International Journal of Management Science
Vol 16, No 1, May 2010

A Novel Integer Programming Approach to the One-Dimensional Facility Layout Problem

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(Received: December 14, 2009 / Revised: January 11, 2010 / Accepted: January 13, 2010)

ABSTRACT

We present a new integer programming formulation and a class of valid inequalities for solving the one-dimensional facility layout problem, which leads to a very efficient solution method for the problem.

Keywords: Facility Layout, Single-row Layout, Integer Programming

1. Introduction

The one-dimensional facility location problem (ODFLP) is defined as follows. We are given n departments, each of which has a length l_i, for all $i \in N$, where $N = \{1, 2, \cdots, n\}$. Also there is given an $n \times n$ symmetric matrix $C = [c_{ij}]$, where c_{ij} is the average daily traffic requirement between two departments i and j. The problem appears in applications such as room arrangement problem on a corridor in building in hospitals and supermarkets, see [10]. Also, there are other applications of the problem, for example, the arrangement of books in a shelf [9] and the machine layout problem [6].

ODFLP is very similar to the linear ordering problem [4], except that the objective function is quadratic in this case. The problem is known to be NP-hard [1], so

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most previous research has focused on developing approximate solution methods, for example, see [6], [7], and [2]. Recently, Amaral [1] proposed an exact approach to the problem using an IP formulation. The computational results reported there show that it can solve the test problems found in various literatures to the optimality. However, the proposed formulation needs too many branch-and-bound nodes with long computational time for rather large sized problem instances, which indicates that it might not be easily applied for solving large-sized problem instances. The formulation uses traditional formulation for the linear ordering problem together with linearization of quadratic variables which are needed to represent the distances between pairs of departments. The linearized formulation requires $O(n^3)$ variables.

In this paper, we propose a new formulation for (ODFLP) which also requires $O(n^3)$ variables. The formulation uses three-index variables each of which represents a partial ordering among three distinct departments. The proposed formulation is shown to yield a stronger LP relaxation than that in [1]. Also the LP relaxation of the proposed formulation, together with a set of strong valid inequalities, can provide optimal solutions to all the benchmark problems in [1] within a very short computational time, which is a very interesting result. Hence it can be served as a very promising tool to resolve large-sized problem instances found in practice. In addition, the proposed formulation gives an alternative formulation for the linear ordering problem. Though the number of variables and constraints are larger than the traditional formulation, it can yield a stronger LP relaxation. Hence the results can also be used to develop an alternative efficient solution approach to the linear ordering problem.

In the next section, we will present the new formulation and compare it with the formulation presented in [1]. In section 3, a class of strong valid inequalities is presented, which is used to further strengthen the formulation. In section 4, computational results are reported which is followed by the concluding remarks.

2. Formulation

Before presenting the formulation, for the convenience of exposition, let us define the relation $i \prec j$, for two distinct pair $i, j \in N$. The relation $i \prec j$ means that department i is located at the left of department j. Note that $\prec$ defines a partial ordering between a pair of departments. Also without loss of generality, we will assume $n \geq 4$.

As presented in section 1, ODFLP seeks to find a linear arrangement (that is, a complete linear ordering) of n departments which minimizes the sum of average daily traffic weighted distances between all pairs of departments. More precisely, for a given linear arrangement, let $d(i, j)$ be the distance between the pair of departments i and j. Then if $i \prec j$ in the given linear arrangement, $d(i, j) = \frac{1}{2}(l_i + l_j) + \sum_{k \in N \setminus \{i,j\}, i \prec k \prec j} l_k$, where the distance is measured between the centroid of departments i and j. So the objective function is to minimize $\sum_{i,j \in N, i<j} c_{ij} d(i, j)$.

