TRICOMI PROBLEM FOR THE ELLIPTIC-HYPERBOLIC EQUATION OF THE SECOND KIND

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Abstract. We prove the uniqueness solvability of the Tricomi problem for the elliptic – hyperbolical equation of the second type by using a new representation of the solution in the generalized class R.

1. Introduction

We consider the equation

$$(1.1) \qquad y^m U_{xx} + U_{yy} + aU_x + bU_y + cU = 0, \qquad m > 0$$

which coefficients are given continuous functions in finite part of the plane of the variables x, y. Line $y = 0$ is a line of parabolic degeneration [3]. Let D be a finite domain, which is bounded by section Γ_0 of the axis $y = 0$ and by arc Γ_1 in the half-plane $y > 0$. In the domain D equation (1.1) is an equation of the elliptic type. It was proved that for this equation the classic statement of the Dirichlet problem in the domain D is correct in case $a \equiv b \equiv c \equiv 0$ (F. Tricomi [25], Holmgren [13], S. Gellerstedt [7], [8], P. Germain, R. Bader [11]). Also the Dirichlet problem was studied for the equation (1.1) with coefficients a, b, c in the domain D(K. I. Babenko [1]). Line $y = 0$ is a line of parabolic degeneration for the equation

$$(1.2) \qquad U_{xx} + y^m U_{yy} + aU_x + bU_y + cU = 0, \qquad m > 0$$

which coefficients are given continuous functions in finite part of the plane of the variables x, y. In each point of the line $y = 0$ the characteristic direction coincidence with this line. Let us mean the domain, which is bounded by section Γ_0 of the axis $y = 0$ and by the arc Γ_1 in the half-plane $y > 0$ as D. For the equation (1.2) in the domain D the usual statement of the Dirichlet problem is not always possible. This problem is possible if $C \leqslant 0$ and: (a) for $m < 1$; (b) for $m = 1$, if $b(x, 0) < 1$; (c) for $1 < m < 2$, if $b(x, 0) \leqslant 0$; (d) for $m \geqslant 2$, if $b(x, 0) < 0$.

And in additional hypotheses if $C \leqslant 0$: (a) $b(x, 0) \geqslant 1$ for $m = 1$; (b) $b(x, 0) > 0$ for $1 < m < 2$; (c) $b(x, 0) \geqslant 0$ for $m \geqslant 2$. The unique possible problem in this case is following: find the regular solution of the equation (1.2) in the domain D, continuous in the closed domain $\overline{D}$ and designated the given value only on the arc Γ_1 (M. V. Keldish [16]).

The problem of existence and uniqueness of the bounded solution $U(x, y)$ of the equation (1.2) in the domain D and satisfying following condition on the Γ_1

$$\frac{\partial U}{\partial N} + AU = \varphi,$$

where N is given direction, which forms an acute angle with interior normal Γ_1, A and φ are given functions (O.A. Oleinik [20]). In case when the Dirichlet problem for the equation (1.2) is not possible in the domain D, it is natural to substitute the condition of boundedness $\lim\limits_{y \to 0} U(x, y)$ to the condition

$$\lim\limits_{y \to 0} \varphi(x, y) U(x, y),$$

where $\varphi(x, y)$ is given function, which $\lim\limits_{y \to 0} \varphi(x, y) = 0$. But in this statement of the Dirichlet problem is still not researched. The problem of the Poincare type in the domain D for the equation (1.1) and (1.2) in case when $a \equiv b \equiv c \equiv 0$ is still not researched.

Line $y = 0$ is a line of parabolic degeneration for the equation

(1.3) $y^m U_{xx} - U_{yy} + aU_x + bU_y + cU = 0, \qquad m > 0$

which coefficients are given continuous functions in finite part of the plane of the variables x, y. In the half-plane $y > 0$ equation (1.3) is equation of hyperbolic type. Let D is domain in the half-plane $y > 0$ bounded by section AB of the degeneration line and by characteristics AC and CB of the equation (1.3). The Cauchy problem for the equation

