JACOBI OPERATORS ALONG
THE STRUCTURE FLOW ON REAL HYPERSURFACES
IN A NONFLAT COMPLEX SPACE FORM II

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ABSTRACT. Let $M$ be a real hypersurface of a complex space form with almost contact metric structure $(\phi, \xi, \eta, g)$. In this paper, we study real hypersurfaces in a complex space form whose structure Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ is $\xi$-parallel. In particular, we prove that the condition $\nabla_\xi R_\xi = 0$ characterizes the homogeneous real hypersurfaces of type $A$ in a complex projective space or a complex hyperbolic space when $R_\xi \phi S = R_\xi S \phi$ holds on $M$, where $S$ denotes the Ricci tensor of type $(1,1)$ on $M$.

1. Introduction

Let $(M_n(c), J, \tilde{g})$ be a complex $n$-dimensional complex space form with Kähler structure $(J, \tilde{g})$ of constant holomorphic sectional curvature $4c$ and let $M$ be an orientable real hypersurface in $M_n(c)$. Then $M$ has an almost contact metric structure $(\phi, \xi, \eta, g)$ induced from $(J, \tilde{g})$.

Typical examples of real hypersurfaces in $M_n(c), c \neq 0$, are homogeneous ones. Takagi [12], [13] classified homogeneous real hypersurfaces of a complex projective space $P_n \mathbb{C}$ into six model spaces. On the other hand, Cecil and Ryan [2] extensively studied a Hopf hypersurface, which is realized as tubes over certain submanifolds in $P_n \mathbb{C}$, by using its focal map. By making use of those results and the mentioned work of Takagi, Kimura [7] proved the local classification theorem for Hopf hypersurfaces of $P_n \mathbb{C}$ all of whose principal curvatures are constants. For the case where a complex hyperbolic space $H_n \mathbb{C}$, Berndt [1] proved the classification theorem for Hopf hypersurfaces whose all principal curvatures are constants. Among the several types of real hypersurfaces appeared in model spaces ([1], [12]), a particular type of tubes over totally geodesic $P_k \mathbb{C}$ or $H_k \mathbb{C}$ $(0 \leq k \leq n - 1)$ adding a horosphere in $H_n \mathbb{C}$, which is called type $A$, has a lot of nice geometric properties. For example, Okumura [9] (resp. Montiel and Romero [8]) showed that a real hypersurface in $P_n \mathbb{C}$ (resp.
$H_n \mathbb{C}$) is locally congruent to one of real hypersurfaces of type $A$ if and only if the Reeb flow $\xi$ is isometric or equivalently the structure operator $\phi$ commutes with the shape operator $H$.

The structure Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ has a fundamental role in contact geometry. Ortega, P\'erez and Santos [10] have proved that there are no real hypersurfaces in $P_n \mathbb{C}, n \geq 3$ with parallel structure Jacobi operator $\nabla R_\xi = 0$. More generally, such a result has been extended by [11] due to them. In this paper, motivated by results mentioned above we consider the parallelism of the structure Jacobi operator $R_\xi$ in the direction of the structure vector field, that is $\nabla_\xi R_\xi = 0$. Recently we have some classification theorems for real hypersurfaces in a non-flat complex space form with respect to the parallelism of $R_\xi$ (cf. [3], [4], [5]).

The main purpose of this paper is to classify real hypersurfaces in a non-flat complex space form $M_n(c)$ which satisfies $\nabla_\xi R_\xi = 0$ and at the same time $R_\xi \phi S = R_\xi S \phi$, where $S$ denotes the Ricci tensor of the hypersurface. Our main result is as follows:

**Main Theorem.** Let $M$ be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$ which satisfies $\nabla_\xi R_\xi = 0$. Then $M$ holds $R_\xi \phi S = R_\xi S \phi$ if and only if $H\xi = 0$ or $M$ is locally congruent to one of the following:

(I) in case that $M_n(c) = P_n \mathbb{C}$,

(A1) a geodesic hypersphere of radius $r$, where $0 < r < \pi/2$ and $r \neq \pi/4$,

(A2) a tube of radius $r$ over a totally geodesic $P_k \mathbb{C}$ for some $k \in \{1, \ldots, n-2\}$, where $0 < r < \pi/2$ and $r \neq \pi/4$;

(II) in case that $M_n(c) = H_n \mathbb{C}$,

(A0) a horosphere,

(A1) a geodesic hypersphere or a tube over a complex hyperbolic hyper-plane $H_{n-1} \mathbb{C}$,

(A2) a tube over a totally geodesic $H_k \mathbb{C}$ for some $k \in \{1, \ldots, n-2\}$.

We note that every Hopf hypersurface always satisfies condition $R_\xi \phi S = R_\xi S \phi$.