To present the new formulation, let us begin with the formulation given in Amaral [1]. Let us define binary variable $\alpha_{ij} = 1$ if $i \prec j$ and $\alpha_{ij} = 0$ otherwise. Then the distance between the pair of departments i and j, where $i < j$, can be represented as follows [1].

$$d(i,j) = \frac{1}{2}(l_i + l_j) - \sum_{k \in N \setminus \{i,j\}} l_k \alpha_{ij} \alpha_{jk} + \sum_{k \in N \setminus \{i,j\}} l_k \alpha_{ij} \alpha_{ik} - \sum_{k \in N \setminus \{i,j\}} l_k \alpha_{ik} \alpha_{jk} + \sum_{k \in N \setminus \{i,j\}} l_k \alpha_{jk}$$

Using the above result, Amaral [1] presented the following IP formulation for (ODFLP):

(IPA)

$$\min \quad \sum_{1 \leq i<j \leq n} \frac{1}{2} c_{ij}(l_i + l_j) + \sum_{1 \leq i<j \leq n} c_{ij} \Big(\sum_{k \in N \setminus \{i,j\}} l_k \alpha_{jk} - \sum_{k \in N \setminus \{i,j\}} l_k W_{ijjk} + \sum_{k \in N \setminus \{i,j\}} l_k W_{ijik} - \sum_{k \in N \setminus \{i,j\}} l_k W_{ikjk} \Big) \quad (1)$$

$$\text{s.t} \quad \alpha_{ij} + \alpha_{jk} - W_{ijjk} \leq 1, \quad W_{ijjk} \leq \alpha_{ij}, \quad W_{ijjk} \leq \alpha_{jk}, \quad \text{for all distinct } i, j, k \in N, \quad (2)$$

$$\alpha_{ij} + \alpha_{ik} - W_{ijik} \leq 1, \quad W_{ijik} \leq \alpha_{ij}, \quad W_{ijik} \leq \alpha_{ik}, \quad \text{for all distinct } i, j, k \in N, \quad (3)$$

$$\alpha_{ik} + \alpha_{jk} - W_{ikjk} \leq 1, \quad W_{ikjk} \leq \alpha_{ik}, \quad W_{ikjk} \leq \alpha_{jk}, \quad \text{for all distinct } i, j, k \in N, \quad (4)$$

$$\alpha_{ij} + \alpha_{ji} = 1, \quad \text{for all } 1 \leq i < j \leq n, \quad (5)$$

$$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} \leq 2, \quad \text{for all distinct } i, j, k \in N, \quad (6)$$

$$\alpha_{ij} \in \{0, 1\}, \quad \text{for all } 1 \leq i < j \leq n, \quad (7)$$

In the above formulation (IPA), the variables such as W_{ijjk} are introduced to linearize the quadratic terms such as $\alpha_{ij}\alpha_{jk}$, which are reflected in the constraints (2), (3), and (4). The constraints (5) and (6) ensure that the variables α_{ij}, for all $1 \le i < j \le n$, define a complete ordering among all the departments.

By using the equation (5) and complementing the variables as necessary (for example, by replacing $W_{ijik} = \alpha_{ij}\alpha_{ik} = (1-\alpha_{ji})\alpha_{ik} = \alpha_{ik} - W_{jiik}$) and also by representing the variables such as $W_{ijjk} = x_{ijk}$, the formulation (IPA) can be restated as follows:

(IPA′)

$$\min \quad \sum_{1\le i<j\le n} \frac{1}{2} c_{ij}(l_i + l_j) + \sum_{1\le i<j\le n} c_{ij}\Big(\sum_{k\in N\setminus\{i,j\}} l_k x_{ikj} + \sum_{k\in N\setminus\{i,j\}} l_k x_{jki}\Big) \tag{8}$$

$$\text{s.t} \quad \alpha_{ij} + \alpha_{jk} - x_{ijk} \le 1, \quad x_{ijk} \le \alpha_{ij}, \quad x_{ijk} \le \alpha_{jk}, \qquad \text{for all distinct } i, j, k \in N, \tag{9}$$

$$\alpha_{ij} + \alpha_{ji} = 1, \qquad \text{for all } 1 \le i < j \le n,$$

$$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} \le 2, \qquad \text{for all distinct } i, j, k \in N,$$

$$\alpha_{ij} \in \{0, 1\}, \qquad \text{for all } 1 \le i < j \le n,$$

$$x_{ijk} \in \{0, 1\}, \qquad \text{for all distinct } i, j, k \in N. \tag{10}$$

Note that if $x_{ijk} = \alpha_{ij}\alpha_{jk} = 1$ then it defines a partial ordering among the departments *i*, *j*, and *k* such as $i \prec j$ and $j \prec k$ for three distinct $i, j, k \in N$. The set of constraints (9) replaces the sets of constraints (2)~(4).

