf{N}$, mappings $U_{n1}, U_{n2}, ..., U_{nN}$ by

$$\begin{aligned} &U_{n1} = \lambda_{n1}T_1 + (1 - \lambda_{n1})I, \\ &U_{n2} = \lambda_{n2}T_2U_{n1} + (1 - \lambda_{n2})I, \\ &\vdots \\ &U_{n,N-1} = \lambda_{n,N-1}T_{N-1}U_{n,N-2} + (1 - \lambda_{n,N-1})I, \\ &W_n := U_{n,N} = \lambda_{n,N}T_NU_{n,N-1} + (1 - \lambda_{n,N})I. \end{aligned} \tag{1}$$

Such a mapping W_n is called the W-mapping generated by $T_1, T_2, ..., T_n$ and $\lambda_{n1}, \lambda_{n2}, ..., \lambda_{nN}$.

For finding an element of $F(S) \cap VI(C, A)$, Iiduka and Takahishi [2] proposed a new iterative scheme: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \ n \geq 1$$

and obtained a strong convergence theorem in a Hilbert space.

Very recently Ceng et al [4] introduced a hybrid iterative scheme:$x_0 \in C$ and

$$\begin{cases} \Theta(y_n, x) + \varphi(x) - \varphi(y_n) + \dfrac{1}{r}\langle K'(y_n) - K'(x_n), \eta(x, y_n)\rangle \geq 0, \forall x \in C, \\ x_{n+1} = \alpha_n f(W_n x_n) + \beta_n x_n + \gamma_n W_n y_n. \end{cases} \tag{2}$$

They prove the sequences generated by the hybrid iterative scheme converge strongly to a common element of the set of solution of MEP and the set of common fixed points of finitely many nonexpansive mappings.

Motivated and inspired by the above results, we introduce a new iterative scheme given as follow: $x_0 \in C$ and

$$\begin{cases} \Theta(u_n, x) + \varphi(x) - \varphi(u_n) + \dfrac{1}{r_n}\langle K'(u_n) - K'(x_n), \eta(x, u_n)\rangle \geq 0, \forall x \in C, \\ x_{n+1} = \alpha_n f(W_n x_n) + \beta_n x_n + \gamma_n W_n P_C(u_n - \lambda_n Au_n), \end{cases} \tag{3}$$

for finding a common element of the set of fixed points of finitely many nonexpansive mappings, the set of solutions of a variational inequality for an α-inverse-strongly monotone mapping and the set of solutions of an equilibrium problem in a real Hilbert space. Furthermore, we will prove the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to the unique solution of the variational inequality

$$\langle (f-I)x^*, x-x^*\rangle \leq 0, x \in \cap_{i=1}^N F(T_i) \cap \Omega \cap VI(C,A).$$

2. Preliminaries

Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point $u \in C$ such that

$$\|x-u\| \leq \|x-y\|, \forall y \in C.$$

The mapping $P_C : x \to u$ is called the metric projection of H onto C. It is known that P_C is nonexpansive. Furthermore, for $x \in H$ and $u \in C$,

$$u = P_C(x) \Leftrightarrow \langle x-u, u-y\rangle \geq 0, \forall y \in C.$$

In this paper we assume that an equilibrium bifunction $\Theta : C \times C \to R$ satisfies the following condition:
(H1) Θ is monotone, i.e.,$\Theta(x,y) + \Theta(y,x) \leq 0$ for all $x, y \in C$:
(H2) for each fixed $y \in C, x \mapsto \Theta(x,y)$ is concave and upper semicontinuous;
(H3) for each $x \in C, y \mapsto \Theta(x,y)$ is convex.

Let $F : C \to H$ and $\eta : C \times C \to H$ be two mappings. Then F is called;
(i) η-monotone if $\langle F(x) - F(y), \eta(x,y)\rangle \geq 0, \quad \forall x, y \in C$;
(ii) η-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle F(x) - F(y), \eta(x,y)\rangle \geq \alpha\|x-y\|^2, \quad \forall x, y \in C;$$

(iii) Lipschitz continuous if there exists a constant $\beta > 0$ such that

$$\|F(x) - F(y)\| \leq \beta\|x-y\|, \quad \forall x, y \in C.$$

When $\eta(x,y) = x - y, \quad \forall x, y \in C$, then the definition (i) and (ii) reduce to the definition of monotone and strong monotone, respectively.

A map $\eta : C \times C \to H$ is called Lipschitz continuous, if there exists a constant $\lambda > 0$ such that $\|\eta(x,y)\| \leq \lambda\|x-y\|, \quad \forall x, y \in C$.

