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JACOBI OPERATORS ALONG THE STRUCTURE FLOW ON REAL HYPERSURFACES IN A NONFLAT COMPLEX SPACE FORM II

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Abstract. Let M be a real hypersurface of a complex space form with almost contact metric structure (ϕ, ξ, η, g). In this paper, we study real hypersurfaces in a complex space form whose structure Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ is ξ-parallel. In particular, we prove that the condition $\nabla_\xi R_\xi = 0$ characterizes the homogeneous real hypersurfaces of type A in a complex projective space or a complex hyperbolic space when $R_\xi \phi S = R_\xi S \phi$ holds on M, where S denotes the Ricci tensor of type (1,1) on M.

1. Introduction

Let $(M_n(c), J, \tilde{g})$ be a complex n-dimensional complex space form with Kähler structure $(J, \tilde{g})$ of constant holomorphic sectional curvature $4c$ and let M be an orientable real hypersurface in $M_n(c)$. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from $(J, \tilde{g})$.

Typical examples of real hypersurfaces in $M_n(c), c \neq 0$, are homogeneous ones. Takagi [12], [13] classified homogeneous real hypersurfaces of a complex projective space $P_n\mathbb{C}$ into six model spaces. On the other hand, Cecil and Ryan [2] extensively studied a Hopf hypersurface, which is realized as tubes over certain submanifolds in $P_n\mathbb{C}$, by using its focal map. By making use of those results and the mentioned work of Takagi, Kimura [7] proved the local classification theorem for Hopf hypersurfaces of $P_n\mathbb{C}$ all of whose principal curvatures are constants. For the case where a complex hyperbolic space $H_n\mathbb{C}$, Berndt [1] proved the classification theorem for Hopf hypersurfaces whose all principal curvatures are constants. Among the several types of real hypersurfaces appeared in model spaces ([1], [12]), a particular type of tubes over totally geodesic $P_k\mathbb{C}$ or $H_k\mathbb{C}$ $(0 \leq k \leq n-1)$ adding a horosphere in $H_n\mathbb{C}$, which is called type A, has a lot of nice geometric properties. For example, Okumura [9] (resp. Montiel and Romero [8]) showed that a real hypersurface in $P_n\mathbb{C}$ (resp.

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$H_n\mathbb{C}$) is locally congruent to one of real hypersurfaces of type A if and only if the Reeb flow ξ is isometric or equivalently the structure operator ϕ commutes with the shape operator H.

The structure Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ has a fundamental role in contact geometry. Ortega, Pérez and Santos [10] have proved that there are no real hypersurfaces in $P_n\mathbb{C}, n \geq 3$ with parallel structure Jacobi operator $\nabla R_\xi = 0$. More generally, such a result has been extended by [11] due to them. In this paper, motivated by results mentioned above we consider the parallelism of the structure Jacobi operator R_ξ in the direction of the structure vector field, that is $\nabla_\xi R_\xi = 0$. Recently we have some classification theorems for real hypersurfaces in a nonflat complex space form with respect to the parallelism of R_ξ (cf. [3], [4], [5]).

The main purpose of this paper is to classify real hypersurfaces in a nonflat complex space form $M_n(c)$ which satisfies $\nabla_\xi R_\xi = 0$ and at the same time $R_\xi \phi S = R_\xi S \phi$, where S denotes the Ricci tensor of the hypersurface. Our main result is as follows:

Main Theorem. *Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$ which satisfies $\nabla_\xi R_\xi = 0$. Then M holds $R_\xi \phi S = R_\xi S \phi$ if and only if $H\xi = 0$ or M is locally congruent to one of the following*:

(I) *in case that $M_n(c) = P_n\mathbb{C}$,*
- (A_1) *a geodesic hypersphere of radius r, where $0 < r < \pi/2$ and $r \neq \pi/4$,*
- (A_2) *a tube of radius r over a totally geodesic $P_k\mathbb{C}$ for some $k \in \{1, \ldots, n-2\}$, where $0 < r < \pi/2$ and $r \neq \pi/4$;*

(II) *in case that $M_n(c) = H_n\mathbb{C}$,*
- (A_0) *a horosphere,*
- (A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$,*
- (A_2) *a tube over a totally geodesic $H_k\mathbb{C}$ for some $k \in \{1, \ldots, n-2\}$.*

We note that every Hopf hypersurface always satisfies condition $R_\xi \phi S = R_\xi S \phi$.

All manifolds in this paper are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented.