Now we will introduce two sets of equations which should be satisfied by the variables x_{ijk} for all distinct $i, j, k \in N$. The first equation states that there should be only one partial ordering among three distinct departments $i, j, k \in N$, which can be represented as follows:

$$x_{ijk} + x_{ikj} + x_{jik} + x_{jki} + x_{kij} + x_{kji} = 1, \quad \text{for all distinct } i, j, k \in N. \tag{11}$$

The second equation states that exactly one of the relations $i \prec j$ and $j \prec i$ should hold for each pair of departments *i* and *j*, which can be represented as follows:

$$x_{ijk} + x_{ikj} + x_{kij} + x_{jil} + x_{jli} + x_{lji} = 1,$$

for all distinct $i, j, k, l \in N$, where $i < j$ and $k \neq l$. (12)

Next we will show that the equations (11) and (12), together with the integrality constraints (10) are sufficient for the formulation of the problem (ODFLP). Let us call it (IP1), which is presented as follows:

(IP1)

$$\min \quad \sum_{1 \le i < j \le n} \frac{1}{2} c_{ij}(l_i + l_j) + \sum_{1 \le i < j \le n} c_{ij}\Big(\sum_{k \in N \setminus \{i,j\}} l_k x_{ikj} + \sum_{k \in N \setminus \{i,j\}} l_k x_{jki} \Big)$$

s.t $x_{ijk} + x_{ikj} + x_{jik} + x_{jki} + x_{kij} + x_{kji} = 1,$ for all distinct $i, j, k \in N,$

$x_{ijk} + x_{ikj} + x_{kij} + x_{jil} + x_{jli} + x_{lji} = 1,$

for all distinct $i, j, k, l \in N,$ where $i < j$ and $k \neq l.$

$x_{ijk} \in \{0, 1\},$ for all distinct $i, j, k \in N.$

To prove the result, we need the following lemma.

Lemma 1: For a given pair $i, j \in N$, the following equation is implied by the sets of equations (11) and (12);

$$x_{ijk} + x_{ikj} + x_{kij} = x_{ijl} + x_{ilj} + x_{lij}, \text{ for all } k \neq l, \ k, l \in N \setminus \{i, j\}. \quad (13)$$

Proof: Suppose the equation (13) does not hold. Without loss of generality, we can assume $x_{ijk} + x_{ikj} + x_{kij} > x_{ijl} + x_{ilj} + x_{lij}$. Then we can get $x_{ijk} + x_{ikj} + x_{kij} > 1 - (x_{jil} + x_{jli} + x_{lji})$ by using the equation (11). It can be rearranged as $x_{ijk} + x_{ikj} + x_{kij} + x_{jil} + x_{jli} + x_{lji} > 1$. Hence by the equation (12), this leads to a contradiction.

Note that the above lemma 1 holds for the LP relaxation of (IP1).

Since the equation (13) holds, for any given feasible solution to (IP1), we can define the following equation unambiguously:

$$\alpha_{ij} = x_{ijk} + x_{ikj} + x_{kij}, \text{ for all distinct } i, j \in N \text{ and for any } k \in N \setminus \{i, j\}. \qquad (14)$$

Then we can prove the following lemma.

Lemma 2: The α_{ij}, for all distinct $i, j \in N$, defined in (14) satisfies the constraints (5) and (6).

