http://dx.doi.org/10.5370/JEET.2011.6.5.706

Necessary and Sufficient Conditions for the Existence of Decoupling Controllers in the Generalized Plant Model

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Abstract – Necessary and sufficient conditions for the existence of diagonal, block-diagonal, and triangular decoupling controllers in linear multivariable systems for the most general setting are presented. The plant model in this study is sufficiently general to accommodate non-square plant and non-unity feedback cases with one-degree-of-freedom (1DOF) or two-degree-of-freedom (2DOF) controller configuration. The existence condition is described in terms of rank conditions on the coefficient matrices in partial fraction expansions.

Keywords: Decoupling controller, Generalized plant model, Existence condition, Linear multivariable control, Block decoupling

1. Introduction

The decoupling design in linear multivariable control systems aims at finding stabilizing controllers that eliminate interactions between the reference inputs and the plant outputs. Mathematically, all stabilizing controllers that make the transfer matrix from the reference input to the plant output are found to be diagonal such that one plant output is affected by only one input. Recently, the application of decoupling design in industrial fields has been increasing [1]–[4]. A typical example is the load control of a crane, whose purpose is to minimize the movement of the pendulum-like load. It is reported [1] that the decoupling design between (the trolley, orientation) inputs and (the position, angle) outputs makes it easier to control the crane by minimizing the pendulum movement of the load.

A decoupling controller, however, does not always exist. Existence conditions of decoupling controllers in linear multivariable systems have been studied by many researchers. Vardulakis [5] proposed a sufficient condition wherein a diagonal decoupling controller exists if there is no unstable pole-zero coincidence in the plant. Necessary and sufficient conditions for the existence of decoupling controllers were obtained in various ways. Lin [6], [7] exploited the internal stability requirement as the constraints in constructing diagonal and block-diagonal input–output maps. Youla and Bongiorno [8] took a similar approach with that of [6] for a diagonal decoupling problem, but the class of all stabilizing decoupled transfer matrices was explicitly parameterized, which made it possible to obtain the optimal H_2 decoupling controller. Gómez and Goodwin [9] adopted an algebraic approach based on coprime factorizations to treat diagonal and triangular decoupling designs. In [10], a unifying approach was suggested to treat diagonal, block-diagonal, and triangular decoupling problems. These previous studies, however, considered the conventional model with unity feedback [5]–[7], [9] or state feedback [10]. In [8], the unity feedback constraint was loosened, but the arbitrary non-unity feedback was still not assumed.

In this paper, necessary and sufficient conditions for the existence of decoupling controllers are presented in the generalized plant model, which accommodates non-square plants and non-unity feedback cases with one-degree-of-freedom (1DOF) or two-degree-of-freedom (2DOF) configuration. The approach taken in this paper is direct such that diagonal, block-diagonal, and triangular decoupling problems are treated in a unified frame. The existence condition is described in terms of rank conditions on the coefficient matrices in partial fraction expansions.

Notations – Throughout this paper, only real rational matrices are considered. Notations C, C_+, and $\bar{C}_+$ denote the complex number plane, the open right half plane of C, and the closure of C_+, respectively. C^m denotes the set of $m \times 1$ complex vectors. The notation $T_{ba}(s)$ stands for the transfer matrix from a to b. A rational matrix $G(s)$ is said to be stable if it is analytic in $\bar{C}_+$. The transpose of $G(s)$ is denoted by $G'(s)$, and $G^{(n)}(s)$ denotes the n-th derivative of $G(s)$. The notations $\bar{\xi}$ and ξ^* denote the conjugate and the conjugate transpose of the vector ξ, respectively. When X_i is a matrix, $diag\{X_1, X_2, \cdots, X_n\}$ or $diag\{X_i\}_{i=1}^n$ denotes the block-diagonal matrix whose $i-th$ block diagonal element is X_i; when the block-diagonal elements are all X, it is denoted by $diag^{(n)}\{X\}$. The Kronecker product of two matrices is denoted as $G \otimes R$. The notation $vec(G)$ denotes the vector formed by stacking all the columns of

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Received: September 30, 2010; Accepted: May 12, 2011

the matrix G. The Khatri-Rao product of two matrices is denoted as $G \odot R$ and is the matrix whose i-th column is given by $g_i \otimes r_i$, where g_i and r_i are the i-th column of G and i-th column of R, respectively. A convenient formula is $(A \otimes B)(C \odot D) = (AC \odot BD)$. For a diagonal matrix, $vecd(G)$ denotes the vector formed by stacking all the diagonal elements of the matrix G. When V is a diagonal matrix, $vec(AVD) = (D' \otimes A)vec(V) = (D' \odot A)vecd(V)$ [11]. The notation e_q denotes the $1 \times q$ row vector whose elements are all 1, and iff stands for "if and only if".

