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ERROR ESTIMATES FOR FULLY DISCRETE DISCONTINUOUS GALERKIN METHOD FOR NONLINEAR PARABOLIC EQUATIONS

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ABSTRACT. In this paper, we develop discontinuous Galerkin methods with penalty terms, namaly symmetric interior penalty Galerkin methods to solve nonlinear parabolic equations. By introducing an appropriate projection of u onto finite element spaces, we prove the optimal convergence of the fully discrete discontinuous Galerkin approximations in $\ell^2(L^2)$ normed space.

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1. Introduction

Discontinuous Galerkin methods using interior penalties for elliptic and parabolic equations were introduced by several authors in [1], [2] and [10]. These approaches generalized the method developed by Nitsche [3] for treating Dirichlet boundary condition by the introduction of penalty terms on the boundary of the domain. These methods referred to as interior penalty Galerkin schemes are not locally mass conservative.

A new type of elementwise conservative discontinuous Galerkin method for diffusion problems was introduced and analyzed by Oden, Babuska, and Baumann [4]. The primal discontinuous Galerkin methods consist of four types: Oden-Babuška-Baumann DG method [4], symmetric interior penalty Galerkin (SIPG) method [10], nonsymmetric interior penalty Galerkin (NIPG) method [6, 9] and incomplete interior penalty Galerkin (IIPG) method. Compared to

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the classical Galerkin method the discontinuous Galerkin method is very well suited for the adaptive control of error and can provide high orders of accuracy, provided that the solution of the model problem is sufficiently smooth.

Rivière and Wheeler [8] introduced semidiscrete and fully discrete locally conservative discontinuous Galerkin formulations for nonlinear parabolic equations. They derived a priori $L^\infty(L^2)$ and $L^2(H^1)$ error estimates for the semidiscrete approximations and a priori $\ell^\infty(L^2)$ and $\ell^2(H^1)$ error estimates for the fully discrete approximations. They proved the semidiscrete approximations converges optimally in h in $L^2(H^1)$ normed spaces and the fully discrete approximations converges optimally in h and Δt in $L^2(H^1)$ normed spaces. Rivière and Wheeler [7] constructed semidiscrete discontinuous Galerkin approximations to the solution of the transport problem with nonlinear reaction, and proved that the approximations converge optimally in h and suboptimally in r in H^1 normed space and suboptimally in L^2 normed space. And Ohm, Lee, and Shin [5] constructed a semidiscrete locally conservative discontinuous Galerkin formulation for nonlinear parabolic equations and obtained an optimal error estimate in $L^\infty(L^2)$ norm, which improved the result of [8].

The objectives of this paper are to introduce the fully discrete discontinuous Galerkin approximations for nonlinear parabolic equations and to prove that they converge optimally in both spatial and temporal directions in $\ell^\infty(L^2)$ and $\ell^2(H^1)$ normed spaces.

The model problem and some assumptions are introduced in section 2. In section 3 we introduce several definitions and construct finite element spaces on which we suggest approximation properties. We introduce an appropriate projection onto finite element space and analyze its convergence. In section 4 we formulate the fully discrete discontinuous Galerkin approximations and prove its optimal convergence in $\ell^\infty(L^2)$ and $\ell^2(H^1)$ normed spaces.

2. Model problem

Consider the following nonlinear parabolic partial differential equation

$$u_t - \nabla \cdot (a(x,u)\nabla u) = f(x,u), \qquad (x,t) \in \Omega \times (0,T], \tag{2.1}$$

with the boundary condition

$$a(x,u)\nabla u \cdot n = 0, \qquad (x,t) \in \partial\Omega \times (0,T],, \tag{2.2}$$

and the initial condition

$$u(x,0) = \psi(x), \qquad x \in \Omega, \tag{2.3}$$

where Ω is a bounded convex domain in $\mathbf{R}^d$, $d = 2$ and n is a unit outward normal vector to $\partial\Omega$.

Assume that the followings are satisfied:

1. There exist constants γ and γ^* such that

$$0 < \gamma \leq a(x,p) \leq \gamma^* \quad \text{for} \quad (x,p) \in \Omega \times \mathbf{R}$$

and

$$\left| \frac{\partial}{\partial p} a(x,p) \right|, \quad \left| \frac{\partial^2}{\partial p^2} a(x,p) \right| \leq \gamma^* \quad \text{for} \quad (x,p) \in \Omega \times \mathbf{R}.$$

2. f is uniformly Lipschitz continuous with respect to their second variable.
3. The model problem has a unique solution satisfying the following regularity conditions:

$$u \in W^{2,\infty}\Big([0,T]; H^s(\Omega)\Big) \quad \text{for} \quad s \geq 2 \quad \text{and} \quad \nabla u \in L^\infty(\Omega \times [0,T]).$$

3. Definitions and discontinuous Galerkin method

Let $\mathcal{E}_h = \{E_1, E_2, \cdots, E_{N_h}\}$ be a subdivision of Ω where E_j is a triangle or a quadrilateral if $d = 2$ and E_j is a 3-simplex or 3-rectangle if $d = 3$. Let $h_j = diam(E_j)$ and $h = \max\{h_j : j = 1, 2, \cdots, N_h\}$. We denote the edges of the elements by $\Big\{e_1, e_2, \cdots, e_{P_h}, e_{P_h+1}, \cdots, e_{M_h}\Big\}$, where $e_k \subset \Omega, 1 \leq k \leq P_h$, and $e_k \subset \partial\Omega$, $P_h + 1 \leq k \leq M_h$. For each edge $e_k, P_h + 1 \leq k \leq M_h$, we take n_k the unit outward normal vector to $\partial\Omega$. And if $e_k = \partial E_i \cap \partial E_j, i < j$ for $1 \leq k \leq P_h$ then we take n_k the unit outward normal vector to E_i.