(1.3) in the domain D with initial data on AB

(1.4) $U(x,0) = \tau(x), \qquad U_y(x,0) = \nu(x)$

where $\tau(x)$ and $\nu(x)$ are given functions is studied in many articles (F. Tricomi [25], S. Gellerstedt [9], F. I. Frankl [6], I. S. Beryezin [2], M. I. Protter [21], G. Helwig [12] and other). The Cauchy problem for the equation (1.3) with initial data (1.4) is not correct if $m > 2$ [2]. But if we put additional conditions for example

$$\lim_{y\to 0} y^{1-\frac{m}{2}} a(x,y) = 0, \quad m > 0,$$

this problem is correct [21]. A.V. Bicadze showed non correctness of the Cauchy problem for the equation (1.3) and

$$U_{xx} - y^m U_{yy} + aU_x + bU_y + cU = 0, \quad m > 0$$

and offered to research this problem with modified initial data

$$\lim_{y\to 0} \varphi(x,y)U(x,y) = \tau(x), \quad \lim_{y\to 0} \psi(x,y)U_y(x,y) = \nu(x)$$

$$\lim_{y\to 0} \varphi(x,y) = 0, \quad \lim_{y\to 0} \psi(x,y) = 0$$

and with non full initial data(i.e. without one of conditions (1.4)). In [10] was proved that Cauchy-Goursat problem for the equation (1.3) has unique solution. A.V. Bicadze [4] offered to research Cauchy problem with modified initial data for the following equation
(1.5)
$$y^m U_{yy} - U_{xx} + a(x,y)U_x + b(x,y)U_y + c(x,y)U = 0, \quad 0 < m < 2, \quad y > 0$$

which is different from the equation (1.3). The Cauchy problem with modified initial data for the equation (1.5) is researched in [5, 24]. In particular, in [24] the equation

(1.6) $L_\alpha U \equiv yU_{yy} + U_{xx} + \alpha U_y = 0$

was considered in the domain D_2 in half-plane $y < 0$ bounded by characteristics of the equation (1.6)

$$AC : x - 2\sqrt{-y} = 0, \qquad BC : x + 2\sqrt{-y} = 1, \quad AB : y = 0.$$

All negative values of α, except integer were considered. In the domain D_2 we consider the modified Cauchy problem with initial data on the degeneration line:

(1.7) $U_\alpha(x,0) = \tau(x),$

$$(1.8) \qquad \lim_{y \to -0} (-y)^\alpha [U_\alpha - A_n^-(\tau)]_y' = \nu(x)$$

where $\tau(x)$, $\nu(x)$ are given functions, $A_n^-(\tau)-$ known operator (defined shortly below), moreover $\tau(x) \in C^{(2(n+1))}[0;1]$, $\nu(x) \in C^{(2)}[0;1]$. The solution of this problem in characteristic variables is defined by the formula

(1.9)

$$U_\alpha(\xi, \eta)$$

$$= \gamma_1 \sum_{k=0}^n N_k(\alpha, n, \delta)(\eta - \xi)^{-2\delta - 1} 4^{-2k} \int_\xi^\eta \tau^{(2k)}(\lambda)(\lambda - \xi)^{k+\delta}(\eta - \lambda)^{k+\delta} d\lambda$$

$$-(-1)^n \gamma_2 4^{2(\alpha-1)} \int_\xi^\eta \nu(\lambda)(\lambda - \xi)^{1/2-\alpha}(\eta - \lambda)^{1/2-\alpha} d\lambda$$

$$\equiv A_n^-(\tau) - (-1)^n \gamma_2 4^{2(\alpha-1)} \int_\xi^\eta \nu(\lambda)(\lambda - \xi)^{1/2-\alpha}(\eta - \lambda)^{1/2-\alpha} d\lambda,$$

where

$$\gamma_1 = \frac{\Gamma(2 + 2\delta)}{\Gamma^2(1 + \delta)}, \quad \gamma_2 = \frac{\Gamma(1 - 2\alpha)}{(1 - \alpha)\Gamma^2(1/2 - \alpha)},$$

$$N_k(\alpha, n, \delta) = \frac{2^{2k} C_n^k \Gamma(1 + \delta)}{\Gamma(1 + \delta + k) \prod_{s=0}^{k-1} (\alpha + s)},$$

$$\delta = \alpha + n - \frac{3}{2}, \ \alpha = -n + \alpha_0, \ 0 < \alpha_0 < 1/2, \ 1/2 < \alpha_0 < 1, \ n = 0, 1, 2, ...,$$

DEFINITION. [15] The function $U(\xi, \eta)$ is called a generalized solution of the Cauchy problem for the equation (1.6) in the domain D_2 from the class R, if it could be represented in the form (1.9) and

$$(1.10) \qquad \tau(x) = \int_0^x (x - t)^{-2\beta} T(t) dt,$$

here $\nu(x)$ and $T(x)$ are continuous on $[0;1]$ functions, $-2\beta = 2n - 2\delta - 2$.

