Approximately Quadratic Derivations and Generalized Homomorphisms

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Abstract. Let $A$ be a unital Banach algebra. If $f : A \rightarrow A$ is an approximately quadratic derivation in the sense of Hyers-Ulam-J.M. Rassias, then $f : A \rightarrow A$ is an exactly quadratic derivation. On the other hands, let $A$ and $B$ be Banach algebras. Any approximately generalized homomorphism $f : A \rightarrow B$ corresponding to Cauchy, Jensen functional equation can be estimated by a generalized homomorphism.

1. Introduction

In 1940, S. M. Ulam\textsuperscript{26} proposed the following question concerning the stability of group homomorphisms:

Let $G_1$ be a group and let $G_2$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \rightarrow G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \rightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In next year, D.H. Hyers\textsuperscript{10} answers the problem of Ulam under the assumption that the groups are Banach spaces: if $\epsilon > 0$ and $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a mapping with $\mathcal{X}$ a normed space, $\mathcal{Y}$ a Banach space such that

$$||f(x + y) - f(x) - f(y)|| \leq \epsilon$$

for all $x, y \in \mathcal{X}$, then there exists a unique additive mapping $T : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$||f(x) - T(x)|| \leq \epsilon$$

for all $x \in \mathcal{X}$.

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A generalized version of the theorem of Hyers for approximately linear mappings was given by Th.M. Rassias [21] by introducing the unbounded Cauchy difference. Since then, the stability problems of several functional equation have been extensively investigated by a number of authors (for instance, [1, 3, 6, 23]).

On the other hand, J.M. Rassias [19] generalized the Hyers’ stability result by presenting a weaker condition controlled by (or involving) a product of different powers of norms (from the right-hand side of assumed conditions). That is, assume that there exist constants \( \varepsilon \geq 0 \) and \( p_1, p_2 \in \mathbb{R} \) such that \( p = p_1 + p_2 \neq 1 \), and \( f : X \to Y \) is a mapping with \( X \) a normed space, \( Y \) a Banach space such that the inequality

\[
\|f(x + y) - f(x) - f(y)\| \leq \varepsilon \|x\|^{p_1} \|y\|^{p_2}
\]

for all \( x, y \in X \), then there exist a unique additive mapping \( T : X \to Y \) such that

\[
\|f(x) - T(x)\| \leq \frac{\varepsilon}{2 - 2p} \|x\|^p
\]

for all \( x \in X \). If, in addition, \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \) in \( X \), then \( T \) is linear.

A counter-example for a singular case of this result was given by P. Găvrută [7].

Particularly, one of the important functional equations studied is the following functional equation:

\[
f(x + y) + f(x - y) = 2f(x) + 2f(y).
\]

The quadratic function \( f(x) = ax^2 \) is a solution of this functional equation, and so one usually is said the above functional equation to be quadratic [1, 13, 20].

The Hyers-Ulam stability problem of the quadratic functional equation was first proved by F. Skof [25] for functions between a normed space and a Banach space. Afterwards, her result was extended by P.W. Cholewa [4] and S. Czerwik [5]:

If \( p \neq 2 \) and \( f : \mathcal{X} \to \mathcal{Y} \) is a mapping with \( \mathcal{X} \) a normed space, \( \mathcal{Y} \) a Banach space such that

\[
\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)
\]

for all \( x, y \in \mathcal{A} \), then there exists a unique quadratic mapping \( Q : \mathcal{X} \to \mathcal{Y} \) such that

\[
\|f(x) - Q(x)\| \leq c + k\varepsilon\|x\|^p
\]

for all \( x \in \mathcal{X} \) if \( p \geq 0 \) and for all \( x \in \mathcal{X} \setminus \{0\} \) if \( p < 0 \), where: when \( p < 2 \), \( c = \frac{\|f(0)\|}{3} \), \( k = \frac{2}{4 - 2p} \) and when \( p > 2 \), \( c = 0 \), \( k = \frac{2}{2p - 4} \).
Let $A$ be an algebra over the real or complex field $F$. An additive mapping $d : A \to A$ is said to be a *ring derivation* if the functional equation $d(xy) = xd(y) + d(x)y$ holds for all $x, y \in A$.

T. Miura *et al.* [18] investigated the stability of ring derivations on Banach algebras:

*Suppose that $A$ is a Banach algebra, $p \geq 0$ and $\varepsilon \geq 0$. If $p \neq 1$ and $f : A \to A$ is a mapping such that

$$
\|f(x + y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)
$$

for all $x, y \in A$ and

$$
\|f(xy) - xf(y) - f(x)y\| \leq \varepsilon\|x\|^p\|y\|^p
$$

for all $x, y \in A$, then there exists a unique ring derivation $d : A \to A$ such that

$$
\|f(x) - d(x)\| \leq \frac{2\varepsilon}{|2 - 2^p|} \|x\|^p
$$

for all $x \in A$. In particular, if $A$ is a Banach algebra without order, then $f$ is an ring derivation.*

Several results for the stability of derivations have been obtained by many authors (for instances, [2, 16, 17, 24]).