For an $s \geq 0$ we let

$$H^s(\mathcal{E}_h) = \Big\{v \in L^2(\Omega) | \ v|_{E_j} \in H^s(E_j), j = 1, 2, \cdot, N_h\Big\}.$$

We now define the average and the jump for $\phi \in H^s(\mathcal{E}_h), s > \frac{1}{2}$. For $e_k = \partial E_i \cap \partial E_j, i < j$, for $1 \leq k \leq P_h$, we set

$$\{\phi\} = \frac{1}{2}(\phi|_{E_i})|_{e_k} + \frac{1}{2}(\phi|_{E_j})|_{e_k}, \quad [\phi] = (\phi|_{E_i})|_{e_k} - (\phi|_{E_j})|_{e_k}.$$

The L^2 inner product is denoted by $(\cdot,\cdot)$ and the usual Sobolev norm on $E \subset \mathbf{R}^d$ is denoted by $\|\cdot\|_{m.E}$ for a postive integer m. Simply denote $\|\cdot\|_{m.\Omega}$ by $\|\cdot\|_m$ and $\|\cdot\|_{0.\Omega}$ by $\|\cdot\|$. We define the following broken norms:

$$\|\phi\|_m^2 = \sum_{j=1}^{N_h} \|\phi\|_{m.E_j}^2, \quad |||\phi|||^2 = \sum_{j=1}^{N_h} \Big(\|\phi\|_{1.E_j}^2 + h_j^2|\phi|_{2.E_j}^2\Big) + J^\sigma(\phi,\phi),$$

where $J^{\sigma}(\phi,\psi) = \sum_{k=1}^{P_h} \frac{\sigma}{|e_k|} \int_{e_k} [\phi][\psi] ds$ denotes the interior penalty term. Here, $|e_k|$ denotes the length of e_k and σ is a nonnegative real number.

Let r be a positive integer. The finite element space is taken to be

$$D_r(\mathcal{E}_h) = \prod_{j=1}^{N_h} P_r(E_j)$$

where $P_r(E_j)$ denotes the set of all polynomials of total degree less than or equal to r on E_j.

The following lemma is given in [5, 8]. Notice that throughout this paper C denotes a generic positive constant.

Lemma 3.1. *Let $u \in H^s(\Omega)$ for $s \geq 2$ and let $r \geq 2$. Let $\bar{a}$ be a piecewise positive constant defined on Ω. Then there is $\hat{u} \in D_r(\mathcal{E}_h)$, interpolant of u, such that*

$$\int_{e_k} \{\bar{a}\nabla(\hat{u}-u)\cdot n_k\} ds = 0 \quad \forall k = 1, \cdots, P_h, \tag{3.1}$$

$$\|\hat{u}-u\|_{\infty,E_j} \leq C\frac{h^{\mu}}{r^{s-1}}\|u\|_{s,E_j}, \tag{3.2}$$

$$\|\nabla(\hat{u}-u)\|_{0,E_j} \leq C\frac{h^{\mu-1}}{r^{s-1}}\|u\|_{s,E_j}, \tag{3.3}$$

$$\|\nabla^2(\hat{u}-u)\|_{0,E_j} \leq C\frac{h^{\mu-2}}{r^{s-2}}\|u\|_{s,E_j}, \tag{3.4}$$

$$\|\hat{u}-u\|_{0,E_j} \leq C\frac{h^{\mu}}{r^{s-1}}\|u\|_{s,E_j}, \tag{3.5}$$

where $\mu = \min(r+1, s)$. Moreover for $e_k = \partial E_i \cap \partial E_j$

$$\|\nabla\hat{u}\|_{\infty,e_k} \leq C\|\nabla u\|_{\infty,E_i\cup E_j} \tag{3.6}$$

The weak formulation of the model problem (2.1)-(2.3) is defined by

$$\begin{aligned}
&(u_t, v) + \sum_{j=1}^{N_h} \int_{E_j} a(u)\nabla u\cdot\nabla v dx - \sum_{k=1}^{P_h}\int_{e_k}\{a(u)\nabla u\cdot n_k\}[v]ds \\
&\quad - \sum_{k=1}^{P_h}\int_{e_k}\{a(u)\nabla v\cdot n_k\}[u]ds + J^{\sigma}(u,v) \\
&= (f(u), v), \quad v \in H^2(\mathcal{E}_h),\ t > 0,
\end{aligned} \tag{3.7}$$

and

$$u(x,0) = \psi(x). \tag{3.8}$$

We define a bilinear function $B(\rho;\cdot,\cdot)$ on $H^2(\mathcal{E}_h)\times H^2(\mathcal{E}_h)$ such that

$$\begin{aligned} B(\rho : v, w) = & \sum_{j=1}^{N_h}\int_{E_j} a(\rho)\nabla v\cdot\nabla w dx \\ & - \sum_{k=1}^{P_h}\int_{e_k}\{a(\rho)\nabla v\cdot n_k\}[w]ds \\ & - \sum_{k=1}^{P_h}\int_{e_k}\{a(\rho)\nabla w\cdot n_k\}[v]ds + J^\sigma(v,w). \end{aligned} \tag{3.9}$$