A differentiable function $K : C \to R$ on a convex set C is called:
(i) η-convex if $K(y) - K(x) \geq \langle K'(x), \eta(y,x)\rangle, \quad \forall x, y \in C$, where $K'(x)$ is the Fréchet derivative of K at x;
(ii) η-strongly convex if there exists a constant $\mu > 0$ such that

$$K(y) - K(x) - \langle K'(x), \eta(y,x)\rangle \geq \frac{\mu}{2}\|x-y\|^2, \quad \forall x, y \in C.$$

A mapping $F : C \to R$ is called sequentially continuous at x_0, if $F(x_n) \to F(x_0)$ for each sequence x_n satisfying $x_n \to x_0$. A mapping F is called sequentially continuous on C if it is sequentially continuous at each point of C.

Let $S_r : C \to C$ be the mapping such that for each $x \in C$, $S_r(x)$ is the solution set of MEP(x, r), i.e.,

$$S_r(x) = \{y \in C : \Theta(y,z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z,y)\rangle \geq 0, \\ \forall z \in C, \forall x \in C\}$$

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H, f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H, \langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A is an inverse-strongly monotone mapping of C into H and let $N_C v$ be normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}$, and define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$,[5].

Lemma 2.1. [4] *Let C be a nonempty closed convex subset of a real Hilbert space H, and $\varphi : C \to R$ be a lower semicontinuous and convex functional. Let $\Theta : C \times C \to R$ be an equilibrium bifunction satisfying conditions (H1)-(H3). Assume that*
(i) $\eta : C \times C \to H$ is a Lipschitz continuous with constant $\lambda > 0$ such that
(a) $\eta(x, y) + \eta(y, x) = 0, \ \forall\, x, y \in C$,
(b) $\eta(.,.)$ is affine in the first variable,
(c) for each fixed $y \in C, x \mapsto \eta(y, x)$ is sequentially continuous from the weak topology to the weak topology;
(ii) $K : C \to R$ is η-strongly convex with constant $\mu > 0$ and its derivative K' is sequentially continuous from the weak topology to the strong topology;
(iii) for each $x \in C$, there exists a bounded subset $D_x \subseteq C$ and $z_x \in C$ such that for any $y \in C \backslash D_x$,

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x, y)\rangle < 0.$$

Then the following results hold:
(i) S_r is single-valued;
(ii)(a)

$$\langle K'(x_1) - K'(x_2), \eta(u_1, u_2)\rangle \geq \langle K'(u_1) - K'(u_2), \eta(u_1, u_2)\rangle \\ \forall\ (x_1, x_2) \in C \times C,$$

where $u_i = S_r x_i, \quad i = 1, 2$;
(b) S_r is a nonexpansive if K' is Lipschitz continuous with constant $\nu > 0$ such that $\mu \geq \lambda\nu$;
(iii) $F(S_r) = \Omega$;

(iv) Ω *is closed and convex.*

We remark that from Lemma 2.1 in particular, whenever $K(x) = \frac{\|x\|^2}{2}$ and $\eta(x, y) = x - y$ for each $(x, y) \in C \times C$, Then S_r is firmly nonexpanxive, i.e.,

$$\langle x_1 - x_2, S_r(x_1) - S_r(x_2)\rangle \geq \|S_r(x_1) - S_r(x_2)\|^2, \forall (x_1, x_2) \in C \times C.$$

Lemma 2.2. [7] *Let* C *be a nonempty closed convex subset of a Banach space* X*, Let* $T_1, T_2, ..., T_N$ *be a finite family of nonexpansive mappings of* C *into itself such that* $\cap_{i=1}^N F(T_i)$ *is nonempty, and let* $\lambda_{n_1}, \lambda_{n_2}, ..., \lambda_{n_N}$ *be real numbers such that* $0 < \lambda_{n_i} \leq b < 1$ *for any* $i \in \boldsymbol{N}$*. For any* $n \in \boldsymbol{N}$*, let* W_n *the be W-mapping of* C *into itself generated by* $\lambda_{n_1}, \lambda_{n_2}, ..., \lambda_{n_N}$ *and* $T_1, T_2, ..., T_N$*. Then* W_n *is nonexpansive. Further if* X *is strictly convex, then* $F(W_n) = \cap_{i=1}^N F(T_i)$.

Lemma 2.3. [4] *If the sequences* $\{x_n\}$ *and* $\{y_n\}$ *generated iteratively by (1) are bounded, then the following estimates hold:*

$$\|W_{n+1}x_{n+1} - W_n x_n\| \leq \|x_{n+1} - x_n\| + 2M\Sigma_{i=1}^N|\lambda_{n+1,i} - \lambda_{n,i}|, \quad \forall n \geq 0 \quad (4)$$

and

$$\|W_{n+1}y_{n+1} - W_n y_n\| \leq \|y_{n+1} - y_n\| + 2M\Sigma_{i=1}^N|\lambda_{n+1,i} - \lambda_{n,i}|, \quad \forall n \geq 0 \quad (5)$$

for some constant $M > 0$.