All manifolds in this paper are assumed to be connected and of class $C^\infty$ and the real hypersurfaces are supposed to be oriented.

2. Preliminaries

We denote by $M_n(c)$, $c \neq 0$, be a non-flat complex space form with the Fubini-Study metric $\tilde{g}$ of constant holomorphic sectional curvature $4c$ and Levi-Civita connection $\tilde{\nabla}$. For an immersed $(2n - 1)$-dimensional Riemannian manifold $\tau : M \rightarrow M_n(c)$, the Levi-Civita connection $\nabla$ of induced metric and the shape operator $H$ of the immersion are characterized

$$\tilde{\nabla}_X Y = \nabla_X Y + g(HX, Y)\nu, \quad \tilde{\nabla}_X \nu = -HX$$
for any vector fields $X$ and $Y$ on $M$, where $g$ denotes the Riemannian metric of $M$ induced from $\tilde{g}$ and $\nu$ a unit normal vector on $M$. In the sequel the indices $i, j, k, l, \ldots$ run over the range $\{1, 2, \ldots, 2n - 1\}$ unless otherwise stated. For a local orthonormal frame field $\{e_i\}$ of $M$, we denote the dual 1-forms by $\{\theta_i\}$. The connection forms $\theta_{ij}$ are defined by

$$d\theta_i + \sum_j \theta_{ij} \wedge \theta_j = 0, \quad \theta_{ij} + \theta_{ji} = 0.$$  

Then we have

$$\nabla_{e_i} e_j = \sum_k \theta_{kij}(e_i)e_k = \sum_k \Gamma_{kij}e_k,$$

where we put $\theta_{ij} = \sum_k \theta_{kij}$. The almost contact metric structure ($\phi = (\phi_{ij}), \xi = \sum_i \xi_i e_i$) is induced on $M$ by following equation:

$$J(e_i) = \sum_j \phi_{ji}e_j + \xi_i \nu.$$  

The structure tensor $\phi$ and the structure vector $\xi$ satisfy

$$\sum_k \phi_{ik} \phi_{kj} = \xi_i \xi_j - \delta_{ij}, \quad \sum_j \xi_j \phi_{ij} = 0, \quad \sum_i \xi_i^2 = 1, \quad \phi_{ij} + \phi_{ji} = 0,$$

$$d\phi_{ij} = \sum_k (\phi_{ik} \theta_{kj} - \phi_{jk} \theta_{ki} - \xi_i h_{jk} \theta_k + \xi_j h_{ik} \theta_k),$$

$$d\xi_i = \sum_j \xi_j \theta_{ji} - \sum_{j,k} \phi_{ji} h_{jk} \theta_k.$$  

We denote the components of the shape operator or the second fundamental tensor $H$ of $M$ by $h_{ij}$. The components $h_{ij,k}$ of the covariant derivative of $H$ are given by $\sum_k h_{ij,k} \theta_k = dh_{ij} - \sum_k h_{ik} \theta_{kj} - \sum_k h_{jk} \theta_{ki}$. Then we have the equation of Gauss and Codazzi

$$R_{ijkl} = c(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) + \phi_{ik} \phi_{jl} - \phi_{jl} \phi_{ik} + 2 \phi_{ij} \phi_{kl}) + h_{ik} h_{jl} - h_{il} h_{jk},$$

$$h_{ij,k} - h_{ik,j} = c(\xi_k \phi_{ij} + \xi_i \phi_{kj} - \xi_j \phi_{ik} - \xi_k \phi_{ij}),$$

respectively.

From (2.2) the structure Jacobi operator $R_{ij} = (\Xi_{ij})$ is given by

$$\Xi_{ij} = \sum_{k,l} h_{ik} h_{jl} \xi_k \xi_l - \sum_{k,l} h_{ij} h_{kl} \xi_k \xi_l + c \xi_i \xi_j - c \delta_{ij},$$

From (2.2) the Ricci tensor $S = (S_{ij})$ is given by

$$S_{ij} = (2n + 1) c \delta_{ij} - 3c \xi_i \xi_j + h_{ij} - \sum_k h_{ik} h_{kj},$$

where $h = \sum_i h_{ii}$.

First we remark:
Lemma 1 ([5]). Let $U$ be an open set in $M$ and $F$ a smooth function on $U$. We put $dF = \sum_i F_i \theta_i$. Then we have

$$F_{ij} - F_{ji} = \sum_k F_k \Gamma_{kij} - \sum_k F_k \Gamma_{kji}.$$ 

Now we retake a local orthonormal frame field $e_i$ in such a way that

- $e_1 = \xi$,
- $e_2$ is in the direction of $\sum_{i=2}^{2n-1} h_{1i} e_i$,
- $e_3 = \phi e_2$.