For a $\lambda > 0$, we define the following function B_λ on $H^2(\mathcal{E}_h)\times H^2(\mathcal{E}_h)$

$$B_\lambda(\rho : v, w) = B(\rho : v, w) + \lambda(v,w). \tag{3.10}$$

Now we state the following lemmas whose proofs are given in [5].

Lemma 3.2. *There exists a positive constant C independent on h such that*

$$|B_\lambda(\rho : v,w)| \le C|||v|||\,|||w|||, \qquad \forall v,w\in H^2(\mathcal{E}_h).$$

Lemma 3.3. *For a sufficiently large σ there exists a positive constant $\widetilde{c}$ independent on h such that*

$$B_\lambda(\rho : v,v) \ge \widetilde{c}|||v|||^2, \quad \forall v\in D_r(\mathcal{E}_h).$$

Now we let $\widetilde{H} = \{\psi\in H^1|\nabla\psi\cdot\eta = 0 \text{ on } \partial\Omega\}$

Lemma 3.4. *Let $t\in[0,T]$ be fixed and suppose that $\phi\in H^2(\mathcal{E}_h)$ satisfies*

$$B_\lambda(u : \phi, v) = F(v), \quad \forall v\in D_r(\mathcal{E}_h),$$

where $F : H^2(\mathcal{E}_h)\to\mathbb{R}$ is a linear function. Let M_1 and M_2 be constants for which

$$|F(w)| \le M_1|||w|||, \quad \forall w\in H^2(\mathcal{E}_h)$$

and

$$|F(\psi)| \le M_2\|\psi\|_{2,\Omega}, \quad \forall\psi\in H^2\cap\widetilde{H}$$

hold. Then we have the following estimation

$$\|\phi\| \le C(|||\phi||| + M_1)h + M_2.$$

Theorem 3.1. *There exists a unique $\tilde{u} \in D_r(\mathcal{E}_h)$ satisfying*

$$B_\lambda(u : u - \tilde{u}, v) = 0, \quad \forall v \in D_r(\mathcal{E}_h)$$

together with the following approximation properties:

$$\begin{aligned}
\|u - \tilde{u}\| + h|\!|\!|u - \tilde{u}|\!|\!| &\le C\frac{h^\mu}{r^{s-2}}\|u\|_s, \\
\|u_t - \tilde{u}_t\| + h|\!|\!|u_t - \tilde{u}_t|\!|\!| &\le C\frac{h^\mu}{r^{s-2}}\Big(\|u\|_s + \|u_t\|_s\Big), \\
\|u_{tt} - \tilde{u}_{tt}\| + h|\!|\!|u_{tt} - \tilde{u}_{tt}|\!|\!| &\le C\frac{h^\mu}{r^{s-2}}\Big(\|u\|_s + \|u_t\|_s + \|u_{tt}\|_s\Big), \\
\|u_{ttt} - \tilde{u}_{ttt}\| + h|\!|\!|u_{ttt} - \tilde{u}_{ttt}|\!|\!| &\le C\frac{h^\mu}{r^{s-2}}\Big(\|u\|_s + \|u_t\|_s + \|u_{tt}\|_s + \|u_{ttt}\|_s\Big),
\end{aligned}$$

where $s \ge 2$ and $\mu = \min(r+1, s)$.

Proof. By Lemma 3.2 and Lemma 3.3, the unique existence of $\tilde{u}$ follows. The proofs of the first two inequalities are given in [5] and the proofs of the last two inequalities can be proved by the similar way from that used in [5]. □

4. Error estimates for fully discrete approximations

For a positive integer N we let $\Delta t = T/N$, $t_j = j\Delta t$, $g_j = g(x, t_j)$, $0 \le j \le N$, and $t_{j,\theta} = \frac{1+\theta}{2}t_{j+1} + \frac{1-\theta}{2}t_j$, $g_{j,\theta} = \frac{1+\theta}{2}g_{j+1} + \frac{1-\theta}{2}g_j$, $0 \le j \le N-1$ where $\theta \in [0,1]$. Define $\|g\|_{\ell^\infty(L^2)} = \max_{j=0,\cdots,N}\|g_j\|$, $\|g\|_{\ell^2(H^1)} = \left(\sum_{j=0}^{N-1}\|g_j\|_1^2\right)^{1/2}$ and $|\!|\!|g|\!|\!|_{\ell^2(|\!|\!|\cdot|\!|\!|)} = \left(\sum_{j=0}^{N-1}|\!|\!|g_{j,\theta}|\!|\!|^2\right)^{1/2}$