2. Preliminaries

We denote by $M_n(c)$, $c \neq 0$, be a nonflat complex space form with the Fubini-Study metric $\tilde{g}$ of constant holomorphic sectional curvature $4c$ and Levi-Civita connection $\tilde{\nabla}$. For an immersed $(2n-1)$-dimensional Riemannian manifold $\tau : M \to M_n(c)$, the Levi-Civita connection ∇ of induced metric and the shape operator H of the immersion are characterized

$$\tilde{\nabla}_X Y = \nabla_X Y + g(HX, Y)\nu, \quad \tilde{\nabla}_X \nu = -HX$$

for any vector fields X and Y on M, where g denotes the Riemannian metric of M induced from $\tilde{g}$ and ν a unit normal vector on M. In the sequel the indices $i, j, k, l, \ldots$ run over the range $\{1, 2, \ldots, 2n-1\}$ unless otherwise stated. For a local orthonormal frame field $\{e_i\}$ of M, we denote the dual 1-forms by $\{\theta_i\}$. Then the connection forms θ_{ij} are defined by

$$d\theta_i + \sum_j \theta_{ij} \wedge \theta_j = 0, \quad \theta_{ij} + \theta_{ji} = 0.$$

Then we have

$$\nabla_{e_i} e_j = \sum_k \theta_{kj}(e_i) e_k = \sum_k \Gamma_{kij} e_k,$$

where we put $\theta_{ij} = \sum_k \Gamma_{ijk}\theta_k$. The almost contact metric structure ($\phi = (\phi_{ij}), \xi = \sum_i \xi_i e_i$) is induced on M by following equation:

$$J(e_i) = \sum_j \phi_{ji} e_j + \xi_i \nu.$$

The structure tensor ϕ and the structure vector ξ satisfy

$$\sum_k \phi_{ik}\phi_{kj} = \xi_i\xi_j - \delta_{ij}, \quad \sum_j \xi_j \phi_{ij} = 0, \quad \sum_i \xi_i^2 = 1, \quad \phi_{ij} + \phi_{ji} = 0,$$

$$d\phi_{ij} = \sum_k (\phi_{ik}\theta_{kj} - \phi_{jk}\theta_{ki} - \xi_i h_{jk}\theta_k + \xi_j h_{ik}\theta_k), \tag{2.1}$$

$$d\xi_i = \sum_j \xi_j \theta_{ji} - \sum_{j,k} \phi_{ji} h_{jk} \theta_k.$$

We denote the components of the shape operator or the second fundamental tensor H of M by h_{ij}. The components $h_{ij;k}$ of the covariant derivative of H are given by $\sum_k h_{ij;k}\theta_k = dh_{ij} - \sum_k h_{ik}\theta_{kj} - \sum_k h_{jk}\theta_{ki}$. Then we have the equation of Gauss and Codazzi

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \phi_{ik}\phi_{jl} - \phi_{il}\phi_{jk} + 2\phi_{ij}\phi_{kl}) + h_{ik}h_{jl} - h_{il}h_{jk}, \tag{2.2}$$

$$h_{ij;k} - h_{ik;j} = c(\xi_k\phi_{ij} + \xi_i\phi_{kj} - \xi_j\phi_{ik} - \xi_i\phi_{jk}), \tag{2.3}$$

respectively.

From (2.2) the structure Jacobi operator $R_\xi = (\Xi_{ij})$ is given by

$$\Xi_{ij} = \sum_{k,l} h_{ik}h_{jl}\xi_k\xi_l - \sum_{k,l} h_{ij}h_{kl}\xi_k\xi_l + c\xi_i\xi_j - c\delta_{ij}. \tag{2.4}$$

From (2.2) the Ricci tensor $S = (S_{ij})$ is given by

$$S_{ij} = (2n+1)c\delta_{ij} - 3c\xi_i\xi_j + hh_{ij} - \sum_k h_{ik}h_{kj}, \tag{2.5}$$

where $h = \sum_i h_{ii}$.

First we remark:

Lemma 1 ([5]). *Let U be an open set in M and F a smooth function on U. We put $dF = \sum_i F_i\theta_i$. Then we have*

$$F_{ij} - F_{ji} = \sum_k F_k \Gamma_{kij} - \sum_k F_k \Gamma_{kji}.$$

Now we retake a local orthonormal frame field e_i in such a way that
- $e_1 = \xi$,
- e_2 is in the direction of $\sum_{i=2}^{2n-1} h_{1i}e_i$,
- $e_3 = \phi e_2$.

Then we have

$$\xi_1 = 1,\ \xi_i = 0\ (i \geq 2),\ h_{1j} = 0\ (j \geq 3) \text{ and } \phi_{32} = 1. \tag{2.6}$$

We put $\alpha := h_{11}, \beta := h_{12}, \gamma := h_{22}, \varepsilon := h_{23}$ and $\delta := h_{33}$.

Hereafter the indices $p, q, r, s, \ldots$ run over the range $\{4, 5, \ldots, 2n-1\}$ unless otherwise stated.