Then we have

$$(2.6) \quad \xi_1 = 1, \quad \xi_i = 0 \ (i \geq 2), \quad h_{1i} = 0 \ (j \geq 3) \quad \text{and} \quad \phi_{32} = 1.$$ 

We put $\alpha := h_{11}, \beta := h_{12}, \gamma := h_{22}, \varepsilon := h_{23}$ and $\delta := h_{33}$.

Hereafter the indices $p, q, r, s, \ldots$ run over the range $\{4, 5, \ldots, 2n - 1\}$ unless otherwise stated.

Since $d\xi_i = 0$, we have

$$(2.7) \quad \theta_{12} = \varepsilon \theta_2 + \delta \theta_3 + \sum_p h_{3p} \theta_p,$$

$$(2.8) \quad \theta_{13} = -\beta \theta_1 - \gamma \theta_2 - \varepsilon \theta_3 - \sum_p h_{2p} \theta_p,$$

$$\theta_{1p} = \sum_q \phi_{qp} h_{q2} \theta_2 + \sum_q \phi_{qp} h_{q3} \theta_3 + \sum_{q,r} \phi_{qp} h_{qr} \theta_r.$$ 

We put

$$(2.9) \quad \theta_{23} = \sum_i X_i \theta_i, \quad \theta_{2p} = \sum_i Y_{pi} \theta_i, \quad \theta_{3p} = \sum_i Z_{pi} \theta_i.$$ 

Then it follows from $d\phi_{2i} = 0$ that $Y_{pi} = -\sum_q \phi_{pq} Z_{qi}$ or $Z_{pi} = \sum_q \phi_{pq} Y_{qi}$.

The equations (2.4) and (2.5) are rewritten as

$$(2.10) \quad S_{ij} = hh_{ij} - \sum_k h_{ik} h_{jk} - 3c \delta_{ij} \delta_{j1} + (2n + 1)c \delta_{ij},$$

respectively.

3. Real hypersurfaces satisfying $\nabla_\xi R_\xi = 0$ and $R_\xi \phi S = R_\xi S \phi$

First we assume that $\nabla_\xi R_\xi = 0$. The components $\Xi_{ij;k}$ of the covariant derivatiation of $R_\xi = (\Xi_{ij})$ is given by

$$\sum_k \Xi_{ij;k} \theta_k = d \Xi_{ij} - \sum_k \Xi_{kj} \theta_k - \sum_k \Xi_{ik} \theta_k.$$
Substituting (2.9) into the above equation we have
\[
\sum_k \xi_{ij,k} \theta_k = - (da)_{ij} - ac_{ij} + (db_{11})_{ij} + c_{ij}
\]
(3.1)
\[
+ \alpha \sum_k h_{kj} \theta_{ki} - \alpha h_{ij} \theta_{ki} - \beta h_{ij} \theta_{ki} - c \delta_{ij} \theta_{ki}
\]
\[
+ \alpha \sum_k h_{ik} \theta_{kj} - \alpha h_{ij} \theta_{kj} - \beta h_{ij} \theta_{kj} - c \delta_{ij} \theta_{kj}.
\]

In the following, we assume that \( \beta \neq 0 \).

Our assumption \( \nabla_{\xi} R_{\xi} = 0 \) is equivalent to \( \xi_{ij,1} = 0 \), which can be stated as follows:

(3.2) \( \varepsilon = 0, \quad \alpha \delta + c = 0, \quad h_{3p} = 0, \)
(3.3) \( (\beta^2 - \alpha \gamma)_1 - 2 \alpha \sum_p h_{2p} Y_{p1} = 0, \)
(3.4) \( (\beta^2 - \alpha \gamma - c) X_1 + \alpha \sum_p h_{2p} Z_{p1} = 0, \)
(3.5) \( (\alpha h_{2p})_1 + \alpha \sum_q h_{pq} Y_{q1} + (\beta^2 - \alpha \gamma) Y_{p1} - \alpha \sum_q h_{2q} \Gamma_{qp1} = 0, \)
(3.6) \( \alpha h_{2p} X_1 - \sum_q (\alpha h_{qp} + c \delta_{pq}) Z_{q1} = 0, \)
(3.7) \( (\alpha h_{pq})_1 - \alpha h_{2q} Y_{p1} - \alpha \sum_r h_{rq} \Gamma_{rp1} - \alpha h_{2p} Y_{q1} - \alpha \sum_r h_{pr} \Gamma_{rq1} = 0. \)

Hereafter we shall use (3.2) without quoting.