The fully discrete discontinuous Galerkin approximation $\{U_j\}_{j=0}^N$ is a sequence in $D_r(\mathcal{E}_h)$ that satisfies

$$\begin{aligned}
&\int_\Omega \frac{U_{j+1} - U_j}{\Delta t} v dx + \int_\Omega a(U_{j,\theta})\nabla U_{j,\theta} \cdot \nabla v dx \\
&\quad - \sum_{k=1}^{P_h}\int_{e_k}\{a(U_{j,\theta})\nabla U_{j,\theta} \cdot n_k\}[v]ds \\
&\quad - \sum_{k=1}^{P_h}\int_{e_k}\{a(U_{j,\theta})\nabla v \cdot n_k\}[U_{j,\theta}]ds + J^\sigma(U_{j,\theta}, v) \\
&= (f(U_{j,\theta}), v), \ \forall v \in D_r(\mathcal{E}_h)
\end{aligned} \tag{4.1}$$

and

$$U_0 = P_h\psi$$

where $P_h\psi$ is an appropriate projection of ψ onto $D_r(\mathcal{E}_h)$ satisfying the following approximation property:

$$\|U_0 - \widetilde{u}(0)\| \leq C\frac{h^\mu}{r^{s-2}}\|\psi\|_s, \;\; \mu = \min(r+1, s). \tag{4.2}$$

Notice that (4.1) corresponds to Crank-Nicolson DGM when $\theta = 0$ and that (4.1) corresponds to backward Euler DGM when $\theta = 1$. And notice that the fully discrete discontinuous Galerkin approximations $\{U_j\}_{j=0}^N$ satisfying (4.1) and (4.2) are well defined if Δt is sufficiently small.

Theorem 4.1. *Let $\eta = \widetilde{u} - u$ and $\zeta = \widetilde{u} - U$. Then there exist constants $C > 0$ and $\beta > 0$ independent on h and Δt satisfying the following statements:*

(i) *If $\theta \in (0,1]$ and u is sufficiently smooth such that $u \in L^\infty(H^s)$, $u_t \in L^\infty(H^s)$ and $u_{tt} \in L^\infty(H^s)$ then*

$$\begin{aligned}
&\|\zeta\|^2_{\ell^\infty(L^2)} + \beta\Delta t\|\!|\zeta|\!\|^2_{\ell^2(\|\!|\cdot|\!\|)} \\
&\leq C\left\{\frac{h^{2\mu}}{r^{2(s-2)}}(\|\psi\|_s^2 + \|u\|^2_{L^\infty(H^s)} + \|u_t\|^2_{L^\infty(H^s)}) + (\Delta t)^2\|u_{tt}\|^2_{L^\infty(H^s)}\right\}
\end{aligned}$$

(ii) *if $\theta = 0$, and u is sufficiently smooth such that $u \in L^\infty(H^s)$, $u_t \in L^\infty(H^s)$, and $u_{ttt} \in L^\infty(H^s)$, then*

$$\begin{aligned}
&\|\zeta\|^2_{\ell^\infty(L^2)} + \beta\Delta t\|\!|\zeta|\!\|^2_{\ell^2(\|\!|\cdot|\!\|)} \\
&\leq C\left\{\frac{h^{2\mu}}{r^{2(s-2)}}(\|\psi\|_s^2 + \|u\|^2_{L^\infty(H^s)} + \|u_t\|^2_{L^\infty(H^s)}) + (\Delta t)^2\|u_{ttt}\|^2_{L^\infty(H^s)}\right\}.
\end{aligned}$$

Proof. Since $\zeta = \widetilde{u} - U$, we obtain from (4.1) that for $t = t_{j,\theta}, 0 \leq j \leq N-1$

$$\begin{aligned}
&\left(\frac{\zeta_{j+1} - \zeta_j}{\Delta t}, v\right) + (u_t(t_{j,\theta}), v) - \left(\frac{\widetilde{u}_{j+1} - \widetilde{u}_j}{\Delta t}, v\right) \\
&+ B(u_{j,\theta} : u_{j,\theta}, v) - B(U_{j,\theta} : U_{j,\theta}, v) \\
&= (f(u_{j,\theta}) - f(U_{j,\theta}), v), \;\; \forall v \in D_r(\mathcal{E}_h).
\end{aligned} \tag{4.3}$$

Hence

$$\begin{aligned}
&\left(\frac{\zeta_{j+1} - \zeta_j}{\Delta t}, v\right) + (u_t(t_{j,\theta}), v) - \left(\frac{\widetilde{u}_{j+1} - \widetilde{u}_j}{\Delta t}, v\right) \\
&+ B(u_{j,\theta} : u_{j,\theta}, v) + B(U_{j,\theta} : \zeta_{j,\theta}, v) - B(U_{j,\theta} : \widetilde{u}_{j,\theta}, v) \\
&= (f(u_{j,\theta}) - f(U_{j,\theta}), v).
\end{aligned} \tag{4.4}$$

