ON THE WEAK NATURAL NUMBER OBJECT OF THE WEAK TOPOS FUZ

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Abstract. Category $\text{Fuz}$ of fuzzy sets has a similar function to the Category $\text{Set}$. But it forms a weak topos. We study a natural number object and a weak natural number object in the weak topos $\text{Fuz}$. Also we study the weak natural number object in $\text{Fuz}^C$.

1. Introduction

Category $\text{Fuz}$ of fuzzy sets has a similar function to the topos $\text{Set}$. $\text{Fuz}$ has finite products, middle object, equalizers, exponentials and weak subobject classifier. But $\text{Fuz}$ is not a topos, it forms a weak topos. There are some comparisons between weak topos $\text{Fuz}$ and topos $\text{Set}$. A natural number object in a topos means an object together with morphisms. An important characterization of natural number objects in a topos was given by P. Freyd. Natural number object applied to define the order structure and retains a certain amount of Booleanness. In this paper, first we show that $\text{Fuz}$ has no nontrivial natural number object. So we define a weak natural number object in a weak topos $\text{Fuz}$ and $\text{Fuz}^C$.

2. Preliminaries

In this section, we state some definitions and properties which will serve as the basic tools for the arguments used to prove our results.

**Definition 2.1.** An *elementary topos* is a category $\mathcal{E}$ that satisfies the following:

(T1) $\mathcal{E}$ is finitely complete,
(T2) $\mathcal{E}$ has exponentiation,
(T3) $\mathcal{E}$ has a subobject classifier.

(T2) means that for every object $A$ in $\mathcal{E}$, the endofunctor $(-) \times A$ has its right adjoint $(-)^A$. Hence for every object $A$ in $\mathcal{E}$, there exists an object $B^A$, and a morphism $ev_A : B^A \times A \to B$, called the evaluation map of $A$, such that for any $Y$ and $f : Y \times A \to B$ in $\mathcal{E}$, there exists a unique morphism $g$ such that $ev_A \circ (g \times id) = f$;

\[
\begin{array}{c}
Y \times A \xrightarrow{f} B \\
\downarrow g \times id \quad \downarrow id \\
B^A \times A \xrightarrow{ev_A} B
\end{array}
\]

And subobject classifier in (T3) is an $\mathcal{E}$-object $\Omega$, together with a morphism $\top : 1 \to \Omega$ such that for any monomorphism $h : D \to C$, there is a unique morphism $\chi_h : C \to \Omega$, called the character of $h : D \to C$ which makes the following diagram a pull-back;

\[
\begin{array}{c}
D \xrightarrow{l} 1 \\
\downarrow h \quad \downarrow \top \\
C \xrightarrow{\chi_h} \Omega
\end{array}
\]

Example 2.2. Category $\text{Set}$ is a topos. $\{\ast\}$ is a terminal object. $\Omega = \{0, 1\}$ and $\top : \{\ast\} \to \Omega$ with $\top(\ast) = 1$ is a subobject classifier. If we define

- $\chi_h = 1$ if $c = h(d)$ for some $d \in D$,
- $\chi_h = 0$ otherwise

then $\chi_h$ is a characteristic function of $D$.

Category $\text{Fuz}$ of fuzzy sets is a category whose object is $(A, \alpha_A)$ where $A$ is an object and $\alpha_A : A \to I$ is a morphism with $I = (0, 1]$ in $\text{Set}$ and morphism from $(A, \alpha_A)$ to $(B, \alpha_B)$ is a function $f : A \to B$ in $\text{Set}$ such that $\alpha_A(a) \leq \alpha_B \circ f(a)$ [3].

Definition 2.3. We say that an object $(I, \alpha_I)$ is a middle object of $\text{Fuz}$ if there exists a unique morphism $f : A \to I$ such that $\alpha_A(a) = \alpha_I \circ f(a)$ for all $(A, \alpha_A)$ and $a \in A$.

Definition 2.4. We say that $((J, \alpha_J), i)$ is a weak subobject classifier of $\text{Fuz}$ if there exists a unique morphism $\alpha_J : (A, \alpha_A) \to (J, \alpha_J)$ for all monomorphism $f : (B, \alpha_B) \to (A, \alpha_A)$ where $J = [0, 1]$ and $\alpha_J(j) = 1$ for all $j \in J$ such that $\alpha_f(a) \leq$
\[ \alpha_A(a) \text{ and the following diagram} \]

\[
\begin{array}{c}
(B, \alpha_B) \xrightarrow{\alpha_B} (I, \alpha_I) \\
\downarrow f \quad \downarrow i \\
(A, \alpha_A) \xrightarrow{\alpha_f} (J, \alpha_J)
\end{array}
\]

is a pull-back.