We here introduce the following mapping:

A quadratic mapping $D : A \to A$ is said to be a *quadratic derivation* if the functional equation $D(xy) = x^2D(y) + D(x)y^2$ holds for all $x, y \in A$. As a simple example, let us consider the algebra of $2 \times 2$ matrices

$$
A = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : \ a, b \in \mathbb{C} \right\},
$$

where $\mathbb{C}$ is a complex field. Then it is easy to see that the mapping $D : A \to A$ defined by

$$
D\left( \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & a^2 \\ 0 & 0 \end{bmatrix}
$$

is a quadratic derivation. Here it is natural to ask that there exists an approximately quadratic derivation which is not an exactly quadratic derivation. The following example is a slight modification of an example due to [18].

**Example.** Let $X$ be a compact Hausdorff space and let $C(X)$ be the commutative Banach algebra of real-valued continuous functions on $X$ under pointwise operations
and the supremum norm $\| \cdot \|_\infty$. We define $f : C(X) \to C(X)$ by

$$f(a)(x) = \begin{cases} a(x)^2 \log |a(x)| & \text{if } a(x) \neq 0, \\ 0 & \text{if } a(x) = 0 \end{cases}$$

for all $a \in C(X)$ and $x \in X$. It is easy to see that

$$f(ab) = a^2 f(b) + f(a)b^2$$

for all $a, b \in C(X)$.

Note that the following inequality holds for all $u, v \in \mathbb{R} \setminus \{0\}$ with $u + v \neq 0$, where $\mathbb{R}$ is a real field,

$$|(u + v)^2 \log |u + v| + (u - v)^2 \log |u - v| - 2u^2 \log |u| - 2v^2 \log |v|| \leq 4|u| |v|$$

In fact, fix $u, v \in \mathbb{R} \setminus \{0\}$, $u + v \neq 0$ arbitrarily. Since $\log(1 + x) \leq x$ for all $x \geq 0$,

$$\begin{align*}
|(u + v)^2 \log |u + v| + (u - v)^2 \log |u - v| - 2u^2 \log |u| - 2v^2 \log |v| & \leq |(u + v)^2 \log(|u| + |v|) + (u - v)^2 \log(|u| - |v|) - 2u^2 \log |u| - 2v^2 \log |v|| \\
& = |2(u^2 + v^2) \log(|u| + |v|) - 2u^2 \log |u| - 2v^2 \log |v|| \\
& \leq 2|u|^2 \log \frac{|u| + |v|}{|u|} + 2|v|^2 \log \frac{|u| + |v|}{|v|} \\
& \leq 2|u|^2 \log \left(1 + \frac{|v|}{|u|}\right) + 2|v|^2 \log \left(1 + \frac{|u|}{|v|}\right) \\
& \leq 2|u|^2 \frac{|v|}{|u|} + 2|v|^2 \frac{|u|}{|v|} = 4|uv|
\end{align*}$$

which gives

$$\|f(a + b) + f(a - b) - 2f(a) - 2f(b)\|_\infty \leq 4\|ab\|_\infty$$

for all $a, b \in C(X)$. Hence we may regard $f$ as an approximately quadratic derivation on $C(X)$.

Let $A$ and $B$ be Banach algebras and let $h : A \to B$ be a linear mapping. Define the bilinear mapping $H$ by $H(x, y) = h(xy) - h(x)h(y)$ for all $x, y \in A$. We say that $h$ is a generalized homomorphism if $H$ is continuous in $x \in A$ for each fixed $y \in A$ and in $y \in A$ for each fixed $x \in A$, respectively. The mapping was introduced by B.E. Johnson [12].

By an approximately generalized homomorphism corresponding to a functional equation $\mathcal{E}(f) = 0$, we mean a mapping $f : A \to B$ such that

$$\|\mathcal{E}(f)\| \leq \varepsilon$$
and the mapping $F : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{B}$ defined by
\begin{equation}
F(x, y) = f(xy) - f(x)f(y)
\end{equation}
for all $x, y \in \mathcal{A}$, is continuous in $x \in \mathcal{A}$ for each fixed $y \in \mathcal{A}$ and in $y \in \mathcal{A}$ for each fixed $x \in \mathcal{A}$, respectively.

In Section 2, we prove the stability in the sense of Hyers-Ulam-J.M. Rassias and the superstability of quadratic derivations on Banach algebras as in the case of ring derivations. In Section 3 and 4, the stability of generalized homomorphisms on Banach algebras via Cauchy, Jensen equations is established, respectively.

2. Stability of Quadratic Derivations

In this section, $\mathbb{Q}$ and $\mathbb{N}$ will denote the set of the rational and the natural numbers, respectively.