Since $d\xi_i = 0$, we have

$$\begin{aligned} \theta_{12} &= \varepsilon\theta_2 + \delta\theta_3 + \sum_p h_{3p}\theta_p, \\ \theta_{13} &= -\beta\theta_1 - \gamma\theta_2 - \varepsilon\theta_3 - \sum_p h_{2p}\theta_p, \\ \theta_{1p} &= \sum_q \phi_{qp}h_{q2}\theta_2 + \sum_q \phi_{qp}h_{q3}\theta_3 + \sum_{q,r} \phi_{qp}h_{qr}\theta_r. \end{aligned} \tag{2.7}$$

We put

$$\theta_{23} = \sum_i X_i\theta_i, \quad \theta_{2p} = \sum_i Y_{pi}\theta_i, \quad \theta_{3p} = \sum_i Z_{pi}\theta_i. \tag{2.8}$$

Then it follows from $d\phi_{2i} = 0$ that $Y_{pi} = -\sum_q \phi_{pq}Z_{qi}$ or $Z_{pi} = \sum_q \phi_{pq}Y_{qi}$. The equations (2.4) and (2.5) are rewritten as

$$\Xi_{ij} = -\alpha h_{ij} + h_{1i}h_{1j} + c\delta_{i1}\delta_{j1} - c\delta_{ij}, \tag{2.9}$$

$$S_{ij} = hh_{ij} - \sum_k h_{ik}h_{jk} - 3c\delta_{i1}\delta_{j1} + (2n+1)c\delta_{ij}, \tag{2.10}$$

respectively.

3. Real hypersurfaces satisfying $\nabla_\xi R_\xi = 0$ and $R_\xi\phi S = R_\xi S\phi$

First we assume that $\nabla_\xi R_\xi = 0$. The components $\Xi_{ij;k}$ of the covariant derivativation of $R_\xi = (\Xi_{ij})$ is given by

$$\sum_k \Xi_{ij;k}\theta_k = d\Xi_{ij} - \sum_k \Xi_{kj}\theta_{ki} - \sum_k \Xi_{ik}\theta_{kj}.$$

Substituting (2.9) into the above equation we have

$$\begin{aligned}\sum_k \Xi_{ij;k}\theta_k = &-(d\alpha)h_{ij} - \alpha dh_{ij} + (dh_{1i})h_{1j} + h_{1i}(dh_{1j}) \\ &+ \alpha\sum_k h_{kj}\theta_{ki} - \alpha h_{1j}\theta_{1i} - \beta h_{1j}\theta_{2i} - c\delta_{j1}\theta_{1i} \\ &+ \alpha\sum_k h_{ik}\theta_{kj} - \alpha h_{1i}\theta_{1j} - \beta h_{1i}\theta_{2j} - c\delta_{i1}\theta_{1j}.\end{aligned} \tag{3.1}$$

In the following, we assume that $\beta \neq 0$.

Our assumption $\nabla_\xi R_\xi = 0$ is equivalent to $\Xi_{ij;1} = 0$, which can be stated as follows:

$$\varepsilon = 0, \quad \alpha\delta + c = 0, \quad h_{3p} = 0, \tag{3.2}$$

$$(\beta^2 - \alpha\gamma)_1 - 2\alpha\sum_p h_{2p}Y_{p1} = 0, \tag{3.3}$$

$$(\beta^2 - \alpha\gamma - c)X_1 + \alpha\sum_p h_{2p}Z_{p1} = 0, \tag{3.4}$$

$$(\alpha h_{2p})_1 + \alpha\sum_q h_{pq}Y_{q1} + (\beta^2 - \alpha\gamma)Y_{p1} - \alpha\sum_q h_{2q}\Gamma_{qp1} = 0, \tag{3.5}$$

$$\alpha h_{2p}X_1 - \sum_q(\alpha h_{qp} + c\delta_{pq})Z_{q1} = 0, \tag{3.6}$$

$$(\alpha h_{pq})_1 - \alpha h_{2q}Y_{p1} - \alpha\sum_r h_{rq}\Gamma_{rp1} - \alpha h_{2p}Y_{q1} - \alpha\sum_r h_{pr}\Gamma_{rq1} = 0. \tag{3.7}$$

Hereafter we shall use (3.2) without quoting.