Putting $v = \zeta_{j,\theta}$ in (4.4), we obtain

$$\begin{aligned}
&\left(\frac{\zeta_{j+1}-\zeta_j}{\Delta t}, \zeta_{j,\theta}\right) + B_\lambda(U_{j,\theta} : \zeta_{j,\theta}, \zeta_{j,\theta}) \\
&= \left[-(u_t(t_{j,\theta}), \zeta_{j,\theta}) + \left(\frac{\widetilde{u}_{j+1}-\widetilde{u}_j}{\Delta t}, \zeta_{j,\theta}\right)\right] + [B(U_{j,\theta} : \widetilde{u}_{j,\theta}, \zeta_{j,\theta}) \\
&\quad -B(u_{j,\theta} : u_{j,\theta}, \zeta_{j,\theta})] + [(f(u_{j,\theta}) - f(U_{j,\theta}), \zeta_{j,\theta}) + \lambda(\zeta_{j,\theta}, \zeta_{j,\theta})] \\
&\equiv I_1 + I_2 + I_3.
\end{aligned}$$

Notice that

$$\begin{aligned}
&\left(\frac{\zeta_{j+1}-\zeta_j}{\Delta t}, \zeta_{j,\theta}\right) \\
&= \frac{1}{\Delta t}\left(\zeta_{j+1}-\zeta_j, \frac{1}{2}(1+\theta)\zeta_{j+1} + \frac{1}{2}(1-\theta)\zeta_j\right) \\
&= \frac{1}{2\Delta t}\left[(1+\theta)\|\zeta_{j+1}\|^2 - (1-\theta)\|\zeta_j\|^2 - 2\theta(\zeta_j, \zeta_{j+1})\right] \\
&\geq \frac{1}{2\Delta t}\left[(1+\theta)\|\zeta_{j+1}\|^2 - (1-\theta)\|\zeta_j\|^2 - \theta\|\zeta_j\|^2 - \theta\|\zeta_{j+1}\|^2\right] \\
&= \frac{1}{2\Delta t}\left[\|\zeta_{j+1}\|^2 - \|\zeta_j\|^2\right]
\end{aligned}$$

and

$$B_\lambda(U_{j,\theta} : \zeta_{j,\theta}, \zeta_{j,\theta}) \geq \widetilde{c}|||\zeta_{j,\theta}|||^2.$$

Thus we have

$$\frac{1}{2\Delta t}\left[\|\zeta_{j+1}\|^2 - \|\zeta_j\|^2\right] + \widetilde{c}|||\zeta_{j,\theta}|||^2 \leq I_1 + I_2 + I_3.$$

Now we estimate the bounds for I_1, I_2 and I_3. Using the Taylor's expansions of $\widetilde{u}_{j+1}$ and $\widetilde{u}_j$ about $t = t_{j,\theta}$, we obatin

$$\begin{aligned}
&\frac{\widetilde{u}_{j+1}-\widetilde{u}_j}{\Delta t} \\
&= \widetilde{u}_t(t_{j,\theta}) + \frac{1}{2}\left[\left(\frac{1-\theta}{2}\right)^2 - \left(\frac{1+\theta}{2}\right)^2\right](\Delta t)\widetilde{u}_{tt}(t_{j,\theta}) \\
&\quad + \frac{1}{6}\left[\left(\frac{1-\theta}{2}\right)^3(\Delta t)^2\widetilde{u}_{ttt}(t^*) + \left(\frac{1+\theta}{2}\right)^3(\Delta t)^2\widetilde{u}_{ttt}(t^{**})\right],
\end{aligned}$$

for some $t^*, t^{**} \in (t_j, t_{j+1})$. Therefore we get

$$\begin{aligned}
I_1 &= \left(\frac{\widetilde{u}_{j+1}-\widetilde{u}_j}{\Delta t}, \zeta_{j.\theta}\right) - (u_t(t_{j.\theta}), \zeta_{j.\theta}) \\
&= (\widetilde{u}_t(t_{j.\theta}) - u_t(t_{j.\theta}), \zeta_{j.\theta}) + \Delta t(\rho_{j.\theta}, \zeta_{j.\theta})
\end{aligned}$$

where

$$\rho_{j,\theta} = \frac{1}{2}\left[\left(\frac{1-\theta}{2}\right)^2 - \left(\frac{1+\theta}{2}\right)^2\right]\widetilde{u}_{tt}(t_{j,\theta})$$
$$+ \frac{1}{6}\left[\left(\frac{1-\theta}{2}\right)^3 (\Delta t)\widetilde{u}_{ttt}(t^*) + \left(\frac{1+\theta}{2}\right)^3 (\Delta t)\widetilde{u}_{ttt}(t^{**})\right].$$

Since $\eta_t = \widetilde{u}_t - u_t$, we have

$$\begin{aligned} |I_1| &\le \|\eta_t(t_{j,\theta})\|\|\zeta_{j,\theta}\| + \Delta t\|\rho_{j,\theta}\|\|\zeta_{j,\theta}\| \\ &\le C\Big(\|\eta_t(t_{j,\theta})\|^2 + \|\zeta_{j,\theta}\|^2 + (\Delta t)^2\|\rho_{j,\theta}\|^2\Big). \end{aligned}$$

Notice that

$$\begin{aligned} I_2 &= B(U_{j,\theta} : \widetilde{u}_{j,\theta}, \zeta_{j,\theta}) - B(u_{j,\theta} : u_{j,\theta}, \zeta_{j,\theta}) \\ &= B(U_{j,\theta} : \widetilde{u}_{j,\theta}, \zeta_{j,\theta}) - B(u_{j,\theta} : \widetilde{u}_{j,\theta}, \zeta_{j,\theta}). \end{aligned}$$