Lemma 2.1. Suppose that $\mathcal{A}$ is a Banach algebra. Let $\delta, \varepsilon \geq 0$ and let $p, q \geq 0$ with either $p < 1$, $q < 2$ or $p > 1$, $q > 2$. If $f : \mathcal{A} \rightarrow \mathcal{A}$ is a mapping such that
\begin{equation}
\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \delta \|x\|^p \|y\|^p
\end{equation}
for all $x, y \in \mathcal{A}$ and
\begin{equation}
\|f(xy) - x^2 f(y) - f(x)y^2\| \leq \varepsilon \|x\|^q \|y\|^q
\end{equation}
for all $x, y \in \mathcal{A}$, then there exists a unique quadratic derivation $D : \mathcal{A} \rightarrow \mathcal{A}$ such that
\begin{equation}
\|f(x) - D(x)\| \leq k\delta \|x\|^{2p}
\end{equation}
for all $x \in \mathcal{A}$, where $k = \frac{1}{4} - \frac{1}{4p}$ if $p < 1$ and $k = \frac{1}{4} - \frac{1}{4q}$ if $p > 1$.

Proof. Assume that either $p < 1$, $q < 2$ or $p > 1$, $q > 2$. Set $\tau = 1$ if $p < 1$, $q < 2$ and $\tau = -1$ if $p > 1$, $q > 2$. In (2), put $x = y = 0$ to see that $f(0) = 0$. Hence, following Czerwik’s process [5] using the direct method, we obtain from (2)
\begin{equation}
\|4^{-n}f(2^n x) - f(x)\| \leq \varepsilon \|x\|^{2p} \sum_{k=1}^{n} 2^{2(k-1)p} 4^{-k}
\end{equation}
for all $x \in \mathcal{A}$ and all $n \in \mathbb{N}$ if $p < 1$, and
\begin{equation}
\|f(x) - 4^n f(2^{-n} x)\| \leq \left(\frac{\varepsilon}{4}\right) \|x\|^{2p} \sum_{k=1}^{n} 2^{-2k(p-1)}
\end{equation}
for all \( x \in \mathcal{A} \) and all \( n \in \mathbb{N} \) if \( p > 1 \). Using these inequalities and Czerwik’s process, we see that there exists a unique quadratic mapping \( D : \mathcal{A} \to \mathcal{A} \) defined by
\[
D(x) = \lim_{n \to \infty} 4^{-\tau n} f(2^{\tau n} x)
\]
for all \( x \in \mathcal{A} \) such that
\[
\| f(x) - D(x) \| \leq k\delta \| x \|^{2p}
\]
for all \( x \in \mathcal{A} \), where \( k = \frac{1}{4^p - 4} \) if \( p < 1 \) and \( k = \frac{1}{4^p - 4} \) if \( p > 1 \).

We claim that
\[
D(xy) = x^2 D(y) + D(x) y^2
\]
for all \( x, y \in \mathcal{A} \). Since \( D \) is quadratic, we see that \( D(x) = 4^{-\tau n} D(2^{\tau n} x) \) for all \( x \in \mathcal{A} \) and all \( n \in \mathbb{N} \). First, it follows from (4) that
\[
\| 4^{-\tau n} f(2^{\tau n} x) - D(x) \| = 4^{-\tau n} \| f(2^{\tau n} x) - D(2^{\tau n} x) \|
\]
\[
\leq 4^{-\tau n} k\delta \| 2^{\tau n} x \|^{2p}
\]
\[
= 4^{(p-1)n} k\delta \| x \|^{2p}
\]
for all \( x \in \mathcal{A} \) and all \( n \in \mathbb{N} \). Since \( \tau(p - 1) < 0 \), we have
\[
(5) \quad \| 4^{-\tau n} f(2^{\tau n} x) - D(x) \| \to 0 \quad \text{as} \quad n \to \infty.
\]

Following the similar argument as the above, we obtain
\[
\| 4^{-2\tau n} f(2^{2\tau n} xy) - D(xy) \| \leq 4^{\tau(p-1)n} k\delta \| xy \|^{2p}
\]
for all \( x, y \in \mathcal{A} \) and all \( n \in \mathbb{N} \), and so
\[
(6) \quad \| 4^{-2\tau n} f(2^{2\tau n} xy) - D(xy) \| \to 0 \quad \text{as} \quad n \to \infty.
\]

Since \( f \) satisfies (3), we get
\[
\| 4^{-2\tau n} f(2^{2\tau n} xy) - 4^{-\tau n} x^2 f(2^{\tau n} y) - f(2^{\tau n} x) 4^{-\tau n} y^2 \|
\]
\[
= 4^{-2\tau n} \| f((2^{\tau n} x)(2^{\tau n} y)) - (2^{\tau n} x)^2 f(2^{\tau n} y) - f(2^{\tau n} x)(2^{\tau n} y)^2 \|
\]
\[
\leq 4^{-2\tau n} \varepsilon \| 2^{\tau n} x \|^q \| 2^{\tau n} y \|^q = 2^{\tau n(q-2)} \varepsilon \| x \|^q \| y \|^q
\]
for all \( x, y \in \mathcal{A} \) and all \( n \in \mathbb{N} \). Invoking \( \tau(q - 2) < 0 \), we obtain
\[
(7) \quad \| 4^{-2\tau n} f(2^{2\tau n} xy) - 4^{-\tau n} x^2 f(2^{\tau n} y) - f(2^{\tau n} x) 4^{-\tau n} y^2 \| \to 0 \quad \text{as} \quad n \to \infty.
\]