Lemma 2.4. [6] *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* X *and* β_n *be a sequence in [0,1] with*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$*for all* $n \geq 0$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$*. Then* $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.5. [3] *Let* $\{s_n\}$ *be a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1 - \lambda_n)s_n + \beta_n, n \geq 0,$$

where $\{\lambda_n\}$ *is a sequence in* $(0, 1)$*, and* $\{\beta_n\}$ *is a sequence in* $\boldsymbol{R}$ *such that*
(i) $\sum_{n=1}^\infty \lambda_n = \infty$,
(ii) $\limsup_{n\to\infty} \frac{\beta_n}{\lambda_n} \leq 0$ *or* $\sum_{n=1}^\infty |\beta_n| < \infty$*. Then* $\lim_{n\to\infty} s_n = 0$.

3. Main result

Theorem 3.1. *Let* H *be a real Hilbert space, let* C *be a nonempty closed convex subset of* H*, and* $\varphi : C \to R$ *be a lower semicontinuous and convex functional. Let* $\Theta : C \times C \to R$ *be an equilibrium bifunction satisfying conditions (H1)-(H3) and let* $\{T_i\}_{i=1}^N$ *be a finite family of nonexpansive mappings of* C *into itself. Let* $\lambda_{n_1}, \lambda_{n_2}, ..., \lambda_{n_N}$ *be real numbers such that* $\lim_{n\to\infty}(\lambda_{n+1,i} - \lambda_{n,i}) = 0$ *for all* $i = 1, 2..., N$*. Let* A *is* α*-inverse-strongly monotone mapping of* C *into* H *such that* $\cap_{i=1}^N F(T_i) \cap \Omega \cap VI(CA) \neq \emptyset$*. let* f *be a contraction of* C *into itself with* $\alpha \in [0, 1)$*. Suppose that* $\{\alpha_n\}$*,* $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are three sequences in* $[0, 1)$ *with* $\alpha_n + \beta_n + \gamma_n = 1, \forall n$*. Assume that*

(1) Let $\eta : C \times C \to H$ *is Lipschitz continuous with constant* $\lambda > 0$ *such that*
(a) $\eta(x, y) + \eta(y, x) = 0, \quad \forall\, x, y \in C$,
(b)$\eta(.,.)$ *is affine in the first variable,*
(c) for each fixed $y \in C, x \mapsto \eta(y, x)$ *is sequentially continuous from the weak topology to the weak topology;*
(2) $K : C \to R$ *is* η*-strongly convex with constant* $\mu > 0$ *and its derivative* K' *is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with constant* $\nu > 0$ *such that* $\mu \geq \lambda\nu$*;*
(3) for each $x \in C$*, there exists a bounded subset* $D_x \subseteq C$ *and* $z_x \in C$ *such that for any* $y \in C \backslash D_x$,

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r_n}\langle K'(y) - K'(x), \eta(z_x, y)\rangle < 0.$$

(4) $\lambda_n \subset [a, b]$ *for some* $a, b \in (0, 2\alpha)$ *and* $\lim_{n\to\infty} |\lambda_n - \lambda_{n+1}| = 0$*;*
(5) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, *and* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*;*
(6) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_n - r_{n+1}| = 0$*. Then the sequences* $\{x_n\}$ *and* $\{u_n\}$ *generated by (3) converge strongly to the unique solution of the variational inequality:*

$$\langle (f - I)x^*, x - x^*\rangle \leq 0, x \in \bigcap_{i=1}^{N} F(T_i) \cap \Omega \cap VI(CA) = \Gamma$$

provided S_{r_n} *is a firmly nonexpansive.*

Proof. Let $Q = P_\Gamma$. Then Qf is a contraction of H into C. In fact, there exists a constant $\alpha \in [0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|, \forall x, y \in H$. So, we have that

$$\|Qf(x) - Qf(y)\| \leq \|f(x) - f(y)\| \leq \alpha\|x - y\|$$

for all $x, y \in H$. So, Qf is a contraction of H into C. Since H is complete, there exists a unique element of C, such that $x^* = Qf(x^*)$. Such a $x^* \in H$ is an element of C. For all $x, y \in C$ and $\lambda > 0$,

$$\begin{aligned}\|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|(x - y) - \lambda(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2\lambda\langle x - y, Ax - Ay\rangle + \lambda^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2.\end{aligned}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping of C into H.