2. Internal Stability and Realizability Condition

The generalized plant model under consideration is given by the following equations:

$$\begin{bmatrix} v \\ y \end{bmatrix} = P(s)\begin{bmatrix} r \\ u \end{bmatrix}, \quad P = \begin{bmatrix} P_{00}(s) & P_{02}(s) \\ P_{20}(s) & P_{22}(s) \end{bmatrix}, \quad u = C(s)y \quad (1)$$

The variables u and y are the control input and the measured variable, respectively. The variable r is an exogenous input, while the variable v is the target variable. The variables r and v are the ones to be decoupled by the transfer matrix T_{vr}. In most cases, r is the reference input and v is the plant output. The variables v and r have the same dimension of $m \times 1$. The variables u and y have the dimensions $m_1 \times 1$ and $m_2 \times 1$, respectively. The following assumption is necessary and sufficient for the existence of a stabilizing controller [12]. Let Ψ_P denote the characteristic denominator [13] of the rational matrix $P(s)$ and Ψ_P^+ the monic polynomial that absorbs all the zeros of Ψ_P in $\bar{C}_+$.

Assumption 1: The general plant block $P(s)$ is free of hidden modes in $\bar{C}_+$, and $\Psi_P^+ = \Psi_{P_{22}}^+$.

The hidden modes of a block whose transfer matrix is $P(s)$ arise when the characteristic denominator of $P(s)$ does not include all the internal modes of the block [13]. We consider the polynomial coprime fractional expressions, $P_{22} = A^{-1}(s)B(s) = B_1(s)A_1^{-1}(s)$. There always exist polynomial matrices $X(s), Y(s), X_1(s)$ and $Y_1(s)$ such that

$$\begin{bmatrix} X_1 & Y_1 \\ -B & A \end{bmatrix}\begin{bmatrix} A_1 & -Y \\ B_1 & X \end{bmatrix} = \begin{bmatrix} A_1 & -Y \\ B_1 & X \end{bmatrix}\begin{bmatrix} X_1 & Y_1 \\ -B & A \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad (2)$$

with $\det X(s) \det X_1(s) \not\equiv 0$ (adopting proper stable rational coprime fractions does not affect the remaining results of this paper). It is well known [14] that the condition $\Psi_P^+ = \Psi_{P_{22}}^+$ in Assumption 1 is equivalent to the one wherein $P_{00} - P_{02}A_1Y_1P_{20}$, $P_{02}A_1$, and AP_{20} are stable. The transfer matrix $T_{vr}(s)$ is the one to be decoupled and is given by the equation below:

$$T_{vr}(s) = P_{00} + P_{02}(I - CP_{22})^{-1}CP_{20} \quad (3)$$

In the following, we define the rational matrix $T(s)$ that can be obtained as $T_{vr}(s)$ by a stabilizing controller.

Definition 1: A stable rational matrix $T(s)$ is said to be realizable for the given plant $P(s)$ if there exists a stabilizing controller $C(s)$ that realizes the transfer matrix $T_{vr}(s)$ of the system as the matrix $T(s)$.

From (1), it follows that $v = P_{00}r + P_{02}u$; when the variable v is the plant output, it does not usually contain a direct term of the reference input r. Hence, in almost all cases, P_{00} becomes a null matrix and in this case $T = T_{vr} = P_{02}(I - CP_{22})^{-1}CP_{20}$. In decoupling design, the transfer matrix T is required to be of full normal rank together with the diagonal requirement. When $P_{00} = 0$, it is necessary that $m \le m_1$ and $rank(P_{02}) = m$ for the full rank requirement of T. Similarly, it is required that $m \le m_2$ and $rank(P_{20}) = m$. Although we presume that $P_{00} = 0$, we do not assume this to keep the plant model as general as possible; instead, we assume only the following rank conditions.

Assumption 2: $m \le m_1$, $m \le m_2$, and $rank(P_{02}) = rank(P_{20}) = m$.