Using (5), (6) and (7), we now see that
\[
\| D(xy) - x^2 D(y) - D(x) y^2 \|
\]
\[
\leq \| D(xy) - 4^{-2\tau n} f(2^{2\tau n} xy) \|.
\]
+ \|4^{-2}r_n f(2^{2}r_n x^2 f(2^{2}r_n y) - 4^{-2}r_n f(2^{2}r_n x^2 (2^{2}r_n y)^2\| \\
+ \|4^{-2}r_n x^2 f(2^{2}r_n y) - x^2 D(y)\| + \|4^{-2}r_n f(2^{2}r_n x^2 y^2 - D(x)y^2\| \\
\leq \|D(xy) - 4^{-2}r_n f(2^{2}r_n xy)\| \\
+ \|4^{-2}r_n f(2^{2}r_n xy) - 4^{-2}r_n f(2^{2}r_n xy - 4^{-2}r_n f(2^{2}r_n xy)^2\| \\
+ \|x^2\|4^{-2}r_n f(2^{2}r_n y) - D(y)\| + \|f(2^{2}r_n x)4^{-2}r_n - D(x)\|y^2\| \to 0 \text{ as } n \to \infty \\
which implies that } D(xy) = x^2 D(y) + D(x)y^2 \text{ for all } x \in A. \text{ Namely, } D \text{ is a quadratic derivation, as claimed and the proof is complete.} \]

**Lemma 2.2.** Suppose that \( \mathcal{A} \) is a unital Banach algebra. Let \( \delta, \varepsilon \geq 0 \) and let \( p, q \geq 0 \) with either \( p < 1, q < 2 \) or \( p > 1, q > 2 \). If \( f : \mathcal{A} \to \mathcal{A} \) is a mapping satisfying (2) and (3), then we have we have 

\[
f(rx) = r^2f(x)
\]

for all \( x \in \mathcal{A} \) and all \( r \in \mathbb{Q} \).

**Proof.** In the case when \( r = 0 \), it is trivial since \( f(0) = 0 \). Let \( e \) be a unit element of \( \mathcal{A} \) and \( r \in \mathbb{Q} \setminus \{0\} \) arbitrarily. Put \( \tau = 1 \) if \( p < 1, q < 2 \) and \( \tau = -1 \) if \( p > 1, q > 2 \). Hence it follows that \( \tau(p - 1) < 0 \) and \( \tau(q - 2) < 0 \). By Lemma 2.1, there exists a unique quadratic derivation \( D : \mathcal{A} \to \mathcal{A} \) such that (4) is true. Recall that \( D \) is quadratic, and hence it is easy to see that \( D(rx) = r^2D(x) \) for all \( x \in \mathcal{A} \). Then we get

\[
\|D((2^{2}r_n e)(rx)) - r^22^{2}r_n e f(x) - f(2^{2}r_n e)r^2 x^2\| \\
\leq r^2\|D(2^{2}r_n e x) - f(2^{2}r_n e x)\| + r^2\|f(2^{2}r_n e x) - 4^{2}r_n e f(x) - f(2^{2}r_n e x^2\| \\
\text{for all } x \in \mathcal{A} \text{ and all } n \in \mathbb{N}. \text{ Now the inequalities (3), (4) and (8) yields that}
\]

\[
\|D((2^{2}r_n e)(rx)) - r^22^{2}r_n e f(x) - f(2^{2}r_n e)r^2 x^2\| \\
\leq r^22^{2}r_n e \|x\|^{2p} + r^22^{2}r_n e \|x\|^{q}
\]

for all \( x \in \mathcal{A} \) and all \( n \in \mathbb{N}. \)