Furthermore we assume that \( R_{\xi} \phi S = R_{\xi} S \phi \). Under the assumption \( \nabla_{\xi} R_{\xi} = 0 \), we have the following additional equations

(3.8) \( \beta^2 - \alpha \gamma - c = 0, \)
(3.9) \( \tilde{R}_{\xi} \tilde{\phi} A = 0, \)
(3.10) \( \tilde{R}_{\xi} \tilde{S} \tilde{\phi} = \tilde{R}_{\xi} S \tilde{\phi}, \)

where \( A = (h_{24}, h_{25}, \ldots, h_{22n-1}) \), \( \tilde{R}_{\xi} = (\tilde{\xi}_{pq}), \quad \tilde{\phi} = (\phi_{pq}), \quad \tilde{S} = (S_{pq}). \) From (3.2) and (3.8) we note

(3.11) \( \delta \neq \gamma. \)

Now, properly speaking, we denote the equation (2.3) by \( (ijk) \) simply. Then we have the following equations (112)–(132).

(112) \( \alpha_2 - \beta_1 = 0, \)
(212) \( \beta_2 - \gamma_1 - 2 \sum_p h_{2p} Y_{p1} = 0, \)
\[(\alpha - \delta) \gamma - \beta X_2 + (\gamma - \delta) X_1 - \beta^2 - \sum_p h_{2p} Z_{p1} = -c,\]

\[(\alpha_3 + 3\beta \delta - \alpha \beta + \beta X_1 = 0,\]

\[(\beta_3 + \gamma \delta + (\gamma - \delta) X_1 - \beta^2 - \sum_p h_{2p} Z_{p1} = 0,\]

\[\beta X_3 + \delta_1 = 0,\]

\[\gamma_3 - 2\beta \delta + 2 \sum_p h_{2p} Y_{p3} + (\gamma - \delta) X_2 - \beta \gamma - \sum_p h_{2p} Z_{p2} = 0,\]

\[\sum_p h_{2p} Z_{p3} - \delta_2 - (\gamma - \delta) X_3 = 0,\]

\[\alpha_p + \beta Y_{p1} = 0,\]

\[\beta_p + 2 \sum_{q,r} h_{2q} \phi_{rq} h_{rp} + \beta Y_{p2} + \alpha \sum_q \phi_{qp} h_{2q} = 0,\]

\[2\delta h_{2p} - \beta Y_{p3} - \alpha h_{2p} + \beta X_p = 0,\]

\[\gamma_p + 2 \sum_q h_{2q} Y_{qp} - h_{2p2} - \sum_q h_{qp} Y_{q2} + \beta \sum_q \phi_{qp} h_{2q} + \gamma Y_{p2}\]
\[+ \sum_q h_{2q} \Gamma_{qp2} = 0,\]

\[\delta X_p + \beta h_{2p} - \gamma X_p + \sum_q h_{2q} Z_{qp} - h_{2p3} - \sum_q h_{qp} Y_{q3}\]
\[+ \gamma Y_{p3} + \sum_q h_{2q} \Gamma_{qp3} = 0,\]

\[\delta_p + h_{2p} X_3 - \sum_q h_{qp} Z_{q3} + \delta Z_{p3} = 0,\]

\[\beta_p + \sum_{q,r} h_{2q} \phi_{rq} h_{rp} - h_{2p1} - \sum_q h_{qp} Y_{q1} + \gamma Y_{p1} + \sum_q h_{2q} \Gamma_{qp1} = 0,\]

\[\delta h_{2p} - \alpha h_{2p} + \beta X_p - h_{2p} X_1 + \sum_q h_{qp} Z_{q1} - \delta Z_{p1} = 0,\]

\[\delta X_p + \beta h_{2p} - \gamma X_p + \sum_q h_{2q} Z_{qp} + h_{2p} X_2 - \sum_q h_{pq} Z_{q2} + \delta Z_{p2} = 0,\]

\[2 \sum_{r,s} h_{rp} \phi_{sr} h_{sq} - \alpha \sum_r \phi_{rp} h_{r} + \alpha \sum_r \phi_{rq} h_{rp}\]
\[- \beta Y_{pq} + \beta Y_{ap} = -2c \phi_{pq},\]

\[h_{2pq} + \sum_r h_{rp} Y_{rq} - \beta \sum_r \phi_{rp} h_{rq} - \gamma Y_{pq} - \sum_r h_{2r} \Gamma_{rpq} - h_{2qp}\]
\[ -\sum_{r} h_{rq} Y_{rp} + \beta \sum_{r} \phi_{r} q h_{rp} + \gamma Y_{qp} + \sum_{r} h_{2r} \Gamma_{rqp} = 0, \]

\((q1p)\)

\[ \sum_{r,s} h_{rq} \phi_{sr} h_{sp} - \alpha \sum_{r} \phi_{r} h_{rp} - \beta Y_{qp} - h_{pq1} + h_{2q} Y_{p1} + \sum_{r} h_{rq} \Gamma_{r} + h_{2p} Y_{q1} + \sum_{r} h_{rp} \Gamma_{r} = c\phi_{pq}, \]