Next, we consider the class of all stabilizing controllers characterized by the following formula:

$$C(s) = -(X_1 - KB)^{-1}(Y_1 + KA) \quad (4)$$

where $K(s)$ is an arbitrary real rational stable matrix such that $\det(X_1 - KB) \not\equiv 0$. Inserting this formula to (3) will obtain the characterized formula for the realizable $T(s)$, as follows:

$$T = T_0 - P_{02}A_1KAP_{20}, \quad T_0 = P_{00} - P_{02}A_1Y_1P_{20} \quad (5)$$

As can be observed, T is stable since T_0, $P_{02}A_1$, AP_{20}, and K are stable. Since $rank(P_{02}) = rank(P_{20}) = m$, the ranks of $P_{02}A_1$ and AP_{20} are also m; in this case, it is known that there exist unimodular matrices V_1 and V_2 [15] such that

$$P_{02}A_1V_1 = \begin{bmatrix} R_{10} & 0 \end{bmatrix} \text{ and } V_2AP_{20} = \begin{bmatrix} R_{20} \\ 0 \end{bmatrix} \quad (6)$$

with $rank(R_{10}) = rank(R_{20}) = m$. Therefore, it follows that

$$T = T_0 - P_{02}A_1KAP_{20} = T_0 - \begin{bmatrix} R_{10} & 0 \end{bmatrix}\hat{K}\begin{bmatrix} R_{20} \\ 0 \end{bmatrix}, \quad (7)$$

where $\hat{K} = V_1^{-1} K V_2^{-1}$. We then consider the following partition:

$$\hat{K} = V_1^{-1} K V_2^{-1} = \begin{bmatrix} \hat{K}_{11} & \hat{K}_{12} \\ \hat{K}_{21} & \hat{K}_{22} \end{bmatrix} \quad (8)$$

where the dimensions of $\hat{K}_{11}$ and $\hat{K}_{22}$ are $m \times m$ and $(m_1 - m) \times (m_2 - m)$, respectively. It then follows that

$$T = T_0 - R_{10} \hat{K}_{11} R_{20} \quad (9)$$

and hence

$$\hat{K}_{11} = R_{10}^{-1} T_0 R_{20}^{-1} - R_{10}^{-1} T R_{20}^{-1}. \quad (10)$$

In view of (4) and (10), a stable rational matrix T is realizable for $P(s)$ iff it makes $R_{10}^{-1} T_0 R_{20}^{-1} - R_{10}^{-1} T R_{20}^{-1}$ stable. As can be seen, a realizable T determines only $\hat{K}_{11}$, a part of $\hat{K}$, and the other parts of $\hat{K}$ can be obtained by other criterions of the control system design.

3. Decoupling Problems

In the previous section, we have derived the realizability condition for a stable rational matrix T that guarantees the existence of a stabilizing controller C. When an additional requirement is added to the transfer matrix T, we need to add this constraint on T in solving the realizability problem. In the following section, we will show that when a decoupling constraint is added, the realizability problem can be transformed into the following standard problem by appropriate vector operations, and this invokes the introduction of the following standard problem.

Standard Problem for Decoupling Design (SPDD): For a given $\tilde{m} \times 1$ vector $\phi(s)$ and a given $\tilde{m} \times \tilde{n}$ matrix $\Psi(s)$, we find a stable $\tilde{n} \times 1$ vector $h(s)$ that makes ϕ_k stable, where

$$\phi_k = \phi - \Psi h . \quad (11)$$

In the following, we will consider the three decoupling problems and explain the procedure of getting the SPDD for a given decoupling constraint. Taking vector operation on both sides of (10), we obtain $\phi_k(s) = \phi(s) - (R_{20}^{-1}{}' \otimes R_{10}^{-1}) vecT(s)$, where

$$\phi_k = vec\hat{K}_{11}, \quad \phi = vec(R_{10}^{-1} T_0 R_{20}^{-1}) = (R_{20}^{-1}{}' \otimes R_{10}^{-1}) vecT_0 . \quad (12)$$

There is also a need to determine $\Psi(s)$ and $h(s)$ from $(R_{20}^{-1}{}' \otimes R_{10}^{-1}) vecT(s)$ depending on the decoupling structures of $T(s)$.

Diagonal Decoupling: We suppose that $T(s)$ in (10) is diagonal. In this case, since $(R_{20}^{-1}{}' \otimes R_{10}^{-1}) vecT(s) = (R_{20}^{-1'} \odot R_{10}^{-1}) vecdT(s)$, it follows that

$$\Psi = R_{20}^{-1'} \odot R_{10}^{-1} \text{ and } h = vecd(T) . \quad (13)$$

Triangular Decoupling: We consider only the lower triangular case. The formula for the upper triangular case can be obtained by minor modification. We suppose that the $m \times m$ matrix T in (10) is a lower triangular form of $T = [t_{ij}]$, $t_{ij} = 0$ for $i < j$. In this case, we can show that