It follows from (4) and (9) that

\[
\|f((2^{2}r_n e)(rx)) - r^22^{2}r_n e f(x) - f(2^{2}r_n e)r^2 x^2\| \\
\leq \|f((2^{2}r_n e)(rx)) - D((2^{2}r_n e)(rx))\| \\
+ \|D((2^{2}r_n e)(rx)) - r^22^{2}r_n e f(x) - f(2^{2}r_n e)r^2 x^2\| \\
\leq k\delta 4^{2}r_n e (r^2 p + r^2)\|x\|^{2p} + r^22^{2}r_n e \|x\|^{q}
\]
for all $x \in \mathcal{A}$ and all $n \in \mathbb{N}$. That is, we have
\[
\|f((2^{2n}e)(rx)) - r^2 2^{2n}ef(x) - f(2^{2n}e)r^2x^2\| \leq k\delta 4^{\tau n}(r^{2p} + r^2)\|x\|^{2p} + r^2 2^{\tau n}q\|x\|^q
\]
for all $x \in \mathcal{A}$ and all $n \in \mathbb{N}$. From (3) and (10), we obtain
\[
\|4^{\tau n}\{f(rx) - r^2 f(x)\}\|
\leq \|2^{2\tau n}ef(rx) + f(2^{\tau n}e)r^2x^2 - f((2^{\tau n}e)(rx))\|
+ \|f((2^{\tau n}e)(rx)) - r^2 2^{2\tau n}ef(x) - f(2^{\tau n}e)r^2x^2\|
\leq \varepsilon\|2^{\tau n}e\|\|rx\|^q + k\delta 4^{\tau np}(r^{2p} + r^2)\|x\|^{2p} + r^2 2^{\tau n}q\|x\|^q
= 2^{\tau n}(r^q + r^2)\varepsilon\|x\|^q + k\delta 4^{\tau np}(r^{2p} + r^2)\|x\|^{2p}
\]
for all $x \in \mathcal{A}$ and all $n \in \mathbb{N}$. This means that
\[
\|f(rx) - r^2 f(x)\|
\leq 2^{\tau (q-2)n}(r^q + r^2)\varepsilon\|x\|^q + k\delta 4^{\tau (p-1)n}(r^{2p} + r^2)\|x\|^{2p}
\]
for all $x \in \mathcal{A}$ and all $n \in \mathbb{N}$. If we take $n \to \infty$ in (11), then we arrive at
\[
f(rx) = r^2 f(x)
\]
for all $x \in \mathcal{A}$. This completes the proof since $r \in \mathbb{Q} \setminus \{0\}$ was arbitrary. \(\square\)

Now we are ready to prove the main result in this section.

**Theorem 2.3.** Suppose that $\mathcal{A}$ is a unital Banach algebra. Let $\delta, \varepsilon \geq 0$ and let $p, q \geq 0$ with either $p < 1$, $q < 2$ or $p > 1$, $q > 2$. If $f : \mathcal{A} \to \mathcal{A}$ is a mapping satisfying (2) and (3), then $f : \mathcal{A} \to \mathcal{A}$ is a quadratic derivation.

**Proof.** Let $D$ be a unique quadratic derivation as in Lemma 2.2. Put $\tau = 1$ if $p < 1$, $q < 2$ and $\tau = -1$ if $p > 1$, $q > 2$. Since $f(2^{\tau n}x) = 4^{\tau n}f(x)$ for all $x \in \mathcal{A}$ and all $n \in \mathbb{N}$ by Lemma 2.2, it follows from (4) that
\[
\|f(x) - D(x)\| = \|4^{\tau n}f(2^{\tau n}x) - 4^{\tau n}D(2^{\tau n}x)\|
\leq 4^{\tau n}k\delta\|2^{\tau n}x\|^{2p}
= 4^{\tau (p-1)n}k\delta\|x\|^{2p}
\]
for all $x \in \mathcal{A}$ and all $n \in \mathbb{N}$. Namely,
\[
\|f(x) - D(x)\| \leq 4^{\tau (p-1)n}k\delta\|x\|^{2p}
\]
for all \( x \in \mathcal{A} \) and all \( n \in \mathbb{N} \). Since \( \tau(p - 1) < 0 \), if we let \( n \to \infty \) in (12), then we conclude that \( f(x) = D(x) \) for all \( x \in \mathcal{A} \) which implies that \( f \) is a quadratic derivation. 

\[ \square \]

3. Stability of Generalized Homomorphisms via Cauchy Equation

We begin with our investigation establishing the stability of generalized homomorphisms via Cauchy equation. From now on, \( \mathcal{A} \) and \( \mathcal{B} \) denote Banach algebras.

**Theorem 3.1.** Let \( \varepsilon \geq 0 \). For each approximately generalized homomorphism \( f : \mathcal{A} \to \mathcal{B} \) corresponding to the Cauchy inequality

\[ ||f(\alpha x + \beta y) - \alpha f(x) - \beta f(y)|| \leq \varepsilon, \]

for all \( x, y \in \mathcal{A} \) and all \( \alpha, \beta \in \mathbb{U} = \{ \mu \in \mathbb{C} : |\mu| = 1 \} \), there exists a unique generalized homomorphism \( h : \mathcal{A} \to \mathcal{B} \) such that

\[ ||f(x) - h(x)|| \leq \varepsilon \]

for all \( x \in \mathcal{A} \).

**Proof.** Let us the second variable of \( F \) be fixed. Then, by hypothesis, for each fixed \( z \in \mathcal{A} \), the mapping \( F : \mathcal{A} \times \mathcal{A} \to \mathcal{B} \) satisfies the inequality

\[ ||F(\alpha x + \beta y, z) - \alpha F(x, z) - \beta F(y, z)|| \leq ||f(\alpha xz + \beta yz) - f(\alpha x + \beta y)f(z) - \alpha f(xz)f(z) + \alpha f(x)f(z) - \beta f(yz)f(z) + \beta f(y)f(z)|| \]

\[ \leq (1 + ||f(z)||)\varepsilon, \]

that is, we obtain the inequality

\[ ||F(\alpha x + \beta y, z) - \alpha F(x, z) - \beta F(y, z)|| \leq (1 + ||f(z)||)\varepsilon \]

for all \( x, y \in \mathcal{A} \) and all \( \alpha, \beta \in \mathbb{U} \).