Proof: By using the equation (12), it is trivial to show that the constraint (5) holds. Hence we only need to consider the constraint (6). By using the relation (14), we can get the following equations:

$$\alpha_{ij} = x_{ijk} + x_{ikj} + x_{kij}, \quad \alpha_{jk} = x_{jki} + x_{jik} + x_{ijk}, \quad \alpha_{ki} = x_{kij} + x_{kji} + x_{jki}.$$

By summing the above equations and also by using the equation (11), we can get

$$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} = 1 + x_{ijk} + x_{jki} + x_{kij} \leq 2.$$

This completes the proof.

Note also that the above lemma 2 holds for the LP relaxation of (IP1).

Hence we can derive a complete ordering from any feasible solution of (IP1), which shows that the formulation (IP1) is correct.

Theorem 1: The formulation (IP1) is correct.

Moreover, we can show that (IP1) gives a stronger LP relaxation than (IPA).

Theorem 2: The LP relaxation of (IP1) is stronger than the LP relaxation of (IPA).

Proof: In the above, we showed that (IPA) can be rewritten as (IPA′). Thus, it is sufficient to show that the LP relaxation of (IP1) is stronger than the LP relaxation of (IPA′). Note that in the proof of Lemma 2, we already show that the constraints (5) and (6) are satisfied if we define the variable α_{ij} as (14). Also it is easy to show that the constraint (9) is satisfied by noting that $\alpha_{ij} + \alpha_{jk} - x_{ijk} = (x_{ijk} + x_{ikj} + x_{kij}) + (x_{jki} + x_{jik} + x_{ijk}) - x_{ijk} = x_{ijk} + x_{ikj} + x_{kij} + x_{jki} + x_{jik} \le 1$. So the feasible set of the LP relaxation of (IP1) is contained in that of (IPA'). Now consider the following feasible solution to the LP relaxation of (IPA'):

$$\alpha_{ij} = 0.5, \text{ for all } 1 \le i < j \le n; \text{ and } x_{ijk} = 0.5, \text{ for all distinct } i, j, k \in N.$$

The above solution is infeasible to the LP relaxation of (IP1). So the result follows.

We want to mention that Amaral [1] presented some classes of strong valid inequalities which are appended to the formulation (IPA). A further comparison of the strength of the LP relaxations for the strengthened formulation with the valid inequalities is left as a further study. Rather, we only mention that the computational study given in section 4 shows that the strengthened formulation produces the same LP bounds to all the test problems as (IP1).

3. A Class of Strong Valid Inequalities

To derive a valid inequality, consider a partial ordering $i \prec j \prec k$ which means $x_{ijk} = 1$ for a distinct $i, j, k \in N$. Assume the reverse ordering which means $k \prec j \prec i$. Now let us choose $l \in N \setminus \{i, j, k\}$. Then there are four possible positions which l can choose as presented in Figure 1.

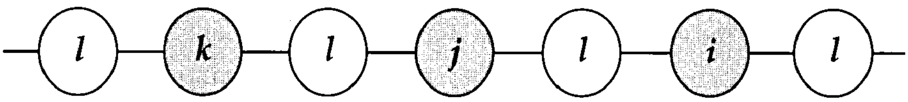

Figure 1. Four Possible Locations of 1

From Figure 1, we can derive four variables x_{lkj}, x_{klj}, x_{jli}, and x_{jil}, each of which is incompatible with the others. Also, each of the four variables is incompatible with the variable x_{ijk}. Hence we can prove that the following theorem holds:

Theorem 3: The following inequality is valid for (IP1):

$$x_{ijk} + x_{lkj} + x_{klj} + x_{jli} + x_{jil} \le 1, \quad \text{for all distinct } i, j, k, l \in N \tag{15}$$

We can show that the above inequality (15) can cut some fractional solutions to the LP relaxation of (IP1). Consider the following point:

$$x_{ijk} = x_{kji} = x_{ilk} = x_{kli} = x_{jil} = x_{lij} = x_{lkj} = x_{jkl} = 0.5 \tag{16}$$

It can be easily verified that the point (16) satisfies all the constraints of the LP relaxation of (IP1). However, the point can be removed by adding the inequality (15). Hence, we can conclude that the inequality (15), if appended to (IP1), can strengthen the formulation. Let us denote the strengthened formulation as (IP2).