Therefore

$$\begin{aligned} |I_2| &= |B(U_{j,\theta} : \widetilde{u}_{j,\theta}, \zeta_{j,\theta}) - B(u_{j,\theta} : \widetilde{u}_{j,\theta}, \zeta_{j,\theta})| \\ &\le \left|\sum_{j=1}^{N_h}\int_{E_j} (a(U_{j,\theta}) - a(u_{j,\theta}))\nabla\widetilde{u}_{j,\theta}\cdot\nabla\zeta_{j,\theta}\right| \\ &\quad + \left|\sum_{k=1}^{P_h}\int_{e_k} \{(a(U_{j,\theta}) - a(u_{j,\theta}))\nabla\widetilde{u}_{j,\theta}\cdot n_k\}[\zeta_{j,\theta}]\right| \\ &\quad + \left|\sum_{k=1}^{P_h}\int_{e_k} \{(a(U_{j,\theta}) - a(u_{j,\theta}))\nabla\zeta_{j,\theta}\cdot n_k\}[\widetilde{u}_{j,\theta}]\right| \\ &\equiv I_{21} + I_{22} + I_{23}. \end{aligned}$$

Using the trace theorem, the inverse estimate, and the boundedness of $\|\nabla\widetilde{u}\|_{\infty,E_j}$ and $\|\nabla\widetilde{u}\|_{\infty,e_k}$, we obtain the following estimates for I_{21}, I_{22} and I_{23}:

$$\begin{aligned} I_{21} &\le \sum_{j=1}^{N_h}\int_{E_j} |(a(U_{j,\theta}) - a(u_{j,\theta}))\nabla\widetilde{u}_{j,\theta}\cdot\nabla\zeta_{j,\theta}| \\ &\le C\sum_{j=1}^{N_h}\Big(\|\eta_{j,\theta}\|_{0,E_j} + \|\zeta_{j,\theta}\|_{0,E_j}\Big)\|\nabla\zeta_{j,\theta}\|_{0,E_j}\|\nabla\widetilde{u}_{j,\theta}\|_{\infty,E_j} \\ &\le C\sum_{j=1}^{N_h}(\|\eta_{j,\theta}\|_{0,E_j}^2 + \|\zeta_{j,\theta}\|_{0,E_j}^2) + \varepsilon_1 |||\zeta_{j,\theta}|||^2, \end{aligned}$$

$$
\begin{aligned}
I_{22} &\leq \sum_{k=1}^{P_h} \left[\|\eta_{j,\theta}\|_{0,e_k} + \|\zeta_{j,\theta}\|_{0,e_k} \right] \|\nabla \widetilde{u}_{j,\theta}\|_{\infty,e_k} \|[\zeta_{j,\theta}]\|_{0,e_k} \\
&\leq C(J^\sigma(\zeta_{j,\theta},\zeta_{j,\theta}))^{1/2} \left(\sum_{k=1}^{P_h} \frac{|e_k|}{\sigma} (\|\eta_{j,\theta}\|_{0,e_k}^2 + \|\zeta_{j,\theta}\|_{0,e_k}^2) \right)^{1/2} \\
&\leq C \| \! | \zeta_{j,\theta} | \! \| \left(\sum_{j=1}^{N_h} h_j \left[h_j^{-1} \|\eta_{j,\theta}\|_{0,E_j}^2 + h_j \|\nabla \eta_{j,\theta}\|_{0,E_j}^2 + h_j^{-1} \|\zeta_{j,\theta}\|_{0,E_j}^2 \right] \right)^{1/2} \\
&\leq \varepsilon_2 \| \! | \zeta_{j,\theta} | \! \|^2 + C \sum_{j=1}^{N_h} \left(\|\eta_{j,\theta}\|_{0,E_j}^2 + h_j^2 \|\nabla \eta_{j,\theta}\|_{0,E_j}^2 + \|\zeta_{j,\theta}\|_{0,E_j}^2 \right),
\end{aligned}
$$

and

$$
\begin{aligned}
I_{23} &\leq \sum_{k=1}^{P_h} \|\nabla \zeta_{j,\theta}\|_{\infty,e_k} (\|\eta_{j,\theta}\|_{0,e_k} + \|\zeta_{j,\theta}\|_{0,e_k}) \|[\eta_{j,\theta}]\|_{0,e_k} \\
&\leq \sum_{j=1}^{N_h} h_j^{-2} \|\nabla \zeta_{j,\theta}\|_{0,E_j} \left(\|\eta_{j,\theta}\|_{0,E_j} + h_j \|\nabla \eta_{j,\theta}\|_{0,E_j} + \|\zeta_{j,\theta}\|_{0,E_j} \right) \\
&\qquad \cdot \left(\|\eta_{j,\theta}\|_{0,E_j} + h_j \|\nabla \eta_{j,\theta}\|_{0,E_j} \right) \\
&\leq \sum_{j=1}^{N_h} \|\nabla \zeta_{j,\theta}\|_{0,E_j} \left(\|\eta_{j,\theta}\|_{0,E_j} + h_j \|\nabla \eta_{j,\theta}\|_{0,E_j} + \|\zeta_{j,\theta}\|_{0,E_j} \right) \|u\|_2 \\
&\leq \varepsilon_3 \| \! | \zeta_{j,\theta} | \! \|^2 + \sum_{j=1}^{N_h} \left(\|\eta_{j,\theta}\|_{0,E_j}^2 + h_j^2 \|\nabla \eta_{j,\theta}\|_{0,E_j}^2 + \|\zeta_{j,\theta}\|_{0,E_j}^2 \right)
\end{aligned}
$$