Putting \( \alpha = \beta = 1 \) in (15) and utilizing the Hyers’ direct method [10], there is an additive mapping in the first variable \( S : \mathcal{A} \times \mathcal{A} \to \mathcal{B} \) such that

\[ ||F(x, z) - S(x, z)|| \leq (1 + ||f(z)||)\varepsilon \]
for all \(x \in \mathcal{A}\), where
\[
S(x, z) = \lim_{n \to \infty} \frac{F(2^n x, z)}{2^n}
\]
for all \(x \in \mathcal{A}\). Replacing \(x, y\) by \(2^n x, 2^n y\) in (15), we get
\[
\|2^{-n}F(2^n(\alpha x + \beta y), z) - \alpha 2^{-n}F(2^n x, z) - \beta 2^{-n}F(2^n y, z)\| \leq 2^{-n}(1 + \|f(z)\|)\varepsilon
\]
for all \(x, y \in \mathcal{A}\) and all \(\alpha, \beta \in \mathbb{U}\). Taking limits as \(n \to \infty\), we obtain
\[
S(\alpha x + \beta y, z) = \alpha S(x, z) + \beta S(y, z)
\]
for all \(x, y \in \mathcal{A}\) and all \(\alpha, \beta \in \mathbb{U}\).

Clearly, \(S(0x, z) = 0 = 0S(x, z)\) for all \(x \in \mathcal{A}\). Now, let \(\lambda \in \mathbb{C}\) (\(\lambda \neq 0\)), and let \(M \in \mathbb{N}\) greater than \(|\lambda|\). By applying a geometric argument, we see that there exists \(\alpha_1, \alpha_2 \in \mathbb{U}\) such that \(2\frac{\lambda}{M} = \alpha_1 + \alpha_2\). By the additivity of \(S(\cdot, z)\), we get
\[
S(\frac{1}{2}x, z) = \frac{1}{2}S(x, z)
\]
for all \(x \in \mathcal{A}\). Therefore
\[
S(\lambda x, z) = S\left(\frac{M}{2} \cdot 2 \cdot \frac{\lambda}{M} x, z\right) = MS\left(\frac{M}{2} \cdot \frac{\lambda}{M} x, z\right) = \frac{M}{2}S((\alpha_1 + \alpha_2)x, z)
\]
\[
= \frac{M}{2}(\alpha_1 + \alpha_2)S(x, z) = \frac{M}{2} \cdot 2 \cdot \frac{\lambda}{M} S(x, z) = \lambda S(x, z)
\]
for all \(x \in \mathcal{A}\), so that the mapping \(S : \mathcal{A} \times \mathcal{A} \to \mathcal{B}\) is \(\mathbb{C}\)-linear in the first variable.

From the Hyers’ theorem [10], the inequality (13) with \(\alpha = \beta = 1\) guarantees that there exists a unique additive mapping \(h : \mathcal{A} \to \mathcal{B}\) defined by
\[
h(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}
\]
for all \(x \in \mathcal{A}\) satisfying the inequality (14). Applying a similar approach of (15)~(19) to (13), we see that \(h\) is \(\mathbb{C}\)-linear.

For each fixed \(x \in \mathcal{A}\), we note that the mapping \(F : \mathcal{A} \times \mathcal{A} \to \mathcal{B}\) satisfies the inequality
\[
\|2^{-n}F(2^n x, \alpha y + \beta z) - \alpha 2^{-n}F(2^n x, y) - \beta 2^{-n}F(2^n x, z)\|
\leq \|2^{-n}f(\alpha 2^n(xy) + \beta 2^n(z)) - 2^{-n}f(2^n x)f(\alpha y + \beta z)
\]
\[- \alpha 2^{-n}f(2^n(xy)) + \alpha 2^{-n}f(2^n x)f(y) - \beta 2^{-n}f(2^n(z)) + \beta 2^{-n}f(2^n x)f(z)\|
\leq 2^{-n}\|f(\alpha 2^n(xy) + \beta 2^n(z)) - \alpha f(2^n(xy)) - \beta f(2^n(z))\|
\]
\[+ 2^{-n}\|f(2^n x)\|\|f(\alpha x + \beta y) - \alpha f(x) - \beta f(y)\|
\leq 2^{-n}\varepsilon + 2^{-n}\|f(2^n x)\|\varepsilon,
\]
Letting \( n \to \infty \) in this inequality, it follows from (17) that the inequality
\[
\|S(x, \alpha y + \beta z) - \alpha S(x, y) - \beta S(x, z)\| \leq \|h(x)\|\varepsilon
\]
holds for all \( y, z \in \mathcal{A} \). Following the same process as (15)∼(19) with (20), it follows that the mapping \( H : \mathcal{A} \times \mathcal{A} \to \mathcal{B} \) defined by
\[
H(x, y) = \lim_{n \to \infty} \frac{S(x, 2^n y)}{2^n}
\]
for all \( z \in \mathcal{A} \), is \( \mathbb{C} \)-linear in second variable. Since \( S \) was \( \mathbb{C} \)-linear in first variable, \( H \) is also \( \mathbb{C} \)-linear in first variable. Hence, we conclude that \( H \) is \( \mathbb{C} \)-bilinear.