\((q3p)\)

\[ h_{3q} - \delta Z_{qp} - \sum_{r} h_{3r} \Gamma_{rqp} - h_{2q} X_{p} + \sum_{r} h_{qr} Z_{rp} - h_{qp3} + h_{q2} Y_{p3} + h_{q3} Z_{p3} + \sum_{r} h_{qr} \Gamma_{rp3} + h_{p2} Y_{q3} + h_{p3} Z_{q3} + \sum_{r} h_{pr} \Gamma_{r} = 0. \]

Remark 1. We did not write \((p2q), (3pq)\) and \((pqr)\) since we need not use them.

4. Formulas and lemmas

In this section we study the crucial case where \(\beta \neq 0\). By (3.6) and (31p) we have

\[(4.1) \quad \beta X_{p} = (\alpha - \delta) h_{2p}.\]

This and (13p) imply that

\[(4.2) \quad \beta Y_{p3} = \delta h_{2p}.\]

The equation (3.9) can be rewritten as

\[(4.3) \quad \sum_{q,r} (\alpha h_{pq} + c\delta_{pq}) \phi_{qr} h_{r2} = 0,\]

which, together with (4.2), implies

\[ \beta \sum_{q} (h_{pq} - \delta_{pq}) Z_{q3} = \delta \sum_{q,r} (h_{pq} - \delta_{pq}) \phi_{qr} Y_{r3} = 0.\]

Hence it follows from (33p) and (1p1) that

\[(4.4) \quad \delta_{p} = -h_{2p} X_{3} \text{ and } \alpha_{p} = -\beta Y_{p1}.\]

Thus since (4.4) and \(\alpha_{p} \delta + \alpha \delta_{p} = 0\) obtained from (3.2) we have

\[(4.5) \quad \beta \delta Y_{p1} = -\alpha h_{2p} X_{3},\]

and so \(\sum_{p} h_{2p} Z_{p1} = 0\). By (4.2), we have

\[(4.6) \quad \sum_{p} h_{2p} Z_{p3} = \sum_{p,q} h_{2p} \phi_{pq} Y_{q3} = \frac{\delta}{\beta} \sum_{p,q} h_{2p} \phi_{pq} h_{2q} = 0.\]
From (3.6), (4.3) and (4.5) we have
\[ h_{2p} X_1 = 0. \] (4.7)

Now we shall prove the following key lemma.

**Lemma 2.** \( H(e_2) \in \text{span}\{e_1, e_2\} \).

**Proof.** Suppose that \( h_{2p} \neq 0 \). Then from (4.7) we have \( X_1 = 0 \). We can select the vector \( e_4 \) so that \( h_{24} \neq 0 \) and \( h_{25} = \cdots = h_{2,2n-1} = 0 \). We put \( e_5 := \phi e_4 \) and \( \rho := h_{24}(\neq 0) \). Note that \( \phi_{54} = 1 \). Then by (4.3) we have
\[ h_{55} = \delta, \quad h_{p5} = 0 \quad (p \neq 5). \]

Put \( p = 5 \) in (32p). Then since above equation and (4.1) we have \( X_5 = 0 \) and so \( Z_{45} = 0 \). Thus we have \( Y_{55} = 0 \). Furthermore, put \( p = q = 5 \) in (q1p). Then, since \( T_{551} = Y_{55} = 0 \), we have
\[ \alpha_1 = \delta_1 = 0. \] (4.8)

Thus, from (313), (323), (4.6) and (112) we have
\[ X_3 = 0, \] (4.9)
\[ \alpha_2 = \delta_2 = 0, \] (4.10)
\[ \beta_1 = 0. \] (4.11)

By (4.4) and (4.9) we have \( \alpha_p = \delta_p = 0 \). Thus it follows from (1p1) that
\[ \alpha_p = \delta_p = Y_{p1} = Z_{p1} = 0. \] (4.12)

Now we put \( F = \alpha, \ i = 1 \) and \( j = p \) in Lemma 1. Then, from (2.7), (4.8), (4.10) and (4.12) we have
\[ 0 = \alpha_{1p} - \alpha_{p1} = \sum_k \alpha_k I'_{k1p} - \sum_k \alpha_k I'_{kp1} = \alpha_3 (I'_{31p} - I'_{3p1}) = \alpha_3 h_{2p}. \]
Thus we have \( \alpha_3 = 0 \). Hence it follows from (4.8), (4.10) and (4.12) that \( \alpha \) and \( \delta \) are constants, which, together with (113), imply
\[ \alpha = 3\delta. \] (4.13)