for sufficiently small $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, and $\varepsilon_3 > 0$. Therefore we obtain

$$
|I_2| \leq (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \| \! | \zeta_{j,\theta} | \! \|^2 + C \sum_{j=1}^{N_h} \left(\|\eta_{j,\theta}\|_{0,E_j}^2 + h_j^2 \|\nabla \eta_{j,\theta}\|_{0,E_j}^2 + \|\zeta_{j,\theta}\|_{0,E_j}^2 \right).
$$

And

$$
|I_3| \leq C \left[(\|\eta_{j,\theta}\| + \|\zeta_{j,\theta}\|) \|\zeta_{j,\theta}\| + \|\zeta_{j,\theta}\|^2 \right] \leq C(\|\eta_{j,\theta}\|^2 + \|\zeta_{j,\theta}\|^2).
$$

Using the bounds for I_1, I_2 and I_3, we have

$$
\begin{aligned}
&\frac{1}{2\Delta t} \left[\|\zeta_{j+1}\|^2 - \|\zeta_j\|^2 \right] + C \| \! | \zeta_{j,\theta} | \! \|^2 \\
&\leq \varepsilon \| \! | \zeta_{j,\theta} | \! \|^2 + C \left[\|\eta_{j,\theta}\|^2 + \|\zeta_{j,\theta}\|^2 + \|\eta_t(t_{j,\theta})\|^2 + (\Delta t)^2 \|\rho_{j,\theta}\|^2 + h^2 \|\nabla \eta_{j,\theta}\|^2 \right]
\end{aligned}
$$

which implies the following

$$\begin{aligned}
&\|\zeta_{j+1}\|^2 - \|\zeta_j\|^2 + \alpha\Delta t |||\zeta_{j,\theta}|||^2 \\
&\leq C\Delta t\Big[\|\eta_{j,\theta}\|^2 + \|\zeta_{j,\theta}\|^2 + h^2\|\nabla\eta_{j,\theta}\|^2 + \|\eta_t(t_{j,\theta})\|^2\Big] + C(\Delta t)^3\|\rho_{j,\theta}\|^2.
\end{aligned}$$

Since $\eta_{j,\theta} = \frac{1}{2}(1+\theta)\eta_{j+1} + \frac{1}{2}(1-\theta)\eta_j$, we have

$$\begin{aligned}
&\|\zeta_{j+1}\|^2 - \|\zeta_j\|^2 + \alpha\Delta t |||\zeta_{j,\theta}|||^2 \\
&\leq C\Delta t\,[\|\eta_{j+1}\|^2 + \|\eta_j\|^2 + \|(\eta_t)_{j+1}\|^2 + \|(\eta_t)_j\|^2 + \|\zeta_j\|^2 \\
&\quad + \|\zeta_{j+1}\|^2 + h^2\|\nabla\eta_{j+1}\|^2 + h^2\|\nabla\eta_j\|^2] + C(\Delta t)^3\|\rho_{j,\theta}\|^2.
\end{aligned} \tag{4.4}$$

Summing the both sides of (4.4) from $j = 0$ to $N-1$, we have

$$\begin{aligned}
&\|\zeta_N\|^2 + \alpha\Delta t\sum_{j=0}^{N-1} |||\zeta_{j,\theta}|||^2 \\
&\leq \|\zeta_0\|^2 + C\Delta t\sum_{j=0}^{N}[\|\eta_j\|^2 + \|(\eta_t)_j\|^2 + h^2\|\nabla\eta_j\|^2 + \|\zeta_j\|^2] \\
&\quad + C(\Delta t)^3\sum_{j=0}^{N-1}\|\rho_{j,\theta}\|^2.
\end{aligned}$$

By applying the Gronwall's Lemma, we have for a sufficiently small Δt,

$$\begin{aligned}
&\|\zeta_N\|^2 + \beta\Delta t\sum_{j=0}^{N-1} |||\zeta_{j,\theta}|||^2 \\
&\leq C\|\zeta_0\|^2 + C(\Delta t)^3\sum_{j=0}^{N-1}\|\rho_{j,\theta}\|^2 \\
&\quad + C(\Delta t)\sum_{j=0}^{N}[\|(\eta_t)_j\|^2 + \|\eta_j\|^2 + h^2\|\nabla\eta_j\|^2].
\end{aligned} \tag{4.5}$$

Applying (4.2) to (4.5), we obtain

$$\begin{aligned}
\|\zeta_N\|^2 + \beta\Delta t\sum_{j=0}^{N-1} |||\zeta_{j,\theta}|||^2 &\leq C\frac{h^{2\mu}}{r^{2(s-2)}}\|\psi\|_s^2 + C(\Delta t)^3\sum_{j=0}^{N-1}\|\rho_{j,\theta}\|^2 \\
&\quad + C\Delta t\sum_{j=0}^{N}\left[\frac{h^{2\mu}}{r^{2(s-2)}}\left(\|u_j\|_{s,E_j}^2 + \|u_{t_j}\|_{s,E_j}^2\right)\right].
\end{aligned}$$