Furthermore we assume that $R_\xi\phi S = R_\xi S\phi$. Under the assumption $\nabla_\xi R_\xi = 0$, we have the following additional equations

$$\beta^2 - \alpha\gamma - c = 0, \tag{3.8}$$

$$\tilde{R}_\xi\tilde{\phi}A = 0, \tag{3.9}$$

$$\tilde{R}_\xi\tilde{\phi}\tilde{S} = \tilde{R}_\xi\tilde{S}\tilde{\phi}, \tag{3.10}$$

where $A = {}^t(h_{24}, h_{25}, \ldots, h_{2,2n-1})$, $\tilde{R}_\xi = (\Xi_{pq})$, $\tilde{\phi} = (\phi_{pq})$, $\tilde{S} = (S_{pq})$. From (3.2) and (3.8) we note

$$\delta \neq \gamma. \tag{3.11}$$

Now, properly speaking, we denote the equation (2.3) by (ijk) simply. Then we have the following equations (112)–$(q3p)$.

$$\alpha_2 - \beta_1 = 0, \tag{112}$$

$$\beta_2 - \gamma_1 - 2\sum_p h_{2p}Y_{p1} = 0, \tag{212}$$

$$(312)\qquad (\alpha-\delta)\gamma-\beta X_2+(\gamma-\delta)X_1-\beta^2-\sum_p h_{2p}Z_{p1}=-c,$$

$$(113)\qquad \alpha_3+3\beta\delta-\alpha\beta+\beta X_1=0,$$

$$(213)\qquad \beta_3+\gamma\delta+(\gamma-\delta)X_1-\beta^2-\sum_p h_{2p}Z_{p1}=0,$$

$$(313)\qquad \beta X_3+\delta_1=0,$$

$$(223)\qquad \gamma_3-2\beta\delta+2\sum_p h_{2p}Y_{p3}+(\gamma-\delta)X_2-\beta\gamma-\sum_p h_{2p}Z_{p2}=0,$$

$$(323)\qquad \sum_p h_{2p}Z_{p3}-\delta_2-(\gamma-\delta)X_3=0,$$

$$(1p1)\qquad \alpha_p+\beta Y_{p1}=0,$$

$$(12p)\qquad \beta_p+2\sum_{q,r} h_{2q}\phi_{rq}h_{rp}+\beta Y_{p2}+\alpha\sum_q \phi_{qp}h_{2q}=0,$$

$$(13p)\qquad 2\delta h_{2p}-\beta Y_{p3}-\alpha h_{2p}+\beta X_p=0,$$

$$(22p)\qquad \gamma_p+2\sum_q h_{2q}Y_{qp}-h_{2p2}-\sum_q h_{qp}Y_{q2}+\beta\sum_q \phi_{qp}h_{2q}+\gamma Y_{p2}$$
$$+\textstyle\sum_q h_{2q}\Gamma_{qp2}=0,$$

$$(23p)\qquad \delta X_p+\beta h_{2p}-\gamma X_p+\sum_q h_{2q}Z_{qp}-h_{2p3}-\sum_q h_{qp}Y_{q3}$$
$$+\gamma Y_{p3}+\sum_q h_{2q}\Gamma_{qp3}=0,$$

$$(33p)\qquad \delta_p+h_{2p}X_3-\sum_q h_{qp}Z_{q3}+\delta Z_{p3}=0,$$

$$(21p)\qquad \beta_p+\sum_{q,r} h_{2q}\phi_{rq}h_{rp}-h_{2p1}-\sum_q h_{qp}Y_{q1}+\gamma Y_{p1}+\sum_q h_{2q}\Gamma_{qp1}=0,$$

$$(31p)\qquad \delta h_{2p}-\alpha h_{2p}+\beta X_p-h_{2p}X_1+\sum_q h_{qp}Z_{q1}-\delta Z_{p1}=0,$$

$$(32p)\qquad \delta X_p+\beta h_{2p}-\gamma X_p+\sum_q h_{2q}Z_{qp}+h_{2p}X_2-\sum_q h_{pq}Z_{q2}+\delta Z_{p2}=0,$$

$$(1pq)\qquad 2\sum_{r,s} h_{rp}\phi_{sr}h_{sq}-\alpha\sum_r \phi_{rp}h_{rq}+\alpha\sum_r \phi_{rq}h_{rp}$$
$$-\beta Y_{pq}+\beta Y_{qp}=-2c\phi_{pq},$$

$$(2pq)\qquad h_{2pq}+\sum_r h_{rp}Y_{rq}-\beta\sum_r \phi_{rp}h_{rq}-\gamma Y_{pq}-\sum_r h_{2r}\Gamma_{rpq}-h_{2qp}$$

$$-\sum_r h_{rq}Y_{rp} + \beta\sum_r \phi_{rq}h_{rp} + \gamma Y_{qp} + \sum_r h_{2r}\Gamma_{rqp} = 0,$$

$$(q1p) \qquad \sum_{r,s} h_{rq}\phi_{sr}h_{sp} - \alpha\sum_r \phi_{rq}h_{rp} - \beta Y_{qp} - h_{pq1}$$