$$vecT(s) = (D \odot E)\, t_l(s) , \quad (14)$$

where $t_l(s)$ is the vector formed by stacking columns of $T(s)$ excluding the upper triangular zero elements. The constant matrices D and E are defined by $D = [D_{mm}\, D_{m(m-1)} \cdots D_{m1}]$ and $E = [E_{mm}\, E_{m(m-1)} \cdots E_{m1}]$, where D_{mk} is the $m \times k$ matrix whose $(m-k+1)$-th row is e_k and the other rows are zeros and $E_{mk} = [0\ I_k]'$ whose size is $m \times k$. Hence, the following holds true:

$$\Psi = (R_{20}^{-1'} \otimes R_{10}^{-1})(D \odot E) = (R_{20}^{-1'} D) \odot (R_{10}^{-1} E),$$
$$h(s) = t_l(s) \quad (15)$$

Block Decoupling: We suppose that T in (10) is a block-diagonal matrix of the form $T = diag\{T_i\}_{i=1}^{k}$, where T_i is an $n_i \times n_i$ matrix and $n_1 + n_2 + \cdots + n_k = m$. In this case, we can show that

$$vecT(s) = (F \odot G) t_b(s) \quad (16)$$

where $t_b(s)$ is the vector formed by stacking columns of $T(s)$ excluding the off-block diagonal zero elements. The constant matrices F and G are defined by $F = diag\{F_i\}_{i=1}^{k}$, $F_i = diag^{(n_i)}\{e_{n_i}\}$, $G = diag\{G_i\}_{i=1}^{k}$, and $G_i = [I_{n_i}\, I_{n_i} \cdots I_{n_i}]$, whose size is $n_i \times n_i^2$. Hence, the following holds true:

$$\Psi = (R_{20}^{-1'} \otimes R_{10}^{-1})(F \odot G) = (R_{20}^{-1'} F) \odot (R_{10}^{-1} G),$$
$$h(s) = t_b(s) \quad (17)$$

4. Solvability Condition of the SPDD

In the previous section, we have shown that the realizability problems associated with the three decoupling constraints are reduced to the SPDD. We now determine the necessary and sufficient condition for the existence of a solution to the SPDD. Let $s_i \in \overline{C}_+$, $i = 1, 2, \cdots, \nu$ be the distinct unstable poles of ϕ or Ψ in (11) and $p_i = \max(p_{\phi i}, p_{\Psi i})$, where $p_{\phi i}$ and $p_{\Psi i}$ are the multiplicities of s_i as a pole of ϕ and Ψ, respectively. Since we treat only real rational matrices, $\overline{s_i}$ is also the pole of ϕ or Ψ . Then ϕ and Ψ are expressed as follows:

$$\phi = \sum_{i=1}^{\nu} \sum_{k=1}^{p_i} \frac{r_i^k}{(s-s_i)^k} + \phi_0(s),$$

$$\Psi = \sum_{i=1}^{\nu} \sum_{k=1}^{p_i} \frac{R_i^k}{(s-s_i)^k} + \Psi_0(s) \tag{18}$$

where $\phi_0(s)$ and $\Psi_0(s)$ are stable. From (11), it follows that

$$\phi_k = \sum_{i=1}^{\nu} \phi_{si} + \phi_0(s) - \Psi_0(s)h(s) \tag{19}$$

where

$$\phi_{si} = \sum_{k=1}^{p_i} \frac{r_i^k - R_i^k h(s)}{(s-s_i)^k} \tag{20}$$

Since $\phi_0(s)$, $\Psi_0(s)$, and $h(s)$ are stable, ϕ_k is stable iff ϕ_{si} is stable for each, $i=1,2,\cdots,\nu$. We now determine the partial fraction expansion of ϕ_{si} at the pole s_i . For ease of presentation, we will consider the case $p_i = 3$. After straightforward calculation, we get the following results:

$$\phi_{si} = \frac{\rho_i^3}{(s-s_i)^3} + \frac{\rho_i^2}{(s-s_i)^2} + \frac{\rho_i^1}{s-s_i} + \phi_{si0}(s) \tag{21}$$

where

$$\rho_i^3 = r_i^3 - R_i^3 h(s_i), \rho_i^2 = -R_i^3 h^{(1)}(s_i) + r_i^2 - R_i^2 h(s_i) \tag{22}$$

$$\rho_i^1 = -(1/2)R_i^3 h^{(2)}(s_i) - R_i^2 h^{(1)}(s_i) + r_i^1 - R_i^1 h(s_i) \tag{23}$$

and ϕ_{si0} is a stable vector. Since ϕ_{si0} is stable, ϕ_{si} is stable iff $\rho_i^k = 0$ for $k = 1,2,3$, which results in the linear equation $\tilde{R}_i \tilde{h}_i = \tilde{r}_i$, with the following:

$$\tilde{R}_i = \begin{bmatrix} R_i^3 & 0 & 0 \\ R_i^2 & R_i^3 & 0 \\ R_i^1 & R_i^2 & R_i^3 \end{bmatrix}, \ \tilde{r}_i = \begin{bmatrix} r_i^3 \\ r_i^2 \\ r_i^1 \end{bmatrix} \text{ and } \tilde{h}_i = \begin{bmatrix} h(s_i) \\ h^{(1)}(s_i) \\ (1/2)h^{(2)}(s_i) \end{bmatrix} \tag{24}$$

Hence, the condition that $\rho_i^k = 0$, $k = 1 \to 3$ is satisfied iff there exists a solution $\tilde{h}_i$ for the equation $\tilde{R}_i \tilde{h}_i = \tilde{r}_i$, and this leads to the condition $rank(\tilde{R}_i) = rank([\tilde{R}_i \vdots \tilde{r}_i])$. We suppose that the above rank condition is met and let $\tilde{\eta}_i = [(\eta_i^0)' \ (\eta_i^1)' \ 1/2(\eta_i^2)']'$ be a solution for $\tilde{h}_i$ such that

$$h(s_i) = \eta_i^0, \ h^{(1)}(s_i) = \eta_i^1 \ \ and \ \ h^{(2)}(s_i) = \eta_i^2. \tag{25}$$

We can show that there exists a stable rational vector $h(s)$ satisfying the above interpolation conditions [16]. Therefore, the equality $rank(\tilde{R}_i) = rank([\tilde{R}_i \vdots \tilde{r}_i])$ for each i is a necessary and sufficient condition for the SPDD to have a solution. We now state the main theorem.

Theorem 1: Let $s_i \in \overline{C}_+$, $i = 1,2,\cdots,\nu$ be the distinct unstable poles of ϕ or Ψ in (11) with multiplicity $p_{\phi i}$ and $p_{\Psi i}$, respectively, and define $p_i = \max(p_{\phi i}, p_{\Psi i})$. The SPDD has a solution iff

$$rank(\tilde{R}_i) = rank([\tilde{R}_i \vdots \tilde{r}_i]) \ \text{ for } \ i = 1,2,\cdots,\nu, \tag{26}$$

where

$$\tilde{R}_i = \begin{bmatrix} R_i^{p_i} & 0 & 0 & \cdots & 0 & 0 \\ R_i^{p_i-1} & R_i^{p_i} & 0 & \ddots & \ddots & 0 \\ \vdots & R_i^{p_i-1} & \ddots & \ddots & \ddots & \vdots \\ R_i^3 & \ddots & \ddots & \ddots & 0 & \vdots \\ R_i^2 & R_i^3 & \ddots & R_i^{p_i-1} & R_i^{p_i} & 0 \\ R_i^1 & R_i^2 & R_i^3 & \cdots & R_i^{p_i-1} & R_i^{p_i} \end{bmatrix}, \ \tilde{r}_i = \begin{bmatrix} r_i^{p_i} \\ r_i^{p_i-1} \\ \vdots \\ r_i^2 \\ r_i^1 \end{bmatrix}. \tag{27}$$

Here, r_i^q and R_i^q are the coefficients of the term $1/(s-s_i)^q$ in partial fraction expansions of ϕ and Ψ , respectively.

To sum up, checking the existence of a decoupling controller is reduced to checking the rank conditions in (26). A realizable decoupling matrix T can be constructed from a stable $h(s)$ satisfying the interpolation constraints as in (25). Note that $\tilde{R}_i$ in (27) is a lower triangular block Toeplitz matrix.

5. Special Cases

5.1 Simple Transmission Zero Case

The results in Theorem 1 completely describe the existence condition of a decoupling controller in the generalized plant model. It seems, however, that checking such existence condition in Theorem 1 may be troublesome due to the calculation of matrix inverses and the inflation of matrix dimensions when the dimension of the plant is high. In the following corollary, we present a simple existence condition that requires neither matrix inverses nor inflation of matrix dimensions for the following special case.