From (13), we obtain
\[
F(2^n x, y) = f(2^n (xy)) - f(y)
\]
for all \( x, y \in \mathcal{A} \), and so taking \( n \to \infty \) in (22) yields
\[
S(x, y) = h(xy) - h(x)f(y)
\]
for all \( x, y \in \mathcal{A} \). Replacing \( y \) by \( 2^n y \) in (23), we get
\[
S(x, 2^n y) = h(xy) - h(x)f(2^n y)
\]
for all \( x, y \in \mathcal{A} \). Now, setting \( n \to \infty \) in the both sides of (24) gives
\[
H(x, y) = h(xy) - h(x)h(y)
\]
for all \( x, y \in \mathcal{A} \).

To show the continuity of \( H \) in \( x \in \mathcal{A} \) for each fixed \( y \in \mathcal{A} \) we use the way of [10]. Assume that \( F \) is continuous in \( x \in \mathcal{A} \) for each fixed \( y \in \mathcal{A} \). If \( S \) is not continuous at a point \( x \in \mathcal{A} \) for some fixed \( y_0 \in \mathcal{A} \), then there exist a positive integer \( \eta \) and a sequence \( \{x_n\} \) in \( \mathcal{A} \) converging to zero such that
\[
\|S(x_n, y_0)\| > \frac{1}{\eta}
\]
for all \( n \in \mathbb{N} \). Let \( k \) be an integer greater than \( 3\eta\delta \), where \( \delta = (\|1 + f(y_0)\|\varepsilon) \). Then we have
\[
\|S(kx_n, y_0) - S(0, y_0)\| = \|S(kx_n, y_0)\| > 3\delta
\]
for all \( n \in \mathbb{N} \).

But, from (16), we obtain the inequality
\[
\|S(kx_n, y_0) - S(0, y_0)\| \leq \|S(kx_n, y_0) - F(kx_n, y_0)\| + \|F(kx_n, y_0) - F(0, y_0)\| + \|F(0, y_0) - S(0, y_0)\| \leq 3\delta
\]
for sufficiently large \( n \), since \( F(kx_n, y_0) \to F(0, y_0) \) as \( n \to \infty \). This contradiction means that \( S \) is continuous in \( x \in \mathcal{A} \) for each fixed \( y \in \mathcal{A} \). Hence, the relation (21) tells us that \( H \) is continuous in \( x \in \mathcal{A} \) for each fixed \( y \in \mathcal{A} \).

To prove that the mapping \( H \) defined by (25) is continuous in \( y \in \mathcal{A} \) for each fixed \( x \in \mathcal{A} \), let us the first variable of \( F \) be fixed. By the similar one to the manner obtaining the inequality (15), we see that for each fixed \( x \in \mathcal{A} \), the mapping \( F: \mathcal{A} \times \mathcal{A} \to \mathcal{B} \) satisfies the inequality

\[
\| F(x, \alpha y + \beta z) - \alpha F(x, y) - \beta F(x, z) \| \leq (1 + \| f(x) \|) \varepsilon
\]

for all \( y, z \in \mathcal{A} \) and all \( \alpha, \beta \in \mathbb{U} \). Now, the remainder of the proof carries over almost verbatim among (16)~(26). So we conclude that \( H \) is continuous in \( y \in \mathcal{A} \) for each fixed \( x \in \mathcal{A} \). Consequently, \( h \) is a generalized homomorphism. \( \square \)

4. Stability of Generalized Homomorphisms via Jensen Equation

Consider the Jensen equation

\[
2f\left(\frac{x + y}{2}\right) = f(x) + f(y).
\]

It is well known that a function \( f \) between vector spaces with \( f(0) = 0 \) satisfies the Jensen equation if and only if it is additive. In this section, we obtain the stability result of generalized homomorphisms via the Jensen equation.