**Definition 2.5.** A weak topos is a category \( \mathcal{E} \) that satisfies the following:

(WT1) \( \mathcal{E} \) has equalizer, finite product and exponentiation.

(WT2) \( \mathcal{E} \) has a middle object.

(WT3) \( \mathcal{E} \) has a weak subobject classifier.

**Proposition 2.6.** Category Fuz is a weak topos.

For the proof see Yuan and Lee [4].

**Definition 2.7.** A natural number object in a topos \( \mathcal{E} \) means an object \( N \) together with morphisms

\[
\begin{array}{ccc}
1 & \xrightarrow{0} & N & \xrightarrow{s} & N,
\end{array}
\]

where 1 is a terminal object in a topos, such that for any diagram

\[
\begin{array}{ccc}
1 & \xrightarrow{a} & A & \xrightarrow{f} & A,
\end{array}
\]

there exists a unique morphism \( h : N \to A \) such that

\[
\begin{array}{ccc}
1 & \xrightarrow{0} & N & \xrightarrow{s} & N \\
\downarrow id & & \downarrow h & & \downarrow h \\
1 & \xrightarrow{a} & A & \xrightarrow{f} & A
\end{array}
\]

commutes.

**Definition 2.8.** A weak natural number object in a weak topos \( \text{Fuz} \) means an object \( N \) together with morphisms

\[
\begin{array}{ccc}
I & \xrightarrow{0} & N & \xrightarrow{s} & N,
\end{array}
\]

where \( I \) is the middle object in the weak topos \( \text{Fuz} \), such that for any diagram with a normal object \( A \)

\[
\begin{array}{ccc}
I & \xrightarrow{a} & A & \xrightarrow{f} & A
\end{array}
\]
there exists a unique morphism $h : N \to A$ such that

$$
\begin{array}{c}
I \\
\downarrow id
\end{array}
\begin{array}{c}
N \\
\downarrow h
\end{array}
\begin{array}{c}
N \\
\downarrow h
\end{array}
\begin{array}{c}
I \\
\downarrow a
\end{array}
\begin{array}{c}
A \\
\downarrow f
\end{array}
\begin{array}{c}
A
\end{array}
$$

commutes.

3. Main Parts

**Proposition 3.1.** $Fuz$ has no nontrivial natural number object.

**Proof.** In $Fuz$, the terminal object $1$ is a singleton set $(\{\ast\}, \alpha_{\{\ast\}})$ with $\alpha_{\{\ast\}}(\ast) = 1 \in I$. Assume that there exists a natural number object in $Fuz$. That is, there exists an object $(N, \alpha_N)$ together with morphisms

$$
\begin{array}{c}
1 = (\{\ast\}, \alpha_{\{\ast\}}) \\
\downarrow 0
\end{array}
\begin{array}{c}
(\ast) \\
\downarrow s
\end{array}
\begin{array}{c}
(N, \alpha_N) \\
\downarrow 0
\end{array}
$$

such that for any diagram

$$
\begin{array}{c}
1 = (\{\ast\}, \alpha_{\{\ast\}}) \\
\downarrow 0
\end{array}
\begin{array}{c}
(\ast) \\
\downarrow s
\end{array}
\begin{array}{c}
(A, \alpha_A) \\
\downarrow 0
\end{array}
\begin{array}{c}
(N, \alpha_N) \\
\downarrow s
\end{array}
\begin{array}{c}
(N, \alpha_N) \\
\downarrow s
\end{array}
\begin{array}{c}
(A, \alpha_A) \\
\downarrow s
\end{array}
\begin{array}{c}
(A, \alpha_A)
\end{array}
$$

there exists a unique morphism $h : (N, \alpha_N) \to (A, \alpha_A)$ such that

$$
\begin{array}{c}
1 = (\{\ast\}, \alpha_{\{\ast\}}) \\
\downarrow 0
\end{array}
\begin{array}{c}
(\ast) \\
\downarrow s
\end{array}
\begin{array}{c}
(N, \alpha_N) \\
\downarrow s
\end{array}
\begin{array}{c}
(A, \alpha_A) \\
\downarrow s
\end{array}
\begin{array}{c}
(A, \alpha_A)
\end{array}
$$

commutes.