Suppose that R_{10} and R_{20} in (10) have the distinct simple transmission zeros $s_i \in \overline{C}_+$, $i = 1 \to \nu_1$ and $\hat{s}_i \in \overline{C}_+$, $i = 1 \to \nu_2$, respectively, with $s_i \neq \hat{s}_j$ for any i and j . Then there exist nonzero $m \times 1$ vectors ξ_i and μ_i such that $\xi_i^* R_{10}(s_i) = 0$ and $R_{20}(\hat{s}_i)\mu_i = 0$. Let

$\xi_i = [\xi_{i1}\, \xi_{i2} \dots \xi_{im}]'$, $\mu_i = [\mu_{i1}\, \mu_{i2} \dots \mu_{im}]'$, $\xi_i^* T_0(s_i) = \varepsilon_i^* = [\bar{\varepsilon}_{i1}\, \bar{\varepsilon}_{i2} \dots \bar{\varepsilon}_{im}]$, and $T_0(\hat{s}_i)\mu_i = \delta_i = [\delta_{i1}\, \delta_{i2} \dots \delta_{im}]'$.

Corollary 1: For the generalized plant model:

1) A diagonal decoupling controller exists iff $rank(\xi_{ij}) = rank[\xi_{ij}\ \ \varepsilon_{ij}]$ for $j = 1 \to m$, $i = 1 \to \nu_1$ and $rank(\mu_{ij}) = rank[\mu_{ij}\ \ \delta_{ij}]$ for $j = 1 \to m$, $i = 1 \to \nu_2$.
2) A lower triangular decoupling controller exists iff $rank[\xi_{ij}\, \xi_{i(j+1)} \cdots \xi_{im}] = rank[\xi_{ij}\, \xi_{i(j+1)} \cdots \xi_{im}\, \varepsilon_{ij}]$ for $j = 2 \to m$, $i = 1 \to \nu_1$ and $rank[\mu_{i(j-m+1)} \cdots \mu_{i(j-1)}\, \mu_{ij}] = rank[\mu_{i(j-m+1)} \cdots \mu_{i(j-1)}\, \mu_{ij}\, \delta_{ij}]$ for $j = 1 \to m-1$, $i = 1 \to \nu_2$.
3) Consider the block-diagonal decoupling problem of the form $T = diag\{T_i\}_{i=1}^k$, where T_i is an $n_i \times n_i$ matrix. A block diagonal decoupling controller exists iff $rank\, [\xi_{i(q_j+1)}\, \xi_{i(q_j+2)} \cdots \xi_{i(q_j+n_j)}] = rank[\xi_{i(q_j+1)}\, \xi_{i(q_j+2)} \cdots \xi_{i(q_j+n_j)}\, \varepsilon_{i(q_j+l_j)}]$ for $j = 1, 2, \cdots, k$, $q_j = n_0 + n_1 + \cdots + n_{j-1}$, $l_j = 1, 2, \cdots, n_j$, $i = 1 \to \nu_1$ and $rank[\mu_{i(q_j+1)}\, \mu_{i(q_j+2)} \cdots \mu_{i(q_j+n_j)}] = rank[\mu_{i(q_j+1)}\, \mu_{i(q_j+2)} \cdots \mu_{i(q_j+n_j)}\, \delta_{i(q_j+l_j)}]$ for $j = 1, 2, \cdots, k$, $q_j = n_0 + n_1 + \cdots + n_{j-1}$, $l_j = 1, 2, \cdots, n_j$, $i = 1 \to \nu_2$. In the above, $n_0 = 0$ by definition.

Proof: First, we notice that since s_i and $\hat{s}_i$ are simple transmission zeros of R_{10} and R_{20} , respectively, the equality in (26) is reduced to $rank(R_i) = rank([R_i \vdots r_i])$, where r_i and R_i are the residues of ϕ and Ψ at s_i and $\hat{s}_i$, respectively . We can express R_{10}^{-1} and R_{20}^{-1} as follows:

$$R_{10}^{-1} = \sum_{i=1}^{\nu_1} \frac{M_i}{s - s_i} + H_1(s) , \quad R_{20}^{-1} = \sum_{i=1}^{\nu_2} \frac{N_i}{s - \hat{s}_i} + H_2(s) \quad (28)$$

where $H_1(s)$ and $H_2(s)$ are stable matrices. We can show [17] that the residue matrices M_i and N_i are expressed as the following equation:

$$M_i = k_i\, \xi_i^* \quad \text{and} \quad N_i = \mu_i \hat{k}_i \quad (29)$$

where $k_i \in C^m$ and $\hat{k}_i \in C^m$ are nonzero vectors. We prove only for s_i since the proof for $\hat{s}_i$ is similar.