Put $y_n = P_C(u_n - \lambda_n Au_n)$ for every $n \geq 1$. Let $p \in \Gamma$. We have

$$\begin{aligned}\|y_n - p\| &= \|P_C(u_n - \lambda_n Au_n) - P_C(v - \lambda_n Ap)\| \\ &\leq \|(u_n - \lambda_n Au_n) - (p - \lambda_n Ap)\| \\ &\leq \|u_n - p\|.\end{aligned}$$

From $u_n = S_{r_n} x_n$, we have

$$\|u_n - p\| = \|S_{r_n} x_n - S_{r_n} p\| \leq \|x_n - p\|$$

for every $n \geq 1$. Then we compute that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n(f(W_n x_n) - p) + \beta_n(x_n - p) + \gamma_n(W_n y_n - p)\| \\
&\leq \alpha_n\|f(W_n x_n) - f(p)\| + \alpha_n\|f(p) - p\| + \beta_n\|x_n - p\| + \gamma_n\|y_n - p\| \\
&\leq \alpha_n\alpha\|x_n - p\| + (1 - \alpha_n)\|x_n - p\| + \alpha_n\|f(p) - p\| \\
&= (1 - (1 - \alpha)\alpha_n)\|x_n - p\| + \alpha_n\|f(p) - p\| \\
&\leq \max\{\|x_n - p\|, \frac{1}{1 - \alpha}\|f(p) - p\|\}.
\end{aligned}$$

Therefore $\{x_n\}$ is bounded, $\{y_n\}$, $\{u_n\}$, $\{W_n y_n\}, \{W_n x_n\}$ and $\{f(W_n x_n)\}$ are also bounded. Let M denote the possible different constants appearing in the following argument.

Since $I - \lambda_n A$ is nonexpansive and $p = P_C(p - \lambda_n Ap)$, we also have

$$\begin{aligned}
\|y_{n+1} - y_n\| &\leq \|(u_{n+1} - \lambda_{n+1}Au_{n+1}) - (u_n - \lambda_n Au_n)\| \\
&\leq \|(u_{n+1} - \lambda_{n+1}Au_{n+1}) - (u_n - \lambda_{n+1}Au_n)\| + |\lambda_n - \lambda_{n+1}|\|Au_n\| \\
&\leq \|u_{n+1} - u_n\| + |\lambda_n - \lambda_{n+1}|\|Au_n\|.
\end{aligned}$$

Let $x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n$ for all $n \geq 0$. It follows that

$$\begin{aligned}
z_{n+1} - z_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \\
&= \frac{\alpha_{n+1}f(W_{n+1}x_{n+1}) + \gamma_{n+1}W_{n+1}y_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n f(W_n x_n) + \gamma_n W_n y_n}{1 - \beta_n} \\
&= \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(f(W_{n+1}x_{n+1}) - f(W_n x_n)) \\
&\quad + \left(\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}\right)(f(W_n x_n) - W_n y_n) \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}(W_{n+1}y_{n+1} - W_n y_n).
\end{aligned}$$

Then we have

$$\begin{aligned}
\|z_{n+1} - z_n\| &\leq \frac{\alpha_{n+1}\alpha}{1 - \beta_{n+1}}\|W_{n+1}x_{n+1} - W_n x_n\| \\
&\quad + \left(\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}\right)(\|f(W_n x_n)\| + \|W_n y_n\|) \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}\|W_{n+1}y_{n+1} - W_n y_n\|.
\end{aligned}$$

Substituting (4) and (5), we have

$$\begin{aligned}
\|z_{n+1} - z_n\| &\leq \frac{\alpha_{n+1}\alpha}{1 - \beta_{n+1}}\left[\ \|x_{n+1} - x_n\| + 2M\Sigma_{i=1}^N|\lambda_{n+1,i} - \lambda_{n,i}|\ \right] \\
&\quad + \left(\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}\right)(\|f(W_n x_n)\| + \|W_n y_n\|) \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}\left[\ \|y_{n+1} - y_n\| + 2M\Sigma_{i=1}^N|\lambda_{n+1,i} - \lambda_{n,i}|\ \right].
\end{aligned}$$

On the other hand $u_n = S_{r_n} x_n$ and $u_{n+1} = S_{r_{n+1}} x_{n+1}$, we have

$$\Theta(u_n, x) + \varphi(x) - \varphi(u_n) + \frac{1}{r_n}\langle K'(u_n) - K'(x_n), \eta(x, u_n)\rangle \geq 0 \qquad (6)$$

for all $x \in C$, and

$$\Theta(u_{n+1}, x)+\varphi(x)-\varphi(u_{n+1})+\frac{1}{r_{n+1}}\langle K'(u_{n+1})-K'(x_{n+1}), \eta(x, u_{n+1})\rangle \geq 0 \quad (7)$$

for all $x \in C$. Putting $x = u_{n+1}$ in (6) and $x = u_n$ in (7), we have

$$\Theta(u_n, u_{n+1}) + \varphi(u_{n+1}) - \varphi(u_n) + \frac{1}{r_n}\langle K'(u_n) - K'(x_n), \eta(u_{n+1}, u_n)\rangle \geq 0,$$

and

$$\Theta(u_{n+1}, u_n)+\varphi(u_n)-\varphi(u_{n+1})+\frac{1}{r_{n+1}}\langle K'(u_{n+1})-K'(x_{n+1}), \eta(u_n, u_{n+1})\rangle \geq 0.$$