If $n = 0$, $0 < \alpha_0 < 1/2$ the representation of the generalized solution from the class R was defined by M.M. Smirnov [23].

In the present paper we obtain a new representation of a generalized solution of the class R for all integer nonnegative values of α. Derivatives of $\tau(x)$ are absent in this representation and this provides an essential simplification to investigate the Tricomi and Gellerstedt problems for various equations of mixed type. It should be noted that this representation is based on some new relations of hypergeometric functions.

2. Representation of a generalised solution of the class R

Consider (1.10). It follows immediately from it

$$(2.1) \qquad \tau^{(2k)}(x) = \prod_{l=0}^{2k-1}(2n - 2\delta - 2 - l) \int_0^x T(t)(x - t)^{2n-2\delta-2-2k}dt.$$

Substituting (2.1) in (1.9) we have

$$(2.2) \qquad U_\alpha(\xi, \eta) = \gamma_1(\eta - \xi)^{-2\delta-1}J_1 - J_2,$$

where

$$J_1 = \int_0^\xi I_1(\xi; \eta; \zeta)T(\zeta)d\zeta + \int_\xi^\eta I_2(\xi; \eta; \zeta)T(\zeta)d\zeta,$$

$$J_2 = (-1)^n\gamma_2 4^{2(\alpha-1)} \int_\xi^\eta \nu(t)(t - \xi)^{1/2-\alpha}(\eta - t)^{1/2-\alpha}dt,$$

and also

$$(2.3)\ I_1(\xi; \eta; \zeta) = \sum_{k=0}^n N_k(\alpha; n; \delta)4^{-2k}\prod_{l=0}^{2k-1}(2n - 2\delta - 2 - l)$$

$$\times \int_\xi^\eta (t - \xi)^{k+\delta}(\eta - t)^{k+\delta}(t - \zeta)^{2n-2\delta-2-2k}dt$$

$$(2.4) \quad I_2(\xi;\eta;\zeta) = \sum_{k=0}^{n} N_k(\alpha;n;\delta)4^{-2k}\prod_{l=0}^{2k-1}(2n-2\delta-2-l)$$

$$\times \int_{\xi}^{\eta}(t-\xi)^{k+\delta}(\eta-t)^{k+\delta}(t-\zeta)^{2n-2\delta-2-2k}dt.$$

Integrals in (2.3) and (2.4) could be expressed through hypergeometric functions

$$I_1(\xi;\eta;\zeta) = (\eta-\xi)^{2\delta+1}(\eta-\zeta)^{2n-2\delta-2}\sum_{k=0}^{n} N_k(\alpha;n;\delta)4^{-2k}(2n-2\delta-2k-1)_{2k}$$

$$\times\frac{\Gamma^2(k+\delta+1)}{\Gamma(2k+2\delta+2)}Z^{2k}F(k+\delta+1,2\delta+2+2k-2n,2\delta+2k+2;Z), \quad Z=\frac{\eta-\xi}{\eta-\zeta},$$

$$I_2(\xi;\eta;\zeta) = (\eta-\xi)^{n-\delta-1}(\eta-\zeta)^{n+\delta}\sum_{k=0}^{n} N_k(\alpha;n;\delta)4^{-2k}(2\beta)_{2k}$$

$$\times\frac{\Gamma(k+\delta+1)\Gamma(2n-2\delta-1-2k)}{\Gamma(2n-\delta-k)}$$

$$\times Z_1^{-k+n}F(k+\delta+1,-k-\delta,2n-\delta-k;Z_1),$$

$Z_1 = \frac{\eta-\zeta}{\eta-\xi}$.

LEMMA. *The following identity*

$$(2.5)\qquad [2(s-k)+1][2(s-k)+3]...[2(s-k)+(2l-1)]$$
$$= \widehat{P}_l(s) + \widehat{P}_{l-1}(s)k + ... + \widehat{P}_0(s)k(k-1).(k-l+1)$$

is correct. Here $\widehat{P}_i(s), i=\overline{0.l}$, are specific polynomials of the degree i. This lemma is easily checked by induction.

