A SIMPLE PROOF OF THE SION MINIMAX THEOREM

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Abstract. For convex subsets $X$ of a topological vector space $E$, we show that a KKM principle implies a Fan-Browder type fixed point theorem and that this theorem implies generalized forms of the Sion minimax theorem.

The von Neumann-Sion minimax theorem is fundamental in convex analysis and in game theory. von Neumann [8] proved his theorem for simplexes by reducing the problem to the 1-dimensional cases. Sion's generalization [7] was proved by the aid of Helly's theorem and the KKM theorem due to Knaster, Kuratowski, and Mazurkiewicz [5]. In a recent paper, Kindler [4] proved Sion's theorem by applying the 1-dimensional KKM theorem (i.e., every interval in $\mathbb{R}$ is connected), the 1-dimensional Helly theorem (i.e., any family of pairwise intersecting compact intervals in $\mathbb{R}$ has nonempty intersection), and Zorn's lemma (or other method).

In this short note, for convex subsets $X$ of a topological vector space $E$, we show that a KKM principle implies a Fan-Browder type fixed point theorem and that this theorem implies a generalized form of the Sion minimax theorem.

Definition. If a multimap $G : X \to X$ satisfies

$$\text{co } A \subset G(A) := \bigcup_{y \in A} G(y)$$

for all finite subset $A$ of $X$,

then $G$ is called a KKM map.

Definition. A multimap $T : X \to X$ is called a Fan-Browder map provided that

(a) for each $x \in X$, $T(x)$ is convex; and
(b) $X = \bigcup_{y \in N} \text{Int } T^-(y)$ for some finite subset $N$ of $X$.

Here, $\text{Int}$ denotes the interior with respect to $X$ and, for each $y \in X$, $T^-(y) := \{ x \in X \mid y \in T(x) \}$. 

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For a convex subset $X$ of a topological vector space $E$, let us consider the following statements:

(A) **The KKM principle.** For any closed-valued KKM map $G : X \rightarrow X$, the family $\{G(x)\}_{x \in X}$ has the finite intersection property.

(B) **The Fan-Browder fixed point theorem.** Any Fan-Browder map $T : X \rightarrow X$ has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.

Recall that (A) originates from the Knaster-Kuratowski-Mazurkiewicz theorem [5] and holds by Fan’s lemma [3], and (B) from Fan [3] and Browder [1].

**Theorem 1.** The statement (A) implies (B).

**Proof.** Define a map $G : X \rightarrow X$ by $G(x) := X \setminus \text{Int} T^{-}(x)$ for each $x \in X$. Then each $G(x)$ is (relatively) closed, and

$$\bigcap_{y \in N} G(y) = X \setminus \bigcup_{y \in N} \text{Int} T^{-}(y) = X \setminus \emptyset$$

by (b). Therefore, the family $\{G(x)\}_{x \in X}$ does not have the finite intersection property, and hence, $G$ is not a KKM map by (A). Thus, there exists a finite subset $A$ of $X$ such that $\text{co} A \notin G(A) = \bigcup \{X \setminus \text{Int} T^{-}(y) \mid y \in A\}$. Hence, there exists an $x_0 \in \text{co} A$ such that $x_0 \in \text{Int} T^{-}(y) \subseteq T^{-}(y)$ for all $y \in A$; that is, $A \subset T(x_0)$. Therefore, $x_0 \in \text{co} A \subset T(x_0)$ by (a).

**Theorem 2.** Let $X$ and $Y$ be nonempty convex subsets of two topological vector spaces, and $f, s, t, g : X \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$ be four functions,

$$\mu := \inf_{y \in Y} \sup_{x \in X} f(x, y) \quad \text{and} \quad \nu := \sup_{x \in X} \inf_{y \in Y} g(x, y).$$

Suppose that

1. $f(x, y) \leq s(x, y) \leq t(x, y) \leq g(x, y)$ for each $(x, y) \in X \times Y$;
2. for each $r < \mu$ and $y \in Y$, $\{x \in X \mid s(x, y) > r\}$ is convex; for each $r > \nu$ and $x \in X$, $\{y \in Y \mid t(x, y) < r\}$ is convex;
3. for each $r > \nu$, there exists a finite subset $\{x_i\}_{i=1}^n$ of $X$ such that $Y = \bigcup_{i=1}^n \text{Int} \{y \in Y \mid f(x_i, y) > r\}$; and
4. for each $r < \mu$, there exists a finite subset $\{y_j\}_{j=1}^m$ of $Y$ such that $X = \bigcup_{j=1}^m \text{Int} \{x \in X \mid g(x, y_j) < r\}$.