So we have

$$\langle \eta(u_{n+1}, u_n), K'(u_n) - K'(x_n) - \frac{r_n}{r_{n+1}} K'(u_{n+1}) - K'(x_{n+1})\rangle \geq 0,$$

and hence

$$\begin{aligned}&\langle \eta(u_{n+1}, u_n), K'(u_n) - K'(u_{n+1}) + K'(x_{n+1}) - K'(x_n) + \left(1 - \frac{r_n}{r_{n+1}}\right) \\ &\qquad (K'(u_{n+1}) - K'(x_{n+1}))\rangle \geq 0.\end{aligned}$$

Then, by Lemma 2.1 we have

$$\begin{aligned}&\langle \eta(u_{n+1}, u_n), K'(x_{n+1}) - K'(x_n) + \left(1 - \frac{r_n}{r_{n+1}}\right)(K'(u_{n+1}) - K'(x_{n+1}))\rangle \\ &\geq \langle \eta(u_n, u_{n+1}), K'(u_n) - K'(u_{n+1})\rangle \\ &\geq \mu\|u_n - u_{n+1}\|^2,\end{aligned}$$

and hence

$$\begin{aligned}&\mu\|u_n - u_{n+1}\|^2 \\ &\leq \|\eta(u_{n+1}, u_n)\|[\|K'(x_{n+1}) - K'(x_n)\| + \left(1 - \frac{r_n}{r_{n+1}}\right)\|K'(u_{n+1}) - K'(x_{n+1})\|] \\ &\leq \lambda\|u_n - u_{n+1}\|(\nu\|x_n - x_{n+1}\| + \left(1 - \frac{r_n}{r_{n+1}}\right) M).\end{aligned}$$

Without loss of generality, we assume that there exists a real number b such that $r_n > b > 0$ for all $n \in \mathbf{N}$, we have

$$\begin{aligned}\|u_n - u_{n+1}\| &\leq \frac{\lambda\nu}{\mu}\|x_n - x_{n+1}\| + \frac{\lambda}{\mu}\frac{1}{b}|r_n - r_{n+1}|M \\ &\leq \|x_n - x_{n+1}\| + \frac{\lambda}{b\mu}|r_n - r_{n+1}|M.\end{aligned}$$

Hence, we have that

$$\begin{aligned}
&\|z_{n+1} - z_n\| \\
&\le \frac{\alpha_{n+1}\alpha}{1-\beta_{n+1}}[\ \|x_{n+1} - x_n\| + 2M\Sigma_{i=1}^N|\lambda_{n+1,i} - \lambda_{n,i}|\ ] \\
&\quad + \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right)(\|f(W_nx_n)\| + \|W_ny_n\|) \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}}[\|u_{n+1} - u_n\| + |\lambda_n - \lambda_{n+1}|\|Au_n\| + 2M\Sigma_{i=1}^N|\lambda_{n+1,i} - \lambda_{n,i}|\ ] \\
&\le \frac{\alpha_{n+1}\alpha}{1-\beta_{n+1}}[\ \|x_{n+1} - x_n\| + 2M\Sigma_{i=1}^N|\lambda_{n+1,i} - \lambda_{n,i}|\ ] \\
&\quad + \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right)(\|f(W_nx_n)\| + \|W_ny_n\|) \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}}[\|x_n - x_{n+1}\| + \frac{\lambda}{b\mu}|r_n - r_{n+1}|M + |\lambda_n - \lambda_{n+1}|\|Au_n\| \\
&\quad + 2M\Sigma_{i=1}^N|\lambda_{n+1,i} - \lambda_{n,i}|\ ] \\
&\le \|x_{n+1} - x_n\| + 2M\Sigma_{i=1}^N|\lambda_{n+1,i} - \lambda_{n,i}| \\
&\quad + \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right)(\|f(W_nx_n)\| + \|W_ny_n\|) \\
&\quad + \frac{\lambda}{b\mu}|r_n - r_{n+1}|M + |\lambda_n - \lambda_{n+1}|\|Au_n\|.
\end{aligned}$$

This together $\alpha_n \to 0$ and $\lambda_{n+1,i} - \lambda_{n,i} \to 0$, $r_n - r_{n+1} \to 0$ and $\lambda_n - \lambda_{n+1} \to 0$ implies that

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Hence by lemma 2.4, we obtain $\|z_n - x_n\| \to 0$ as $n \to \infty$. Consequently,

$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, $\lim_{n\to\infty}\|u_{n+1} - u_n\| = 0$ and $\lim_{n\to\infty}\|u_{n+1} - u_n\| = 0$.