THEOREM 1. *The following identities are valid:*

$$(2.6)\ \sum_{k=0}^{n}\frac{\Gamma(2\delta+2)\Gamma^2(k+\delta+1)}{\Gamma^2(\delta+1)\Gamma(2k+2\delta+2)}N_k(\alpha,n,\delta)4^{-2k}(2\beta)_{2k}Z^{2k}$$

$$\times F(k+\delta+1,2\delta+2+2k-2n,2\delta+2k+2;Z) = (1-Z)^{-\beta},$$

$$(2.7) \qquad \sum_{k=0}^{n} \frac{\Gamma(-\delta)}{\Gamma^2(1+\delta)\Gamma(-2\delta-1)} N_k(\alpha, n, \delta) 4^{-2k}(2\beta)_{2k} Z_1^{2k}$$

$$\times \frac{\Gamma(k+\delta+1)\Gamma(2n-2\delta-2k-1)}{\Gamma(-k-\delta+2n)} Z_1^{-k+n}$$

$$\times F(k+\delta+1, -k-\delta, -k-\delta+2n; Z_1)$$

$$= (-1)^n (1-Z_1)^{-\beta}.$$

We will prove (2.6) in full details. We write down hypergeometric functions in the left-hand side of (2.6) in the form of series. First, decompose the right-hand side of (2.6) into power series:

$$(2.8) \qquad \sum_{k=0}^{n} \frac{\Gamma(2\delta+2)\Gamma^2(k+\delta+1)}{\Gamma^2(\delta+1)\Gamma(2k+2\delta+2)} N_k(\alpha, n, \delta) 4^{-2k}(2\beta)_{2k} Z^{2k}$$

$$\times \sum_{m=0}^{\infty} \frac{(k+\delta+1)_m (2\beta+2k)_m}{(2\beta+2n+2k)_m m!} Z^m$$

$$= 1 + \beta Z + \frac{\beta(\beta+1)}{2!} z^2 + ... + \frac{\beta(\beta+1)...(\beta+l-1)}{l!} Z^l + ...$$

It is not difficult to check that coefficients in the left and right members of the power series (2.8) at the same degrees Z^l and $[l/2] \leqslant n$ are calculated by the formula

$$(2.9) \qquad \sum_{k=0}^{[l/2]} \frac{\Gamma(2\delta+2)\Gamma^2(k+\delta+1)}{\Gamma^2(\delta+1)\Gamma(2k+2\delta+2)} N_k(\alpha, n, \delta) 4^{-2k}(2\beta)_{2k}$$

$$\times \frac{(k+\delta+1)_{l-2k}(2\beta+2k)_{l-2k}}{(2\beta+2n+2k)_{l-2k}(l-2k)!} = \frac{(\beta)_l}{l!}.$$

Let us show that (2.9) is the identity for any natural n.
For this, we need to consider the following cases.
(case A.)
$l = 2s$. After not complicated transformations (2.9) can be rewrite as

$$(2.10) \qquad \sum_{k=0}^{s} 2^{-k} C_n^k \frac{(\beta)_s (\delta+1+k+s+1)_{s-2k-1}[2\beta+2k]_{s-k}}{[2\delta+2]_s (2s-2k)!} = \frac{(\beta)_{2s}}{(2s)!}.$$

Here $[a]_k = (a+1)(a+3)...(a+2k-1)$.

If, now, we transform (2.10) and substitute $\delta = \beta + n - 1$, we obtain

$$(2.11) \quad P_{2s}(\beta) \equiv \sum_{k=0}^{s} 2^{-k} C_n^k \frac{(n + \beta + s)_{s-k}[2\beta + 2k + 1]_{s-k}}{(2s - 2k)!}$$

$$= \frac{(\beta + s)_s [2n + 2\beta]_s}{(2s)!}.$$

We can consider the left and right members of (2.11) as polynomials with respect to β. Two polynomials are equal if coefficients of the higher degree of β equal and the values of the polynomials coincide in various points, the amount of which is equal to the power of polynomials [19]. The polynomial in the right member equals zero at those values of β for which one of the factors equals zero. Let us next show that for these β the left-hand side of (2.11) also becomes zero. We need to show that for $\beta = -s$

$$(2.12) \qquad\qquad\qquad P_{2s}(-s) \equiv 0.$$

It is correct the formula [22]

$$(2.13) \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{a - m}{k} \binom{a}{k}^{-1} = 0, \quad m < n.$$