On the other hand, taking account of the coefficient of \( \theta_1 \wedge \theta_3 \) in the exterior derivative of \( \theta_{23} \), we have \( X_2 = -2\beta \). Thus, from (312) and (4.13) we have
\[ 2\delta \gamma + \beta^2 = -c. \]
Hence it follows from (3.8) that \( \beta \) and \( \gamma \) are constants. From (3.5) and (4.12) we have
\[ h_{2p1} = \sum_q h_{2p} I'_{qp1} = 0. \]
This, together with (21p), implies \( \rho \delta = 0 \), which is a contradiction. \( \square \)

**Remark 2.** Lemma 2 has been already proved in [5]. However in the case of our condition by using the equation (3.8) we have a short proof.
Owing to Lemma 2 the matrix \( (h_{pq}) \) is diagonalizable, that is, for a suitable choice of an orthonormal frame field \( \{e_i\} \) we can set
\[
h_{pq} = \lambda_p \delta_{pq}.
\]
Then it is easy to see
\[
\tilde{R}_\xi = -((\alpha \lambda_p + c) \delta_{pq}),
\]
\[
\tilde{S} = ((h \lambda_p - (\lambda_p)^2 + (2n + 1)c) \delta_{pq}).
\]
Here we shall sum up all equations obtained from Lemma 2.
From (4.1), (4.2) and (4.4) we have
\[
X_p = Y_{p1} = Z_{p1} = Y_{p3} = Z_{p3} = 0, \quad \alpha_p = \delta_p = 0.
\]
Put \( p = q \) in (3.7). Then we have
\[
(\alpha \lambda_p)_{1} = 0.
\]
Moreover, from (112)–(32p) we have
\[
\begin{align*}
(4.17) & \quad \alpha_2 - \beta_1 = 0, \\
(4.18) & \quad \beta_2 - \gamma_1 = 0, \\
(4.19) & \quad \gamma \delta + \beta X_2 - (\gamma - \delta)X_1 = 0, \\
(4.20) & \quad \alpha_3 + 3 \beta \delta - \alpha \beta + \beta X_1 = 0, \\
(4.21) & \quad \beta_3 - \alpha \delta + \gamma \delta + (\gamma - \delta)X_1 - \beta^2 = c, \\
(4.22) & \quad \delta_1 + \beta X_3 = 0, \\
(4.23) & \quad \gamma_3 - 2 \beta \delta + (\gamma - \delta)X_2 - \beta \gamma = 0, \\
(4.24) & \quad \delta_2 + (\gamma - \delta)X_3 = 0, \\
(4.25) & \quad \beta_p = 0, \\
(4.26) & \quad Y_{p2} = 0, \quad Z_{p2} = 0, \\
(4.27) & \quad \gamma_p = 0.
\end{align*}
\]
It follows from (q1p) and (3.7) that
\[
(4.28) \quad \alpha \beta Y_{qp} = \alpha \lambda_p \lambda_q \phi_{pq} - \alpha^2 \lambda_p \phi_{pq} + \alpha_1 \lambda_p \delta_{pq} - c \alpha \phi_{pq}.
\]
From this, (2pq) and (q3p) we have
\[
\begin{align*}
(4.29) & \quad \beta^2 (\lambda_p + \lambda_q) \phi_{pq} - (\lambda_p - \gamma)(\lambda_p \lambda_q - \alpha \lambda_q - c) \phi_{pq} \\
& \quad - (\lambda_q - \gamma) (\lambda_p \lambda_q - \alpha \lambda_p - c) \phi_{pq} = 0, \\
(4.30) & \quad (\lambda_q - \delta)(\alpha ((\lambda_q)^2 - \alpha \lambda_q - c) \delta_{pq} + \alpha_1 \lambda_q \phi_{pq}) - \alpha \beta (h_{q3} + (\lambda_p - \lambda_q) T_{q3}) = 0.
\end{align*}
\]
If \( p = q \) in above equation, then we have
\[
(4.31) \quad (\lambda_p - \delta)((\lambda_p)^2 - \alpha \lambda_p - c) - \beta (\lambda_p)_3 = 0.
\]
5. Proof of Main Theorem

In this section we prove Main Theorem.
We first prove $\beta = 0$. Suppose that $\beta \neq 0$. It follows from (4.22) that (3.10) is equivalent to
$$\rho_p(\sigma_q - \sigma_p)\phi_{pq} = 0,$$
where $\rho_p = \alpha \lambda_p + c$, $\sigma_p = h\lambda_p - (\lambda_p)^2 + (2n + 1)c$. Therefore if $\phi_{pq} \neq 0$, then we have
$$\lambda_p = \lambda_q.$$

Here we assert that if $\phi_{pq} \neq 0$, then $p = q$. To prove this, we assume that there exist indices $p$ and $q$ such that
$$\phi_{pq} \neq 0, \quad \lambda_p - \lambda_q \neq 0.$$

First we prepare three lemmas.