Since $x_{n+1} = \alpha_n f(W_nx_n) + \beta_n x_n + \gamma_n W_n y_n$, we have

$$\begin{aligned}
\|x_n - W_ny_n\| &\le \|x_{n+1} - x_n\| + \|x_{n+1} - W_ny_n\| \\
&\le \|x_{n+1} - x_n\| + \alpha_n\|f(W_nx_n) - W_ny_n\| + \beta_n\|x_n - W_ny_n\|,
\end{aligned}$$

and thus

$$\|x_n - W_ny_n\| \le \frac{1}{1-\beta_n}\|x_{n+1} - x_n\| + \frac{\alpha_n}{1-\beta_n}\|f(W_nx_n) - W_ny_n\|,$$

which it follows that $\lim_{n\to\infty}\|x_n - W_ny_n\| = 0$.

For $p \in \Gamma$, noting that S_{r_n} is firmly nonexpansive, we have

$$\begin{aligned}
\|u_n - p\|^2 &= \|S_{r_n}x_n - S_{r_n}p\|^2 \le \langle S_{r_n}x_n - S_{r_n}p, x_n - p\rangle \\
&= \langle u_n - p, x_n - p\rangle = \frac{1}{2}(\|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n\|^2),
\end{aligned}$$

and hence

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|u_n - x_n\|^2.$$

Therefore, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n\|f(W_nx_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|W_ny_n - p\|^2\\ &\le \alpha_n\|f(W_nx_n) - v\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|y_n - p\|^2\\ &\le \alpha_n\|f(W_nx_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|u_n - p\|^2\\ &\le \alpha_n\|f(W_nx_n) - p\|^2 + \beta_n\|x_n - p\|^2\\ &\quad + \gamma_n(\|x_n - p\|^2 - \|u_n - x_n\|^2)\\ &\le \alpha_n\|f(W_nx_n) - p\|^2 + \|x_n - p\|^2 - \gamma_n\|u_n - x_n\|^2.\end{aligned}$$

Consequently, we have

$$\begin{aligned}\gamma_n\|u_n - x_n\|^2 &\le \alpha_n\|f(W_nx_n) - p)\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2\\ &\le \alpha_n\|f(W_nx_n) - p)\|^2 + \|x_{n+1} - x_n\|(\|x_n - p\| + \|x_{n+1} - p\|).\end{aligned}$$

Thus we have $\lim_{n\to\infty}\|u_n - x_n\| = 0$. From

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n\|f(W_nx_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|W_ny_n - p\|^2\\ &\le \alpha_n\|f(W_nx_n) - v\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|y_n - p\|^2\\ &\le \alpha_n\|f(W_nx_n) - p\|^2 + \beta_n\|x_n - p\|^2\\ &\quad + \gamma_n\|(u_n - \lambda_nAu_n) - (p - \lambda_nAp)\|^2\\ &\le \alpha_n\|f(W_nx_n) - p\|^2 + \beta_n\|x_n - p\|^2\\ &\quad + \gamma_n[\|u_n - p\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Au_n - Ap\|^2]\\ &\le \alpha_n\|f(W_nx_n) - p\|^2 + \|x_n - p\|^2 + \gamma_na(b - 2\alpha)\|Au_n - Ap\|^2.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&-\gamma_na(b - 2\alpha)\|Au_n - Ap\|^2\\ &\le \alpha_n\|f(W_nx_n) - p)\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2\\ &\le \alpha_n\|f(W_nx_n) - p)\|^2 + \|x_{n+1} - x_n\|(\|x_n - p\| + \|x_{n+1} - p\|).\end{aligned}$$

Since $\alpha_n \to 0(n \to \infty), a, b \in (0, 2\alpha)$, and $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$, we have

$$\|Au_n - Ap\| \to 0, (n \to \infty).$$

From (3), we have

$$\begin{aligned}\|y_n - p\|^2 &= \|P_C(u_n - \lambda_nAu_n) - P_C(p - \lambda_nAp)\|^2\\ &\le \langle (u_n - \lambda_nAu_n) - (p - \lambda_nAp), y_n - p\rangle\\ &= \frac{1}{2}(\|(u_n - \lambda_nAu_n) - (p - \lambda_nAp)\|^2 + \|y_n - p\|^2\\ &\quad - \|(u_n - \lambda_nAu_n) - (p - \lambda_nAp) - (y_n - p)\|^2)\end{aligned}$$

$$\le \frac{1}{2}(\|u_n - p\|^2 + \|y_n - p\|^2 - \|(u_n - y_n) - \lambda_n(Au_n - Ap)\|^2)$$
$$= \frac{1}{2}(\|u_n - p\|^2 + \|y_n - p\|^2 - \|u_n - y_n\|^2 + 2\lambda_n\langle u_n - y_n, Au_n - Ap\rangle - \lambda_n^2\|Au_n - Ap\|^2).$$