Then we have $\mu \leq \nu$, that is,

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} \inf_{y \in Y} g(x, y).$$

**Proof.** Suppose that there exists a real $c$ such that

$$\nu := \sup_{y \in Y} \inf_{x \in X} g(x, y) < c < \inf_{y \in Y} \sup_{x \in X} f(x, y) =: \mu.$$

Define a map $T : X \times Y \rightarrow X \times Y$ by

$$T(x, y) := \{\tilde{x} \in X \mid s(\tilde{x}, y) > c\} \times \{\tilde{y} \in Y \mid t(x, \tilde{y}) < c\}$$
for each \((x, y) \in X \times Y\). Then each \(T(x, y)\) is convex by (2.2). Moreover, for each \((\bar{x}, \bar{y}) \in X \times Y\), we have
\[
T^-(\bar{x}, \bar{y}) = \{x \in X \mid s(x, \bar{y}) > c\} \times \{y \in Y \mid t(\bar{x}, y) < c\}
\]
\[
\sup \{x \in X \mid f(x, \bar{y}) > c\} \times \{y \in Y \mid g(\bar{x}, y) < c\}
\]
\[
\text{Int} \{x \in X \mid f(x, \bar{y}) > c\} \times \text{Int} \{y \in Y \mid g(\bar{x}, y) < c\}.
\]
Therefore, by (2.3) and (2.4), \(X \times Y\) is covered by
\[
\{\text{Int } T^-(x_i, y_j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}.
\]
Hence, \(T\) is a Fan-Browder map. Since \(X \times Y\) is a convex subset of a topological vector space, (A) and (B) hold. Therefore, by (B), we have an \((x_0, y_0) \in X \times Y\) such that \((x_0, y_0) \in T(x_0, y_0)\). Therefore, \(t(x_0, y_0) < c < s(x_0, y_0)\), a contradiction.

Recall that an extended real-valued function \(f : X \to \mathbb{R}\) on a topological space \(X\) is lower [resp., upper] semicontinuous (l.s.c.) [resp., u.s.c.] if \(\{x \in X \mid f(x) > r\}\) [resp., \(\{x \in X \mid f(x) < r\}\)] is open for each \(r \in \mathbb{R}\).

For a convex set \(X\), a extended real-valued function \(f : X \to \mathbb{R}\) is said to be quasiconcave [resp., quasiconvex] if \(\{x \in E \mid f(x) > r\}\) [resp., \(\{x \in E \mid f(x) < r\}\)] is convex for each \(r \in \mathbb{R}\).

**Theorem 3.** Let \(X\) and \(Y\) be compact convex subsets of topological vector spaces, and \(f, s, t, g : X \times Y \to \mathbb{R} \cup \{+\infty\}\) be functions satisfying
\[
\text{(3.1) } f(x, y) \leq s(x, y) \leq t(x, y) \leq g(x, y) \text{ for each } (x, y) \in X \times Y;
\]
\[
\text{(3.2) for each } x \in X, \ f(x, \cdot) \text{ is l.s.c. and } t(x, \cdot) \text{ is quasiconvex on } Y; \text{ and}
\]
\[
\text{(3.3) for each } y \in Y, \ s(\cdot, y) \text{ is quasiconcave and } g(\cdot, y) \text{ is u.s.c. on } X.
\]
Then we have
\[
\min \sup_{y \in Y} f(x, y) \leq \max \inf_{x \in X} g(x, y).
\]

**Proof.** Note that \(y \mapsto \sup_{x \in X} f(x, y)\) is l.s.c. on \(Y\) and \(x \mapsto \inf_{y \in Y} g(x, y)\) is u.s.c. on \(X\). Therefore, the both sides of the inequality exist. Then all the requirements of Theorem 2 are satisfied. \(\square\)

For \(f = s = t = g\) in Theorem 3, we have the following Sion minimax theorem [7]:

**Theorem 4.** Let \(X\) and \(Y\) be compact convex subsets of topological vector spaces and \(f : X \times Y \to \mathbb{R}\) a real function such that
\[
\text{(4.1) for each } x \in X, \ f(x, \cdot) \text{ is l.s.c. and quasiconvex on } Y; \text{ and}
\]
\[
\text{(4.2) for each } y \in Y, \ f(\cdot, y) \text{ is u.s.c. and quasiconcave on } X.
\]
Then
\[
\text{(i) } f \text{ has a saddle point } (x_0, y_0) \in X \times Y; \text{ and}
\]
\[
\text{(ii) we have } \min_{y \in Y} \max_{x \in X} f(x, y) = \max_{x \in X} \min_{y \in Y} f(x, y).
\]
Proof. It is well known and easy to see that the minima and maxima in Theorem 4 exist under our topological assumptions. Hence, there exists an \((x_0, y_0) \in X \times Y\) such that
\[
\min_{y \in Y} \max_{x \in X} f(x, y) = \max_{x \in X} f(x, y_0) \geq \min_{y \in Y} f(x_0, y) = \max_{x \in X} \min_{y \in Y} f(x, y).
\]
Moreover, all the requirements of Theorem 3 with \(f = g\) are satisfied. Therefore, the \(\geq\)’s in the above should be = and we have the conclusion. □

Remark 1. von Neumann [8] obtained Theorem 4 when \(X\) and \(Y\) are subsets of Euclidean spaces and \(f\) is continuous.

2. (A) also holds for open-valued KKM maps, and (B) also holds when \(T^-\) has closed values. In this case, (A) implies (B) also.

3. For other simple proof of the Sion minimax theorem, see [4].

4. Theorem 2 is motivated from [2, Theorem 8], which is for \(f = s = t = g\).

5. For the history of the KKM theory, see [6].

6. All the results in this paper can be extended to abstract convex spaces without assuming any linear structure.

References