$$+h_{2q}Y_{p1} + \sum_r h_{rq}\Gamma_{rp1} + h_{2p}Y_{q1} + \sum_r h_{rp}\Gamma_{rq1} = c\phi_{pq},$$

$$(q3p) \qquad h_{3qp} - \varepsilon Y_{qp} - \delta Z_{qp} - \sum_r h_{3r}\Gamma_{rqp} - h_{2q}X_p + \sum_r h_{qr}Z_{rp} - h_{qp3}$$

$$+h_{q2}Y_{p3} + h_{q3}Z_{p3} + \sum_r h_{qr}\Gamma_{rp3} + h_{p2}Y_{q3} + h_{p3}Z_{q3}$$

$$+\sum_r h_{pr}\Gamma_{rq3} = 0.$$

Remark 1. We did not write $(p2q), (3pq)$ and (pqr) since we need not use them.

4. Formulas and lemmas

In this section we study the crucial case where $\beta \neq 0$. By (3.6) and $(31p)$ we have

$$\beta X_p = (\alpha - \delta)h_{2p}. \tag{4.1}$$

This and $(13p)$ imply that

$$\beta Y_{p3} = \delta h_{2p}. \tag{4.2}$$

The equation (3.9) can be rewritten as

$$\sum_{q,r}(\alpha h_{pq} + c\delta_{pq})\phi_{qr}h_{r2} = 0, \tag{4.3}$$

which, together with (4.2), implies

$$\beta\sum_q (h_{pq} - \delta\delta_{pq})Z_{q3} = \delta\sum_{q,r}(h_{pq} - \delta\delta_{pq})\phi_{qr}Y_{r3} = 0.$$

Hence it follows from $(33p)$ and $(1p1)$ that

$$\delta_p = -h_{2p}X_3 \quad \text{and} \quad \alpha_p = -\beta Y_{p1}. \tag{4.4}$$

Thus since (4.4) and $\alpha_p\delta + \alpha\delta_p = 0$ obtained from (3.2) we have

$$\beta\delta Y_{p1} = -\alpha h_{2p}X_3, \tag{4.5}$$

and so $\sum_p h_{2p}Z_{p1} = 0$. By (4.2), we have

$$\sum_p h_{2p}Z_{p3} = \sum_{p,q} h_{2p}\phi_{pq}Y_{q3} = \frac{\delta}{\beta}\sum_{p,q} h_{2p}\phi_{pq}h_{2q} = 0. \tag{4.6}$$

From (3.6), (4.3) and (4.5) we have

$$h_{2p}X_1 = 0. \tag{4.7}$$

Now we shall prove the following key lemma.

Lemma 2. $H(e_2) \in \mathrm{span}\{e_1, e_2\}$.

Proof. Suppose that $h_{2p} \neq 0$. Then from (4.7) we have $X_1 = 0$. We can select the vector e_4 so that $h_{24} \neq 0$ and $h_{25} = \cdots = h_{2,2n-1} = 0$. We put $e_5 := \phi e_4$ and $\rho := h_{24}(\neq 0)$. Note that $\phi_{54} = 1$. Then by (4.3) we have

$$h_{55} = \delta, \quad h_{p5} = 0 \ (p \neq 5).$$

Put $p = 5$ in $(32p)$. Then since above equation and (4.1) we have $X_5 = 0$ and so $Z_{45} = 0$. Thus we have $Y_{55} = 0$. Furthermore, put $p = q = 5$ in $(q1p)$. Then, since $\Gamma_{551} = Y_{55} = 0$, we have

$$\alpha_1 = \delta_1 = 0. \tag{4.8}$$

Thus, from (313), (323), (4.6) and (112) we have

$$X_3 = 0, \tag{4.9}$$

$$\alpha_2 = \delta_2 = 0, \tag{4.10}$$

$$\beta_1 = 0. \tag{4.11}$$

By (4.4) and (4.9) we have $\alpha_p = \delta_p = 0$. Thus it follows from $(1p1)$ that

$$\alpha_p = \delta_p = Y_{p1} = Z_{p1} = 0. \tag{4.12}$$

Now we put $F = \alpha,\ i = 1$ and $j = p$ in Lemma 1. Then, from (2.7), (4.8), (4.10) and (4.12) we have

$$0 = \alpha_{1p} - \alpha_{p1} = \sum_k \alpha_k \Gamma_{k1p} - \sum_k \alpha_k \Gamma_{kp1} = \alpha_3(\Gamma_{31p} - \Gamma_{3p1}) = \alpha_3 h_{2p}.$$

Thus we have $\alpha_3 = 0$. Hence it follows from (4.8), (4.10) and (4.12) that α and δ are constants, which, together with (113), imply

$$\alpha = 3\delta. \tag{4.13}$$

On the other hand, taking account of the coefficient of $\theta_1 \wedge \theta_3$ in the exterior derivative of θ_{23}, we have $X_2 = -2\beta$. Thus, from (312) and (4.13) we have

$$2\delta\gamma + \beta^2 = -c.$$

Hence it follows from (3.8) that β and γ are constants. From (3.5) and (4.12) we have

$$h_{2p1} - \sum_q h_{2q}\Gamma_{qp1} = 0.$$

This, together with $(21p)$, implies $\rho\delta = 0$, which is a contradiction. □

Remark 2. Lemma 2 has been already proved in [5]. However in the case of our condition by using the equation (3.8) we have a short proof.