So, we have

$$\|y_n - p\|^2 \le \|u_n - p\|^2 - \|u_n - y_n\|^2 + 2\lambda_n\langle u_n - y_n, Au_n - Ap\rangle - \lambda_n^2\|Au_n - Ap\|^2.$$

Hence we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n\|f(W_nx_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|W_ny_n - p\|^2\\ &\le \alpha_n\|f(W_nx_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|y_n - p\|^2\\ &\le \alpha_n\|f(W_nx_n) - p\|^2 + \|x_n - p\|^2 - \|u_n - y_n\|^2\\ &\quad + 2\lambda_n\langle u_n - y_n, Au_n - Ap\rangle - \lambda_n^2\|Au_n - Ap\|^2.\end{aligned}$$

Since $\alpha_n \to 0, \|x_{n+1} - x_n\| \to 0, \|Au_n - Ap\| \to 0$, we obtain

$$\|u_n - y_n\| \to 0.$$

Since $\|W_ny_n - y_n\| \le \|u_n - y_n\| + \|u_n - x_n\| + \|W_ny_n - x_n\|$, we obtain

$$\|W_ny_n - y_n\| \to 0.$$

Next we show that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle \le 0,$$

where $x^* = P_\Gamma f(x^*)$. To show this we can choose a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that

$$\lim_{n\to\infty}\langle f(x^*) - x^*, y_{n_j} - x^*\rangle = \limsup_{n\to\infty}\langle f(x^*) - x^*, y_n - x^*\rangle.$$

Since $\{y_{n_j}\}$ is bounded, there exists a subsequence $\{y_{n_{j_i}}\}$of $\{y_{n_j}\}$ which converges weakly to w. Without loss of generality, we can assume that $y_{n_j} \to w$ weakly. From $\|W_ny_n - y_n\| \to 0$, we have $W_ny_{n_j} \to w$ weakly, Next we show that $w \in \Omega$. Since $u_n = S_{r_n}x_n$, we derive

$$\Theta(u_n, x) + \varphi(x) - \varphi(u_n) + \frac{1}{r_n}\langle K'(u_n) - K'(x_n), \eta(x, u_n)\rangle \ge 0, \quad \forall\ x \in C.$$

From the monotonicity of Θ, we have

$$\varphi(x) - \varphi(u_n) + \frac{1}{r}\langle K'(u_n) - K'(x_n), \eta(x, u_n)\rangle \ge -\Theta(u_n, x) \ge \Theta(x, u_n),$$

and hence

$$\varphi(x) - \varphi(u_{n_j}) + \langle\frac{K'(u_{n_j}) - K'(x_{n_j})}{r_n}, \eta(x, u_{n_j})\rangle \ge \Theta(x, u_{n_j}).$$

Since $\frac{K'(u_{n_j})-K'(x_{n_j})}{r_{n_j}} \to 0$, and $\{u_{n_j}\} \to w$ weakly, from the weak lower semicontinuity of φ and $\Theta(x, y)$ in the second variable y, we have $\Theta(x, w) + \varphi(w) -$

$\varphi(x) \leq 0$, for all $x \in C$. For $0 < t \leq 1$ and $x \in H$, let $x_t = tx + (1-t)w$. Since $x \in C$ and $w \in C$, we have $x_t \in C$ and hence $\Theta(x_t, w) + \varphi(w) - \varphi(x_t) \leq 0$ From the convexity of equilibrium bifunction $\Theta(x, y)$ in the second variable y, we have

$$\begin{aligned} 0 &= \Theta(x_t, x_t) + \varphi(x_t) - \varphi(x_t) \\ &\leq t\Theta(x_t, x) + (1-t)\Theta(x_t, w) + t\varphi(x) + (1-t)\varphi(w) - \varphi(x_t) \\ &\leq t[\Theta(x_t, x) + \varphi(x) - \varphi(x_t)], \end{aligned}$$

and hence $\Theta(x_t, x) + \varphi(x) - \varphi(x_t) \geq 0$. Then, we have $\Theta(w, x) + \varphi(x) - \varphi(w) \geq 0$ for all $x \in C$ and hence $w \in \Omega$.