An alternative way to derive the valid inequality (15) is to view (IP1) as a node packing problem [8]. Specifically, if we relax the equality constraints (11) and (12) to inequality, then the problem reduces to a node packing problem. One can show that the inequality (15) corresponds to a maximal clique inequality which is known to define a facet for the node packing polytope. We can show that there are many other valid inequalities that can be derived in a similar manner, but the results are omitted here.

4. Computational Results

To evaluate the performance of the new formulations (IP1) and (IP2), we used the same problem instances given in [1] and a larger problem instance (denoted as P20 in table 1) which can be found in [6]. The sizes of the problem instances are $4 \le n \le 20$. The following Table 1 shows the computational results.

In Table 1, the results shown under the title of Amaral were replicated from the paper [1], where (MIP3) is a strengthened version of (IPA) with some valid inequalities added which are given there. Though the test environment such as microprocessors and CPLEX versions is somewhat different, we can draw some interesting observations.

Table 1. Computational Results

Name	Amaral (MIP3) and B&B				(IP1)			(IP2)	
	LP	Opt.	B&B Nodes	Time*	LP	Opt.	Time**	LP	Time**
P4	570	638	2	0.02	570	638	0.01	638	0.01
LW5	135	151	2	0.05	135	151	0.01	151	0.00
S8	611	801	22	1.25	611	801	0.31	801	0.22
S8H	2114.5	2324.5	12	0.81	2114.5	2324.5	0.22	2324.5	0.42
S9	1952.5	2469.5	32	3.27	1952.5	2469.5	0.86	2469.5	0.31
S9H	4404.5	4695.5	46	2.36	4404.5	4695.5	1.05	4695.5	0.91
S10	2010.5	2781.5	56	7.92	2010.5	2781.5	1.89	2781.5	0.78
S11	5153.5	6933.5	111	19.19	5153.5	6933.5	5.06	6933.5	2.45
LW11	5153.5	6933.5	58	17.55	5153.5	6933.5	5	6933.5	2.41
P15	3583	6305.5	1759	1267	3583	6305.5	171.5	6305.5	63.7
P17	5119	9254	3369	5446	5119	9254	692.3	9254	181.79
P18	5774.5	10650.5	7515	20396	5774.5	10650.5	1523.1	10650.5	536.09
P20	-	-	-	-	8113	15549	2561.2	15549	1808.22

Note * P4 1.6GHz, CPLEX 8.0.
** P4 3.0GHz, CPLEX 10.0.

Firstly, the new formulation (IP1) gives the same LP bounds as [1] for all the test problem instances. Though we did not thoroughly compare the strengths of the two formulations, the results show that LP relaxation bound of (IP1) is as good as Amaral's [1] at least for the test problem instances.

Secondly, the LP relaxation of formulation (IP2) produces optimal integer solutions to all the test problem instances with no branch-and-bound nodes needed. The results are very interesting when considering the formulation given in [1] requires many branch-and-bound nodes. Also note that the computation time is much smaller. Hence we can conclude that the newly proposed formulation (IP2) has a potential to solve larger-sized problem instances in an efficient manner. Note that the problem instance P20 which is not considered in [1] also has been solved by only using the LP relaxation.

For some large problem instances, the size of (IP2) would be somewhat large. However, in this case, we can use the branch-and-price-and-cut approach which can handle large-sized formulations very efficiently, see [3].

5. Concluding Remarks

This paper presents a new formulation for the one-dimensional facility layout problem. The formulation has some interesting theoretical property and more importantly, it can solve all the test problems only by solving LP relaxation of it. Hence it can be used to solve large-sized problem instances which may be found in practice. In a theoretical aspect, the proposed formulation can be viewed as an alternative approach to formulate the linear ordering problem [4]. It would be very interesting to thoroughly analyze the formulation and compare it with the traditional formulation, which is left as a further research topic.

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