**Lemma 3.** $\alpha \alpha_1 - (\alpha \gamma)_1 = 0$.

**Proof.** From (5.1) we have
$$(\alpha \lambda_p + c)(h - \lambda_p - \lambda_q) = 0,$$
$$(\alpha \lambda_q + c)(h - \lambda_p - \lambda_q) = 0.$$ Thus if $h - \lambda_p - \lambda_q \neq 0$, then we have $\lambda_p = \lambda_q = 0$, which is a contradiction. Hence we have $\alpha h - \alpha \lambda_p - \alpha \lambda_q = 0$. Taking account of the coefficient of $\theta_1$ in the exterior derivative of this, it follows from (4.16) that
$$\alpha h_1 = 0. $$

From (4.26) we have $(\alpha \sum_p \lambda_p)_1 = 0$. Combining this equation with $h = \alpha + \gamma + \delta + \sum_p \lambda_p$, we have
$$(\alpha(h - \alpha - \gamma - \delta))_1 = 0.$$ Eliminating $h$ from this Lemma 3 follows from (5.2).

**Lemma 4.** $\alpha_1 \neq 0$ and $\alpha + \gamma - \delta = 0$.

**Proof.** From (4.24) we have $2\beta \beta_1 - (\alpha \gamma)_1 = 0$. Hence it follows from Lemma 4 that
$$\alpha \alpha_1 + \beta \beta_1 = 0.$$ On the other hand, by (4.32) and (4.34) we have $(\gamma - \delta)\delta_1 - \beta \delta_2 = 0$, and therefore $(\gamma - \delta)\alpha_1 - \beta \alpha_2 = 0$. Thus we have
$$\alpha_1 = 0.$$ Suppose that
$$(\alpha + \gamma - \delta)\alpha_1 = 0 \quad \text{and} \quad (\alpha + \gamma - \delta)\beta_1 = 0.$$ It follows from (3.8) that
$$\beta^2 - \alpha \gamma - c = 2\beta \beta_3 - \gamma \alpha_3 - \alpha \gamma_3 = 0.$$
From (4.19)–(4.21), (4.23) and (3.8) we have the following:

\[ \delta \gamma + \beta X_2 - (\gamma - \delta)X_1 = 0, \]
\[ \alpha_3 + 3\beta\delta - \alpha\beta + \beta X_1 = 0, \]
\[ \beta_3 + (\gamma - \delta)X_1 + \gamma\delta - \alpha\gamma - c = 0, \]
\[ \gamma_3 - 2\beta\delta + (\gamma - \delta)X_2 + \beta\gamma = 0. \]

Substituting (5.8)–(5.10) into (5.6) we have

\[ X_1 = 4\alpha, \]

by virtue of (3.11) and (5.7). Substituting this equation into (5.7)–(5.9) we have

\[ \beta X_2 = 4\alpha(\gamma - \delta) - \delta\gamma, \]
\[ \alpha_3 + 3\beta\delta + 3\alpha\beta = 0, \]
\[ \beta_3 + 3\alpha\gamma - 3\alpha\delta + \gamma\delta = 0. \]

It follows from (4.23), (3.8) and (5.12) that

\[ \alpha\gamma_3 + \beta(3\alpha\gamma - 6\alpha\delta - \gamma\delta) = 0. \]

From (4.22), (5.3) and (5.5) we have \( X_3 = 0 \) and \( \beta_1 = 0 \) and therefore \( \alpha_2 = \delta_2 = 0 \) because of (4.17). Hence, by (3.8) and (5.5), we have \( \gamma_1 = 0 \), and so \( \beta_2 = 0 \).

Now put \( F = \alpha \) and \( \beta \) in Lemma 1. Then we have

\[ \alpha_3(\gamma + X_1) = 0, \]
\[ \beta_3(\gamma + X_1) = 0. \]

If \( \gamma + X_1 \neq 0 \), then we have \( \alpha_3 = \beta_3 = 0 \). It follows from (4.15) and (4.25) that \( \alpha, \beta \) and \( \delta \) are constants and that \( \alpha_1 = \beta_1 = 0 \) for \( i = 1, 2 \). Furthermore, by (3.8) we see that \( \gamma \) is a constant. Thus from (5.13)–(5.15) we have

\[ \alpha + \delta = 0, \]
\[ 3\alpha\gamma - 3\alpha\delta + \gamma\delta = 0, \]
\[ 3\alpha\gamma - 6\alpha\delta - \gamma\delta = 0. \]

Hence, by (3.2) and (3.8) we have \( \alpha^2 - c = 0 \) and \( 2\beta^2 + c = 0 \), which is a contradiction. Therefore \( X_1 = -\gamma \), which, together with (5.11), implies \( \gamma = -X_1 = -4\alpha \). Thus it follows from (5.13) that \( \gamma_3 = -4\alpha_3 = 12\beta(\delta + \alpha) \). Therefore from (5.15) we have a contradiction \( \alpha\delta = 0 \).