Owing to Lemma 2 the matrix (h_{pq}) is diagonalizable, that is, for a suitable choice of a orthonormal frame field $\{e_p\}$ we can set

$$h_{pq} = \lambda_p \delta_{pq}.$$

Then it is easy to see

$$\begin{aligned} \tilde{R}_\xi &= -((\alpha\lambda_p + c)\delta_{pq}), \\ \tilde{S} &= ((h\lambda_p - (\lambda_p)^2 + (2n+1)c)\delta_{pq}). \end{aligned} \tag{4.14}$$

Here we shall sum up all equations obtained from Lemma 2.

From (4.1), (4.2) and (4.4) we have

$$X_p = Y_{p1} = Z_{p1} = Y_{p3} = Z_{p3} = 0, \quad \alpha_p = \delta_p = 0. \tag{4.15}$$

Put $p = q$ in (3.7). Then we have

$$(\alpha\lambda_p)_1 = 0. \tag{4.16}$$

Moreover, from (112)–(32p) we have

$$\alpha_2 - \beta_1 = 0, \tag{4.17}$$

$$\beta_2 - \gamma_1 = 0, \tag{4.18}$$

$$\gamma\delta + \beta X_2 - (\gamma - \delta)X_1 = 0, \tag{4.19}$$

$$\alpha_3 + 3\beta\delta - \alpha\beta + \beta X_1 = 0, \tag{4.20}$$

$$\beta_3 - \alpha\delta + \gamma\delta + (\gamma - \delta)X_1 - \beta^2 = c, \tag{4.21}$$

$$\delta_1 + \beta X_3 = 0, \tag{4.22}$$

$$\gamma_3 - 2\beta\delta + (\gamma - \delta)X_2 - \beta\gamma = 0, \tag{4.23}$$

$$\delta_2 + (\gamma - \delta)X_3 = 0, \tag{4.24}$$

$$\beta_p = 0, \tag{4.25}$$

$$Y_{p2} = 0, \quad Z_{p2} = 0, \tag{4.26}$$

$$\gamma_p = 0. \tag{4.27}$$

It follows from ($q1p$) and (3.7) that

$$\alpha\beta Y_{qp} = \alpha\lambda_p\lambda_q\phi_{pq} - \alpha^2\lambda_p\phi_{pq} + \alpha_1\lambda_p\delta_{pq} - c\alpha\phi_{pq}. \tag{4.28}$$

From this, (2pq) and ($q3p$) we have

$$\begin{aligned} \beta^2(\lambda_p + \lambda_q)\phi_{pq} - (\lambda_p - \gamma)(\lambda_p\lambda_q - \alpha\lambda_q - c)\phi_{pq} \\ - (\lambda_q - \gamma)(\lambda_p\lambda_q - \alpha\lambda_p - c)\phi_{pq} = 0, \end{aligned} \tag{4.29}$$

$$(\lambda_q - \delta)(\alpha((\lambda_q)^2 - \alpha\lambda_q - c)\delta_{pq} + \alpha_1\lambda_q\phi_{pq}) - \alpha\beta(h_{qp3} + (\lambda_p - \lambda_q)\Gamma_{qp3}) = 0. \tag{4.30}$$

If $p = q$ in above equation, then we have

$$(\lambda_p - \delta)((\lambda_p)^2 - \alpha\lambda_p - c) - \beta(\lambda_p)_3 = 0. \tag{4.31}$$

5. Proof of Main Theorem

In this section we prove Main Theorem.

We first prove $\beta = 0$. Suppose that $\beta \neq 0$. It follows from (4.22) that (3.10) is equivalent to

$$\rho_p(\sigma_q - \sigma_p)\phi_{pq} = 0,$$

where $\rho_p = \alpha\lambda_p + c,\ \sigma_p = h\lambda_p - (\lambda_p)^2 + (2n+1)c$. Therefore if $\phi_{qp} \neq 0$, then we have

$$(\lambda_p - \lambda_q)(\alpha\lambda_p + c)(h - \lambda_q - \lambda_p) = 0. \tag{5.1}$$

Here we assert that if $\phi_{pq} \neq 0$, then $\lambda_p = \lambda_q$. To prove this, we assume that there exist indices p and q such that

$$\phi_{pq} \neq 0, \quad \lambda_p - \lambda_q \neq 0.$$

First we prepare three lemmas.