**Theorem 4.1.** Let \( \varepsilon \geq 0 \) and let \( f: \mathcal{A} \to \mathcal{B} \) be an approximately generalized homomorphism corresponding to the Jensen inequality

\[
\left\| 2f\left(\frac{\alpha x + \beta y}{2}\right) - \alpha f(x) - \beta f(y) \right\| \leq \varepsilon,
\]

for all \( x, y \in \mathcal{A} \) and all \( \alpha, \beta \in \mathbb{I} = \{1, i\} \). For each fixed \( z \in \mathcal{A} \) (resp. \( x \in \mathcal{A} \)), there is a positive number \( r_z \) (resp. \( r_x \)) such that the real functions \( t \mapsto \| F(tx, z) \| \) (resp. \( t \mapsto \| F(x, tz) \| \)) is bounded on the interval \([0, r_z]\) (resp. \([0, r_x]\)). Then there exists a unique generalized homomorphism \( h: \mathcal{A} \to \mathcal{B} \) such that

\[
\| f(x) - h(x) \| \leq \varepsilon
\]

for all \( x \in \mathcal{A} \).
Proof. By hypothesis, for each fixed $z \in A$, the mapping $F : A \times A \to B$ satisfies the inequality
\[
\left\| 2F \left( \frac{\alpha x + \beta y}{2}, z \right) - \alpha F(x, z) - \beta F(y, z) \right\| \\
\leq \left\| 2f \left( \frac{\alpha x + \beta y}{2} \right) - 2f \left( \frac{\alpha x + \beta y}{2} \right) f(z) \right. \\
- \alpha f(xz) + \alpha f(x)f(z) - \beta f(yz) + \beta f(y)f(z) \left. \right\| \\
\leq \left\| 2f \left( \frac{\alpha x + \beta y}{2} \right) - \alpha f(xz) - \beta f(yz) \right\| \\
+ \left\| 2f \left( \frac{\alpha x + \beta y}{2} \right) - \alpha f(x) - \beta f(y) \right\| \| f(z) \| \\
\leq (1 + \| f(z) \|) \varepsilon,
\]
that is, we obtain the inequality
\[
\tag{29} \left\| 2F \left( \frac{\alpha x + \beta y}{2}, z \right) - \alpha F(x, z) - \beta F(y, z) \right\| \leq (1 + \| f(z) \|) \varepsilon
\]
for all $x, y \in A$ and all $\alpha, \beta \in I$.

Putting $\alpha = \beta = 1$ in (29) and using the Jung’s result [14], there is an additive mapping in the first variable $S : A \times A \to B$ such that
\[
\tag{30} \| F(x, z) - S(x, z) \| \leq (1 + \| f(z) \|) \varepsilon
\]
for all $x \in A$, where
\[
\tag{31} S(x, z) = \lim_{n \to \infty} \frac{F(2^n x, z)}{2^n}
\]
for all $x \in A$. By replacing $x$ by $2^{n+1}x$ and letting $y = 0$ in (29), we get
\[
2^{-(n+1)} \left\| 2F \left( \frac{2^{n+1} x}{2}, z \right) - iF(2^{n+1} x, z) - F(0, z) \left\| \leq 2^{-(n+1)}(1 + \| f(z) \|) \varepsilon
\]
for all $x \in A$. Taking limits as $n \to \infty$, we obtain
\[
\tag{32} S(\text{i}x, z) = iS(x, z)
\]
for all $x \in A$. To prove the homogeneous property in the first variable of $S$, let us $g \in A^*$, where $A^*$ is the dual of $A$, and define the additive function $\Upsilon : \mathbb{R} \to \mathbb{R}$ by
\[ \Upsilon(t) = g(S(tx, z)). \] The function is bounded since
\[
|\Upsilon(t)| \leq \|g\| \|S(tx, z)\|
\leq \|g\|\|(S(tx, z) - F(tx, z))\| + \|F(tx, z)\|
\leq \|g\|\left( (1 + \|f(z)\|)\varepsilon + \sup\{\|F(tx, z)\| : t \in [0, r_z]\} \right).
\]
(33)

It follows from Corollary 2.5 of [1] that \( \Upsilon(t) = \Upsilon(1)t \) for all \( t \in \mathbb{R} \). Hence we get
\[ g(S(tx, z)) = g(tS(x, z)) \]
for all \( t \in \mathbb{R} \) and all \( g \in \mathcal{A}^* \) which implies that \( S(tx, z) = tS(x, z) \) for all \( t \in \mathbb{R} \).

Now, for each complex number \( \lambda = u + iv \), we have
\[
S(\lambda x, z) = S(ux + ivx, z)
= S(ux, z) + S(ivx, z)
= uS(x, z) + ivS(x, z) = \lambda S(x, z),
\]
(34)
that is, the mapping \( S : \mathcal{A} \times \mathcal{A} \to \mathcal{B} \) is \( \mathbb{C} \)-linear in the first variable.

From Jung’s result [14], the inequality (27) with \( \alpha = \beta = 1 \) implies that there exists a unique additive mapping \( h : \mathcal{A} \to \mathcal{B} \) defined by
\[ h(x) = \lim_{n \to -\infty} \frac{f(2^n x)}{2^n} \]
for all \( x \in \mathcal{A} \) satisfying the inequality (28). Applying a similar approach to (29)~(34) to (27), we see that \( h \) is \( \mathbb{C} \)-linear. The remainder of the proof follows the similar argument as in the proof of Theorem 2.1. Therefore, \( h \) is a generalized homomorphism. \( \square \)

References


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