We shall prove that $w \in F(W_n)$. Assume that $\{y_{n_j}\} \to w$ weakly and $w \neq W_n w$, by Opial's condition, we have

$$\begin{aligned} \liminf_{j\to\infty} \|y_{n_j} - w\| &< \liminf_{j\to\infty} \|y_{n_j} - W_n w\| \\ &\leq \liminf_{j\to\infty} (\|y_{n_j} - W_n y_{n_j}\| + \|W_n y_{n_j} - W_n w\|) \\ &\leq \liminf_{j\to\infty} \|y_{n_j} - w\|, \end{aligned}$$

which is a contradiction. Hence, we get $w \in F(W_n)$.

let us show that $w \in VI(C, A)$. Let

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone. Let $(v, u) \in G(T)$. Since $u - Av \in N_C v$ and $y_n \in C$ we have

$$\langle v - y_n, u - Av \rangle \geq 0.$$

On the other hand, from $y_n = P_C(u_n - \lambda_n A u_n)$, we have $\langle v - y_n, y_n - (u_n - \lambda_n A u_n) \rangle \geq 0$ and hence

$$\langle v - y_n, \frac{y_n - u_n}{\lambda_n} + A u_n \rangle \geq 0.$$

Therefore, we have

$$\begin{aligned} \langle v - y_{n_i}, u \rangle &\geq \langle v - y_{n_i}, Av \rangle \\ &\geq \langle v - y_{n_i}, Av \rangle - \langle v - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} + A u_{n_i} \rangle \\ &= \langle v - y_{n_i}, Av - A u_{n_i} - \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} \rangle \\ &= \langle v - y_{n_i}, Av - A y_{n_i} \rangle + \langle v - y_{n_i}, A y_{n_i} - A u_{n_i} \rangle \\ &\quad - \langle v - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} \rangle \\ &\geq \langle v - y_{n_i}, A y_{n_i} - A u_{n_i} \rangle - \langle v - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} \rangle, \end{aligned}$$

which together with $\|u_n - y_n\| \to 0$ and A is lipschitz continuous implies that $\langle v - w, u\rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we have $w \in T^{-1}0$ and hence $w \in VI(C, A)$. Thus $w \in \Gamma$. Hence

$$\begin{aligned}\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle &= \lim_{j\to\infty}\langle f(x^*) - x^*, x_{n_j} - x^*\rangle \\ &= \lim_{j\to\infty}\langle f(x^*) - x^*, y_{n_j} - x^*\rangle \\ &= \langle f(x^*) - x^*, w - x^*\rangle \leq 0.\end{aligned}$$

Finally, we prove that x_n and u_n converges strongly to x^*.

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &= \|\alpha_n(f(W_n x_n) - x^*) + \beta_n(x_n - x^*) + \gamma_n(W_n y_n - x^*)\|^2 \\ &\leq \|\beta_n(x_n - x^*) + \gamma_n(W_n y_n - x^*)\|^2 + 2\alpha_n\langle f(W_n x_n) - x^*, x_{n+1} - x^*\rangle \\ &\leq (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\langle f(W_n x_n) - f(x^*), x_{n+1} - x^*\rangle \\ &\quad + 2\alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ &\leq (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\alpha\|x_{n+1} - x^*\|\|x_n - x^*\| \\ &\quad + 2\alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ &\leq (1-\alpha_n)^2\|x_n - x^*\|^2 + \alpha_n\alpha(\|x_{n+1} - x^*\|^2 + \|x_n - x^*\|^2) \\ &\quad + 2\alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle,\end{aligned}$$

which implies that

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &\leq \frac{(1-\alpha_n)^2 + \alpha_n\alpha}{1-\alpha_n\alpha}\|x_{n+1} - x^*\|^2 + \frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ &\leq \left[1 - \frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}\right]\|x_{n+1} - x^*\|^2 + \frac{\alpha_n^2}{1-\alpha_n\alpha}\|x_{n+1} - x^*\|^2 \\ &\quad + \frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ &\leq \left[1 - \frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}\right]\|x_{n+1} - x^*\|^2 \\ &\quad + \frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}\left[\frac{\alpha_n M}{2(1-\alpha)} + \frac{1}{1-\alpha}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle\right] \\ &= (1-\delta_n)\|x_{n+1} - x^*\|^2 + \delta_n\sigma_n,\end{aligned}$$

where $\delta_n = \frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}$ and $\sigma_n = [\,\frac{\alpha_n M}{2(1-\alpha)} + \frac{1}{1-\alpha}\langle f(x^*) - Ax^*, x_{n+1} - x^*\rangle]$. It is easy to see that $\Sigma_{n=0}^{\infty} = \infty$ and $\limsup_{n\to\infty}\sigma_n \leq 0$. By Lemma 2.5 we conclude that $x_n \to x^*$ as $n \to \infty$. □

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