We need a condition that $\alpha_N \circ 0(\ast) \geq \alpha_{\{\ast\}}(\ast)$, so that we have $\alpha_N \circ 0(\ast) = \alpha_N(0) = 1$ for $0 \in N$. Since $s : (N, \alpha_N) \to (N, \alpha_N)$ is a morphism in $Fuz$, where $s(n) = n + 1$, it satisfy that $\alpha_N \circ s(0) \geq \alpha_N(0)$. That is, $\alpha_N(1) \geq \alpha_N(0)$. So we get $\alpha_N(1) = 1$. Inductively we get $\alpha_N(n) = 1$ for all $n \in N$. Also, we need a condition that $\alpha_A \circ h \geq \alpha_N$, so that we have $\alpha_A(a) = 1$ for all $a \in A$. □

**Corollary 3.2.** In $Fuz$, there exists an object $(N, \alpha_N)$, where $\alpha_N(n) = 1$ for all $n \in N$, with morphisms

$$
\begin{array}{c}
1 = (\{\ast\}, \alpha_{\{\ast\}}) \\
\downarrow 0
\end{array}
\begin{array}{c}
(\ast) \\
\downarrow s
\end{array}
\begin{array}{c}
(N, \alpha_N) \\
\downarrow s
\end{array}
\begin{array}{c}
(A, \alpha_A) \\
\downarrow f
\end{array}
\begin{array}{c}
(A, \alpha_A)
\end{array}
$$

such that for any diagram

$$
\begin{array}{c}
1 = (\{\ast\}, \alpha_{\{\ast\}}) \\
\downarrow a
\end{array}
\begin{array}{c}
(\ast) \\
\downarrow f
\end{array}
\begin{array}{c}
(A, \alpha_A) \\
\downarrow f
\end{array}
\begin{array}{c}
(A, \alpha_A)
\end{array}
$$


where \(1 = (\{\ast\}, \alpha_{\{\ast\}})\) is a terminal object and \(\alpha_A(a) = 1\) for all \(a \in A\), there exists a unique \(h : (N, \alpha_N) \to (A, \alpha_A)\) such that

\[
\begin{array}{ccc}
1 = (\{\ast\}, \alpha_{\{\ast\}}) & \xrightarrow{0} & (N, \alpha_N) \xrightarrow{\alpha} (N, \alpha_N) \\
& \downarrow{id} & \downarrow{h} \\
& 1 = (\{\ast\}, \alpha_{\{\ast\}}) & \xrightarrow{a} (A, \alpha_A) \xrightarrow{f} (A, \alpha_A)
\end{array}
\]

commutes.

**Lemma 3.3.** Fuz has finite products.

**Proof.** Let \((A, \alpha_A), (B, \alpha_B)\) be two objects in Fuz. Consider \((A \times B, \alpha_{A \times B}), p_A, p_B\) where \(A \times B\) is the cartesian product of a pair \((A, B)\) of the topos Set with \(\alpha_{A \times B} = \min\{\alpha_A, \alpha_B\}\) and projection morphisms \(p_A : (A \times B, \alpha_{A \times B}) \to (A, \alpha_A), p_B : (A \times B, \alpha_{A \times B}) \to (B, \alpha_B)\) satisfying \(\alpha_B \circ p_B \geq \alpha_{A \times B}\) and \(\alpha_A \circ p_A \geq \alpha_{A \times B}\). Then, for any morphisms \(f : (X, \alpha_X) \to (A, \alpha_A)\) and \(g : (X, \alpha_X) \to (B, \alpha_B)\), there exists a unique morphism \(<f, g> : (X, \alpha_X) \to (A \times B, \alpha_{A \times B})\) such that \(p_A \circ <f, g> = f\) and \(p_B \circ <f, g> = g\). Since \(\alpha_A f(x) \geq \alpha_X(x), \alpha_B g(x) \geq \alpha_X(x)\) and \(\alpha_{A \times B}(f(x), g(x)) = \min\{\alpha_A f(x), \alpha_B g(x)\}\), we have that \(\alpha_{A \times B}(f(x), g(x)) \geq \alpha_X(x)\), so \(\alpha_{A \times B} \circ <f, g> \geq \alpha_X\). Thus \(<f, g> : (X, \alpha_X) \to (A \times B, \alpha_{A \times B})\) is a morphism in Fuz. \(\square\)

**Theorem 3.4.** Fuz has a weak natural number object.

**Proof.** Let \((N, \alpha_N)\) be an object with \(\alpha_N(n) = 1\) for all \(n \in N\). Then by Lemma 3.3, there exists an object \(((N \times I), \alpha_{N \times I})\), where \((I, \alpha_I)\) is the middle object in Fuz. Consider the object \(((N \times I), \alpha_{N \times I})\) with morphisms

\[
\begin{array}{ccc}
I & \xrightarrow{0} & N \times I \\
& \xrightarrow{s'} & N \times I
\end{array}
\]

defined by \(0(i) = (0, i)\) and \(s'(n, i) = (n + 1, i)\). Then it satisfy that \(\alpha_{N \times I} \circ 0 \geq \alpha_I\) and \(\alpha_{N \times I} \circ s' \geq \alpha_{N \times I}\).