Lemma 3. $\alpha\alpha_1 - (\alpha\gamma)_1 = 0$.

Proof. From (5.1) we have

$$(\alpha\lambda_p + c)(h - \lambda_p - \lambda_q) = 0,$$
$$(\alpha\lambda_q + c)(h - \lambda_p - \lambda_q) = 0.$$

Thus if $h - \lambda_p - \lambda_q \neq 0$, then we have $\lambda_p = \lambda_q = 0$, which is a contradiction. Hence we have $\alpha h - \alpha\lambda_p - \alpha\lambda_q = 0$. Taking account of the coefficient of θ_1 in the exterior derivative of this, it follows from (4.16) that

$$(\alpha h)_1 = 0. \tag{5.2}$$

From (4.26) we have $(\alpha \sum_p \lambda_p)_1 = 0$. Combining this equation with $h = \alpha + \gamma + \delta + \sum_p \lambda_p$, we have

$$(\alpha(h - \alpha - \gamma - \delta))_1 = 0.$$

Eliminating h from this Lemma 3 follows from (5.2). □

Lemma 4. $\alpha_1 \neq 0$ *and* $\alpha + \gamma - \delta = 0$.

Proof. From (4.24) we have $2\beta\beta_1 - (\alpha\gamma)_1 = 0$. Hence it follows from Lemma 4 that

$$\alpha\alpha_1 + \beta\beta_1 = 0. \tag{5.3}$$

On the other hand, by (4.32) and (4.34) we have $(\gamma - \delta)\delta_1 - \beta\delta_2 = 0$, and therefore $(\gamma - \delta)\alpha_1 - \beta\alpha_2 = 0$. Thus we have

$$(\alpha + \gamma - \delta)\alpha_1 = 0 \quad \text{and} \quad (\alpha + \gamma - \delta)\beta_1 = 0. \tag{5.4}$$

Suppose that

$$\alpha_1 = 0. \tag{5.5}$$

It follows from (3.8) that

$$(\beta^2 - \alpha\gamma - c)_3 = 2\beta\beta_3 - \gamma\alpha_3 - \alpha\gamma_3 = 0. \tag{5.6}$$

From (4.19)–(4.21), (4.23) and (3.8) we have the following:

$$\delta\gamma + \beta X_2 - (\gamma - \delta)X_1 = 0, \tag{5.7}$$

$$\alpha_3 + 3\beta\delta - \alpha\beta + \beta X_1 = 0, \tag{5.8}$$

$$\beta_3 + (\gamma - \delta)X_1 + \gamma\delta - \alpha\gamma - c = 0, \tag{5.9}$$

$$\gamma_3 - 2\beta\delta + (\gamma - \delta)X_2 + \beta\gamma = 0. \tag{5.10}$$

Substituting of (5.8)–(5.10) into (5.6) we have

$$X_1 = 4\alpha, \tag{5.11}$$

by virtue of (3.11) and (5.7). Substituting of this equation into (5.7)–(5.9) we have

$$\beta X_2 = 4\alpha(\gamma - \delta) - \delta\gamma, \tag{5.12}$$

$$\alpha_3 + 3\beta\delta + 3\alpha\beta = 0, \tag{5.13}$$

$$\beta_3 + 3\alpha\gamma - 3\alpha\delta + \gamma\delta = 0. \tag{5.14}$$

It follows from (4.23), (3.8) and (5.12) that

$$\alpha\gamma_3 + \beta(3\alpha\gamma - 6\alpha\delta - \gamma\delta) = 0. \tag{5.15}$$

From (4.22), (5.3) and (5.5) we have $X_3 = 0$ and $\beta_1 = 0$ and therefore $\alpha_2 = \delta_2 = 0$ because of (4.17). Hence, by (3.8) and (5.5), we have $\gamma_1 = 0$, and so $\beta_2 = 0$.