Recalling the following definition of $\rho_{j,\theta}$ for $\theta \in (0,1]$,

$$\rho_{j,\theta} = \frac{1}{2}\left[\left(\frac{1-\theta}{2}\right)^2 - \left(\frac{1+\theta}{2}\right)^2\right]\widetilde{u}_{tt}(t_{j,\theta}) + \frac{1}{6}\Delta t\left[\left(\frac{1-\theta}{2}\right)^3 \widetilde{u}_{ttt}(t^*) + \left(\frac{1+\theta}{2}\right)^3 \widetilde{u}_{ttt}(t^{**})\right]$$

we have

$$\|\rho_{j,\theta}\| \le C(\theta)\|\widetilde{u}_{tt}\|_{L^\infty(t_j,t_{j+1};L^2)} \text{ for } \theta \in (0,1].$$

Hence, for $\theta \in (0,1]$

$$\begin{aligned}
&\|\zeta\|^2_{\ell^\infty(L^2)} + \beta\Delta t \|\!|\zeta|\!\|^2_{\ell^2(\|\!|\cdot|\!\|)} \\
&\le C\frac{h^{2\mu}}{r^{2(s-2)}}\|\psi\|_s^2 + C\Delta t \sum_{j=0}^{N} \frac{h^{2\mu}}{r^{2(s-2)}}(\|u_j\|_s^2 + \|(u_t)_j\|_s^2) \\
&\quad + C(\Delta t)^3 \sum_{j=0}^{N-1} \|u_{tt}\|^2_{L^\infty(t_j,t_{j+1};H^s)} \\
&\le C\left\{\frac{h^{2\mu}}{r^{2(s-2)}}\|\psi\|_s^2 + \frac{h^{2\mu}}{r^{2(s-2)}}(\|u\|^2_{L^\infty(H^s)} + \|u_t\|^2_{L^\infty(H^s)}) \right.\\
&\quad \left. + (\Delta t)^2\|u_{tt}\|^2_{L^\infty(H^s)}\right\}.
\end{aligned}$$

And for $\theta = 0$ we obtain

$$\|\rho_{j,\theta}\| \le C\Delta t\|\widetilde{u}_{ttt}\|_{L^\infty(t_j,t_{j+1};H^s)}$$

and therefore

$$\begin{aligned}
&\|\zeta\|^2_{\ell^\infty(L^2)} + \beta\Delta t \|\!|\zeta|\!\|^2_{\ell^2(\|\!|\cdot|\!\|)} \\
&\le C\frac{h^{2\mu}}{r^{2(s-2)}}\|\psi\|_s^2 + C\Delta t \sum_{j=0}^{N} \frac{h^{2\mu}}{r^{2(s-2)}}(\|u_j\|_s^2 + \|(u_t)_j\|_s^2) \\
&\quad + C(\Delta t)^5 \sum_{j=0}^{N-1} \|u_{ttt}\|^2_{L^\infty(t_j,t_{j+1};H^s)} \\
&\le C\left\{\frac{h^{2\mu}}{r^{2(s-2)}}(\|\psi\|_s^2 + \|u\|^2_{L^\infty(H^s)} + \|u_t\|^2_{L^\infty(H^s)}) + (\Delta t)^2\|u_{ttt}\|^2_{L^\infty(H^s)}\right\}
\end{aligned}$$

which completes the proof. □

Finally by combining the results of Theorem 3.1 and Theorem 4.1, we have the following optimal L^2 error estimations for the fully discrete discontinuous Galerkin approximations.

Theorem 4.2. (i) *For $\theta \in (0,1]$, assume that $u \in L^\infty(H^s)$, $u_t \in L^\infty(H^s)$ and $u_{tt} \in L^\infty(H^s)$. Then we obtain*

$$\|U-u\|^2_{\ell^\infty(L^2)} + h^2 |||U-u|||^2_{\ell^2(|||\cdot|||)} \le C\left\{ \frac{h^{2\mu}}{r^{2(s-2)}} (\|\psi\|_s^2 + \|u\|^2_{L^\infty(H^s)} + \|u_t\|^2_{L^\infty(H^s)}) + (\Delta t)^2 \|u_{tt}\|^2_{L^\infty(H^s)} \right\}$$

where $2 \le \mu \le \min(r+1, s)$.

(ii) *For $\theta = 0$, assume that $u \in L^\infty(H^s)$, $u_t \in L^\infty(H^s)$ and $u_{ttt} \in L^\infty(H^s)$. Then, we obtain*

$$\|U-u\|^2_{\ell^\infty(L^2)} + h^2 |||U-u|||^2_{\ell^2(|||\cdot|||)} \le C\left\{ \frac{h^{2\mu}}{r^{2(s-2)}} (\|\psi\|_s^2 + \|u\|^2_{L^\infty(H^s)} + \|u_t\|^2_{L^\infty(H^s)}) + (\Delta t)^2 \|u_{ttt}\|^2_{L^\infty(H^s)} \right\}$$

where $2 \le \mu \le \min(r+1, s)$.

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