For any normal object \((A, \alpha_A)\), we define a morphism \(a : I \to A\) with \(a(i) = a\) for all \(i \in (0, \alpha_A(a)]\) and \(a(i) = c\) for all \(i \in (\alpha_A(a), 1]\), where \(\alpha_A(c) = 1\). Then \(\alpha_Aa(i) \geq \alpha_I(i)\) making \(a : I \to A\) a morphism in Fuz.

Then for any diagram,

\[
\begin{array}{ccc}
I & \xrightarrow{a} & A \\
& \xrightarrow{f} & A
\end{array}
\]

there exists a unique morphism \(h : N \times I \to A\) defined by \(h(0, i) = a(i)\) and \(f \circ h(n, i) = h(n + 1, i)\) such that
commutes.

Then \(\alpha_A a(i) \geq \alpha_I(i)\) and \(h(0, i) = a(i)\) imply \(\alpha_A h(0, i) \geq \alpha_I(i)\), so \(\alpha_A h(0, i) \geq \alpha_{N \times I}(0, i)\).

And \(\alpha_A \circ f \geq \alpha_A\) implies \(\alpha_A \circ f \circ h(0, i) \geq \alpha_A \circ h(0, i) \geq \alpha_I(i)\).

So we have that \(\alpha_A h(1, i) \geq \alpha_I(i)\). It implies that \(\alpha_A h(1, i) \geq \alpha_{N \times I}(1, i)\). Inductively we show that \(\alpha_A h(n, i) \geq \alpha_{N \times I}(n, i)\).

If there exists another morphism \(k : N \times I \to A\) such that \(k(0, i) = a(i)\) and \(f \circ k(n, i) = k \circ s'(n, i) = k(n + 1, i)\).

Then we have that \(f \circ k(0, i) = k \circ s'(0, i)\) and \(f \circ h(0, i) = h \circ s'(0, i)\).

Also we have \(k \circ 0(i) = a(i)\) and \(h \circ 0(i) = a(i)\). So \(f \circ k(0, i) = k(1, i)\) and \(f \circ h(0, i) = h(1, i)\).

This imply \(k(1, i) = f \circ a(i) = h(1, i)\).

Inductively, \(h : N \times I \to A\) is the unique morphism in \(\text{Fuz}\).

\(\square\)

**Theorem 3.5.** For any small category \(C\), \(\text{Fuz}^C\) has a weak natural number object.

**Proof.** Consider a constant functor \(W : C \to \text{Fuz}\) having \(W(a) = N \times I\) for all \(a \in C\) and \(W(f) = id_{N \times I}\) for all \(f \in C\). Also consider a constant natural transformation \(s' : W \to W\) having \(s'_a(n, i) = (n + 1, i)\) and a constant natural transformation \(0 : J \to W\) having \(0_a(i) = (0, i)\) where \(J : C \to \text{Fuz}\) is a functor defined by \(J(a) = I\) for all \(a \in C\) and \(J(f) = id\). Then for any diagram

\[
\begin{array}{ccc}
J & \xrightarrow{k} & K \\
\downarrow & & \downarrow \\
W & \xrightarrow{s'} & W \\
\end{array}
\]

there exists a unique morphism \(h : W \to K\) defined by \(h_a(0, i) = k_a(i)\) and \(f \circ h(n, i) = h(n + 1, i)\), that is, the following diagram

\[
\begin{array}{ccc}
J & \xrightarrow{0} & W & \xrightarrow{s'} & W \\
\downarrow & & \downarrow h & & \downarrow h \\
J & \xrightarrow{k} & K & \xrightarrow{f} & K \\
\end{array}
\]

commutes.
Since $k : J \to K$ is a natural transformation, the square

$$
\begin{array}{ccc}
J(a) & \xrightarrow{J(\alpha)=\text{id}} & J(b) \\
\downarrow k_a & & \downarrow k_b \\
K(a) & \xrightarrow{K(\alpha)} & K(b)
\end{array}
$$

commutes. And we get $k_a(i) = h_a_0(i) = h_a(0, i)$. These imply $K(\alpha)h_a(0, i) = h_b(0, i)$. We assume that $K(\alpha)h_a(n, i) = h_b(n, i)$. Since $f : K \to K$ is a natural transformation, we have $K(\alpha) \circ f_a \circ h_a(n, i) = f_b \circ h_b(n, i)$. By definition of $h$, this implies $K(\alpha)h_a(n+1, i) = h_b(n+1, i)$. By induction, we get $K(\alpha)h_a(n, i) = h_b(n, i)$ for any $n \in \mathbb{N}$. So $h : W \to K$ is a natural transformation. □

References


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