**Lemma 5.** \( X_1 = 0 \).

**Proof.** Suppose that \( X_1 \neq 0 \). Then from (4.29) we have

\[ (-\lambda_p\lambda_q + 2c)(\lambda_p + \lambda_q) + 2(\alpha + \gamma)\lambda_p\lambda_q - 2c\gamma = 0. \]

Multiply above equation by \( \alpha^3 \) and take account of the coefficient of \( \theta_1 \) in the exterior derivative of this equation. Then, from Lemma 3 and (4.16) we have

\[ \alpha\lambda_p + \alpha\lambda_q - \alpha\gamma + \alpha^2 = 0. \]
Again, taking account of the coefficient of $\theta_1$ in the exterior derivative of above equation, we have $\alpha \alpha_1 = 0$, which is a contradiction.

Consequently, for all $p, q$ such that $\phi_{pq} \neq 0$, we have $\lambda_p = \lambda_q$. We take $p, q$ such that $\phi_{pq} \neq 0$. Then by (4.29) we have

$$\beta^2 \lambda_p - (\lambda_p - \gamma)(\lambda_p^2 - \alpha \lambda_p - c) = 0.$$  

Furthermore, from (q3p), (4.28) and (4.16) we have

$$(\lambda_p)_{1}(\lambda_p - \delta) = 0.$$  

It follows from (4.16) and Lemma 4 that $(\lambda_p)_{1} \neq 0$. Hence we have $\lambda_p = \delta$, which, together with (3.8), (3.11) and (5.16), implies $\alpha + \delta = 0$. Therefore $\alpha$ and $\delta$ are constants, which is a contradiction. \hfill $\square$

Taking account of the coefficient of $\theta_3$ in the exterior derivative of equation $\alpha + \gamma - \delta = 0$ and making use of (4.19), (4.20), (4.23) and Lemma 5, we have

$$\alpha(\gamma - \delta)(\gamma(\alpha - \delta)) + \beta^2(\alpha^2 + \alpha \delta - 3\delta^2) = 0.$$  

Again taking account of the coefficient of $\theta_1$ in the exterior derivative of (5.17) and taking account of (3.2) and (5.3), we have

$$\alpha^4 + 4\alpha^2 c - 2\beta^2 c + 3c^2 + \beta^2(\alpha^2 + 3\delta^2) = 0.$$  

Eliminating $\beta$ from (5.17) and (5.18), we have a polynomial $4\alpha^4 + 7c\alpha^2 + 4c = 0$ of degree four with respect to $\alpha$. This shows that $\alpha$ is a constant, which contradicts Lemma 4. Consequently we proved $\beta = 0$.

Since (2.6) and $\beta = 0$, we see that $\alpha$ is a constant in $M$ (see [6]). Thus from (3.1) our assumption $\Xi_{ij,1} = 0$ is equivalent to $\alpha h_{ij,1} = 0$. Put $j = 1$ in (2.3). Then by above equation we have $\alpha h_{i1;k} = -c \alpha \phi_{ik}$. Therefore since (2.1) and $d \xi_j = 0$, we have

$$\alpha \sum_{k,l} h_{ik} \phi_{lk} h_{kj} + \alpha^2 \sum_{k} \phi_{ki} h_{kj} = -\alpha h_{i1;j} = c \alpha \phi_{ij},$$  

which implies that $\alpha^2 (\phi H - H \phi) = 0$. Owing to Okumura’s work or Montiel and Romero’s work stated in Introduction, we complete the proof of our Main Theorem.

**Remark 3.** In previous papers [4] and [5], we classified real hypersurfaces which satisfy $\nabla_\xi R_\xi = 0$ and some several conditions in $M_n(c), c \neq 0, n \geq 3$. Combining these results we have the following:

Let $M$ be a real hypersurface in a complex space form $M_n(c), c \neq 0, n \geq 3$ which satisfies $\nabla_\xi R_\xi = 0$. Then the following conditions are mutually equivalent.

(i) $M$ is locally congruent to type $A$.
(ii) $R_\xi S = S R_\xi$.
(iii) $R_\xi \phi S = S \phi R_\xi$.
(iv) $R_\xi \phi S = R_\xi S \phi$. 

Remark 4. Recently the second author extended Main Theorem to the case $n = 2$ [8].

References


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