1) Inserting the equalities in (28) into (13) and using (29) yields $r_i = (R_{20}^{-1}(s_i)' \otimes M_i) vec T_0(s_i) = (R_{20}^{-1}(s_i)' \otimes k_i \xi_i^*) vec T_0(s_i) = (R_{20}^{-1}(s_i)' \otimes k_i)(I_m \otimes \xi_i^*) vec T_0(s_i) = (R_{20}^{-1}(s_i)' \otimes k_i) vec(\xi_i^* T_0(s_i)) = (R_{20}^{-1}(s_i)' \otimes k_i)\bar{\varepsilon}_i$ and $R_i = R_{20}^{-1}(s_i) \odot M_i = R_{20}^{-1}(s_i) \odot (k_i \xi_i^*) = (R_{20}^{-1}(s_i)' \otimes k_i)\ (I_m \odot \xi_i^*)$. Since the matrix $R_{20}^{-1}(s_i)' \otimes k_i$ has column rank, the condition $rank(R_i) = rank([R_i \vdots r_i])$ is equivalent to $rank(I_m \odot \xi_j^*) = rank([(I_m \odot \xi_i^*) \vdots \bar{\varepsilon}_i])$. Since $I_m \odot \xi_i^* = diag\{\xi_{ij}\}$, $j = 1 \to m$, we can conclude that the rank condition $rank(R_i) = rank([R_i \vdots r_i])$ is equivalent to $rank(\xi_{ij}) = rank[\xi_{ij}\ \ \varepsilon_{ij}]$ for $j = 1 \to m$.
2) By the similar procedure as in 1), we get $r_i = (R_{20}^{-1}(s_i)' \otimes k_i)\bar{\varepsilon}_i$ and $R_i = (R_{20}^{-1}(s_i)' \otimes k_i)(D_m \odot \xi_i^* E_m)$ from (15). Hence, the condition $rank(R_i) = rank([R_i \vdots r_i])$ is equivalent to $rank(D_m \odot \xi_i^* E_m) = rank([(D_m \odot \xi_i^* E_m) \vdots \bar{\varepsilon}_i])$. In view of the fact that $D_m \odot (\xi_i^* E_m) = diag\{[\bar{\xi}_{i1}\, \bar{\xi}_{i2} \cdots \bar{\xi}_{im}],\ [\bar{\xi}_{i2}\, \bar{\xi}_{i3} \cdots \bar{\xi}_{im}], \cdots, [\bar{\xi}_{i(m-1)}\, \bar{\xi}_{im}], \bar{\xi}_{im}]\}$, the above rank condition is equivalent to the one in 2).
3) By the similar procedure as in 1), we get $r_i = (R_{20}^{-1}(s_i)' \otimes k_i)\bar{\varepsilon}_i$ and $R_i = (R_{20}^{-1}(s_i)' \otimes k_i)(F \odot (\xi_i^* G))$ from (17). Hence, the condition $rank(R_i) = rank([R_i \vdots r_i])$ is equivalent to $rank(F \odot (\xi_i^* G)) = rank([(F \odot (\xi_i^* G) \vdots \bar{\varepsilon}_i])$. In view of the fact that $F \odot (\xi_i^* G) = diag\{Z_j\}_{j=1}^k$, $Z_j = diag^{(n_j)}\{[\bar{\xi}_{i(q_j+1)}\, \bar{\xi}_{i(q_j+2)} \cdots \bar{\xi}_{i(q_j+n_j)}]\}$, the above rank condition is equivalent to the one in 3).

5.2 Square Plant with 1DOF Controller Case

The plant model in (1) is sufficiently general to include the cases of non-square plants, non-unity feedback, 1DOF, and 2DOF controller configurations. When we make some assumptions on the structure of the transfer matrices of $P(s)$, we obtain more specific results. We suppose that $P_{00} = 0$, $m = m_1$ (square plant case), and $P_{20} = I$ (1DOF case). In this case, we can show from (5) that a stable rational matrix $T(s)$ is realizable iff $P_{02}^{-1}T$, $P_{02}^{-1}TP_{22}$, $P_{22}P_{02}^{-1}T$, and $(I + P_{22}P_{02}^{-1}T)P_{22}$ are stable and $\det(I + P_{22}P_{02}^{-1}T) \not\equiv 0$. These five matrices can be compactly described by the following:

$$[\ 0_{2m\times m}\ \ \hat{P}_{22}\ ] + \tilde{P}_{22}P_{02}^{-1}T[\ I_m\ \ \ P_{22}\ ] =: \Phi_s \quad (30)$$

where

$$\hat{P}_{22} = [\,0_{m\times m}\ \ P_{22}'\,]' \quad \text{and} \quad \tilde{P}_{22} = [\,I_m\ \ P_{22}'\,]' \quad (31)$$

Hence, $T(s)$ is realizable iff Φ_s is stable, which leads to the SPDD in (11) with the following:

$$\phi = vec[\ 0_{2m\times m}\ \ \hat{P}_{22}\ ], \Psi = [\,I_m\ \ P_{22}\,]' \otimes (\tilde{P}_{22}P_{02}^{-1}) \ \text{ and }$$
$$h = vec\,T \quad (32)$$

The existence condition of various decoupling controllers for this special case can be checked by the procedures in section 3; notice that in this case, we do not need coprime factorizations.