Now put $F = \alpha$ and β in Lemma 1. Then we have

$$\alpha_3(\gamma + X_1) = 0, \quad \beta_3(\gamma + X_1) = 0.$$

If $\gamma + X_1 \neq 0$, then we have $\alpha_3 = \beta_3 = 0$. It follows from (4.15) and (4.25) that α, β and δ are constants and that $\alpha_i = \beta_i = 0$ for $i = 1, 2$. Furthermore, by (3.8) we see that γ is a constant. Thus from (5.13)–(5.15) we have

$$\begin{aligned} &\alpha + \delta = 0, \\ &3\alpha\gamma - 3\alpha\delta + \gamma\delta = 0, \\ &3\alpha\gamma - 6\alpha\delta - \gamma\delta = 0. \end{aligned}$$

Hence, by (3.2) and (3.8) we have $\alpha^2 - c = 0$ and $2\beta^2 + c = 0$, which is a contradiction. Therefore $X_1 = -\gamma$, which, together with (5.11), implies $\gamma = -X_1 = -4\alpha$. Thus it follows from (5.13) that $\gamma_3 = -4\alpha_3 = 12\beta(\delta + \alpha)$. Therefore from (5.15) we have a contradiction $\alpha\delta = 0$. □

Lemma 5. $X_1 = 0$.

Proof. Suppose that $X_1 \neq 0$. Then from (4.29) we have

$$(-\lambda_p\lambda_q + 2c)(\lambda_p + \lambda_q) + 2(\alpha + \gamma)\lambda_p\lambda_q - 2c\gamma = 0.$$

Multiply above equation by α^3 and take account of the coefficient of θ_1 in the exterior derivative of this equation. Then, from Lemma 3 and (4.16) we have

$$\alpha\lambda_p + \alpha\lambda_q - \alpha\gamma + \alpha^2 = 0.$$

Again, taking account of the coefficient of θ_1 in the exterior derivative of above equation, we have $\alpha\alpha_1 = 0$, which is a contradiction.

Consequently, for all p, q such that $\phi_{pq} \neq 0$, we have $\lambda_p = \lambda_q$. We take p, q such that $\phi_{pq} \neq 0$. Then by (4.29) we have

$$\beta^2\lambda_p - (\lambda_p - \gamma)((\lambda_p)^2 - \alpha\lambda_p - c) = 0. \tag{5.16}$$

Furthermore, from $(q3p)$, (4.28) and (4.16) we have

$$(\lambda_p)_1(\lambda_p - \delta) = 0.$$

It follows from (4.16) and Lemma 4 that $(\lambda_p)_1 \neq 0$. Hence we have $\lambda_p = \delta$, which, together with (3.8), (3.11) and (5.16), implies $\alpha + \delta = 0$. Therefore α and δ are constants, which is a contradiction. □

Taking account of the coefficient of θ_3 in the exterior derivative of equation $\alpha + \gamma - \delta = 0$ and making use of (4.19), (4.20), (4.23) and Lemma 5, we have

$$\alpha(\gamma - \delta)(\gamma(\alpha - \delta)) + \beta^2(\alpha^2 + \alpha\delta - 3\delta^2) = 0. \tag{5.17}$$

Again taking account of the coefficient of θ_1 in the exterior derivative of (5.17) and taking account of (3.2) and (5.3), we have

$$\alpha^4 + 4\alpha^2 c - 2\delta^2 c + 3c^2 + \beta^2(\alpha^2 + 3\delta^2) = 0. \tag{5.18}$$

Eliminating β from (5.17) and (5.18), we have a polynomial $4\alpha^4 + 7c\alpha^2 + 4c = 0$ of degree four with respect to α. This shows that α is a constant, which contradicts Lemma 4. Consequently we proved $\beta = 0$.

Since (2.6) and $\beta = 0$, we see that α is a constant in M (see [6]). Thus from (3.1) our assumption $\Xi_{ij;1} = 0$ is equivalent to $\alpha h_{ij;1} = 0$. Put $j = 1$ in (2.3). Then by above equation we have $\alpha h_{i1;k} = -c\alpha\phi_{ik}$. Therefore since (2.1) and $d\xi_i = 0$, we have

$$\alpha \sum_{k,l} h_{ik}\phi_{lk}h_{kj} + \alpha^2 \sum_k \phi_{ki}h_{kj} = -\alpha h_{i1;j} = c\alpha\phi_{ij},$$

which implies that $\alpha^2(\phi H - H\phi) = 0$. Owing to Okumura's work or Montiel and Romero's work stated in Introduction, we complete the proof of our Main Theorem.

Remark 3. In previous papers [4] and [5], we classified real hypersurfaces which satisfy $\nabla_\xi R_\xi = 0$ and some several conditions in $M_n(c), c \neq 0, n \geq 3$. Combining these results we have the following:

Let M be a real hypersurface in a complex space form $M_n(c), c \neq 0, n \geq 3$ which satisfies $\nabla_\xi R_\xi = 0$. Then the following conditions are mutually equivalent.

(i) M is locally congruent to type A.
(ii) $R_\xi S = SR_\xi$.
(iii) $R_\xi \phi S = S\phi R_\xi$.
(iv) $R_\xi \phi S = R_\xi S\phi$.

Remark 4. Recently the second author extended Main Theorem to the case $n = 2$ [8].

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