When a diagonal decoupling T is sought, we can further parameterize it. From (30), T is realizable iff $\tilde{P}_{22}P_{02}^{-1}T$ and $\hat{P}_{22} + \tilde{P}_{22}P_{02}^{-1}T\,P_{22}$ are stable. For the term $\tilde{P}_{22}P_{02}^{-1}T$ to be stable, T must be of the form $T = \Delta_\theta \Delta$, where Δ is an arbitrary diagonal stable matrix and $\Delta_\theta = diag\{\theta_i\}_{i=1}^m$ with θ_i being the monic polynomial of the minimal degree such that { $i-$column of $\tilde{P}_{22}P_{02}^{-1}$ } $\times \theta_i$ is stable. Hence, T is realizable iff $\hat{P}_{22} + \tilde{P}_{22}P_{02}^{-1}\Delta_\theta \Delta\, P_{22}$ is stable, which leads to the SPDD with the following:

$$\phi = vec(\hat{P}_{22}) , \quad \Psi = (-P_{22}') \odot (\tilde{P}_{22}P_{02}^{-1}\Delta_\theta) \quad (33)$$

$$h(s) = vecd(\Delta(s)) \quad (34)$$

6. Example

Consider the case of the 1DOF controller configuration with the square plant $P_a(s)$ and the non-unity feedback sensor F. In this case the transfer matrices in (1) are given by this equation:

$$P_{00}=0,\ P_{02}=P_a(s),\ P_{20}=I,\ and\ \ P_{22}=-FP_a(s) \qquad (35)$$

Consider the following plant [8] and the non-unity feedback:

$$P_a=\begin{bmatrix}\frac{s-1}{s+2} & \frac{s-1}{s+2}\\ \frac{s+2}{s-1} & \frac{2(s+2)}{s-1}\end{bmatrix} \quad \text{and} \quad F=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} \qquad (36)$$

We can use the formulas in (33) and (34). Since

$$P_a^{-1}=\begin{bmatrix}\frac{2(s+2)}{s-1} & \frac{1-s}{s+2}\\ \frac{s+2}{1-s} & \frac{s-1}{s+2}\end{bmatrix}, \qquad (37)$$

we obtain $\Delta_\theta = diag\{s-1,1\}$ after simple calculations. The vector $\phi(s)$ and the matrix $\Psi(s)$ have a simple pole at $s_1=1$. The residue values at $s_1=1$ are obtained as follows:

$$r_1=[\,0\ 0\ -3\ -3\ 0\ 0\ -6\ \ -6\,]' \qquad (38)$$

$$R_1=\begin{bmatrix}18 & -9 & 0 & 0 & 36 & -18 & 0 & 0\\ 0 & 0 & -3 & -3 & 0 & 0 & -6 & -6\end{bmatrix}' \qquad (39)$$

Since $rank(R_1)=rank([R_1 \vdots r_1])=2,$ a diagonal decoupling solution exists. Taking a solution for the equation $R_1 h(s_1)=r_1$ as $h(1)=[\,0\ 1\,]'$, then a diagonal solution for $T(s)$ is parameterized as follows:

$$T(s)=\begin{bmatrix}(s-1)^2 h_a & 0\\ 0 & 1+(s-1)h_b\end{bmatrix} \qquad (40)$$

where h_a and h_b are arbitrary stable rational functions. The controller C(s) in this case can be obtained from (3).

7. Conclusion

The conditions for the existence of diagonal, block-diagonal, and triangular decoupling controllers are obtained for the generalized plant model. These decoupling problems can be transformed into a solvable standard form SPDD, and procedures to obtain solutions for SPDD by solving interpolation problems are explained. The existence condition of a solution for SPDD is described in terms of the rank condition on a block Toeplitz matrix whose elements are the coefficient matrices in partial fraction expansions.

Possible future research works include the characterization of all solutions $h(s)$ in SPDD, extending the results of this paper to the case where the dimensions of r and v are different, and investigating the algebraic properties of lower-triangular block Toeplitz matrices to treat the sensitivity problem in checking the rank condition in (26).

Acknowledgments

This work was supported by the National Research Foundation of Korea under contract number 313-2008-2-D00325.

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