Rewrite (2.13) in the form

$$(2.14) \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(a - k)!}{(a - m - k)!} = 0, \quad m < n.$$

Set in (2.13) $a = n+s-1, m = a-n$. Since $s-1 < s$, the condition $m < n$ holds: $a-k = n+s-1-k$, $(a-k)! = (n+s-k-1)!$, $(a-k-m)! = (n-k)!$ (2.14) follows from (2.12). We need to show that at $\beta = -s - 1$

$$(2.15) \qquad\qquad\qquad P_{2s}(-s - 1) \equiv 0.$$

Let us show that

$$(2.16) \qquad\qquad \sum_{k=0}^{s} (-1)^k C_n^k \frac{(n + s - k - 2)!}{(n - k)!} = 0$$

and

$$(2.17) \qquad\qquad \sum_{k=0}^{s} (-1)^k C_n^k \frac{(n + s - k - 2)!}{(s - k)!} k = 0.$$

We obtain (2.15) from (2.18) and (2.19). Set in (2.14) $a = n + s - 2, m = s - 2$. Then $(a - k)! = (n + s - k - 2)!, (a - m - k)! = (n - k)!$. Since $s - 2 < s$, the condition $m < n$ holds and (2.14) implies (2.16). Substitute the summation variable in the left member of (2.16) by the formula $k' = k - 1$ and denote $s - 1 = s'$. Then (2.17) follows from (2.14) where it is sufficient to set $a = n + s' - 2, m = s' - 1$. It is necessary to show that $P_{2s}(-s - 2) \equiv 0$ at $\beta = -s - 2$. Represent $P_{2s}(-s - 2)$ in the form

$$P_{2s}(-s - 2)$$
$$= \frac{(-1)^s n!}{3(n - 3)! 2^s} \sum_{k=0}^{s} (-1)^k \frac{(n + s - k - 3)! [2(s - k) + 1][2(s - k) + 3]}{k!(n - k)!(s - k)!}.$$

We have on the base of lemma

$$(-1)^s \frac{3(n - 3)! 2^s}{n!} P_{2s}(-s - 2) = \widehat{P_2}(s)\sigma_{21} + \widehat{P_1}(s)\sigma_{22} + \widehat{P_0}(s)\sigma_{23},$$

where

$$\widehat{P_2}(s) = (2s + 1)(2s + 3), \ \widehat{P_1}(s) = -8s - 4, \ \widehat{P_0}(s) = 4,$$

$$\sigma_{21} = \sum_{k=0}^{s} (-1)^k \frac{(n + s - k - 3)!}{k!(n - k)!(s - k)!},$$

$$\sigma_{22} = \sum_{k=0}^{s} (-1)^k \frac{(n + s - k - 3)!}{k!(n - k)!(s - k)!} k,$$

$$\sigma_{23} = \sum_{k=0}^{s} (-1)^k \frac{(n + s - k - 3)!}{k!(n - k)!(s - k)!} k(k - 1).$$

It is not difficult to show that $\sigma_{21} = \sigma_{22} = \sigma_{23} \equiv 0$. It means that $P_{2s}(-s - 2) \equiv 0$. (2.6) can be analogously proved at $\beta = -s - 3$, $\beta = -s - 4$... The proof of the such series at $\beta = -2s + 1$ is reduced with application of lemma to the proof of identical vanishing of all coefficients of polynomials $P_{2s-i}(s), i = \overline{3, 2s}$. We have

$$\sum_{k=0}^{s}(-1)^k\frac{1}{k!(s-k)!} = \frac{1}{s!}\sum_{k=0}^{s}(-1)^k C_s^k = 0,$$

$$\sum_{k=0}^{s}(-1)^k\frac{k}{k!(s-k)!} = -\frac{1}{s'!}\sum_{k=0}^{s'}(-1)^{k'} C_{s'}^{k'} = 0,$$

$$\sum_{k=0}^{s}(-1)^k\frac{k(k-1)}{k!(s-k)!} = 0,$$

and i.e.,

$$\sum_{k=0}^{s}(-1)^k\frac{k(k-1)...(k-s+2)}{k!(s-k)!} = \sum_{k=s-1}^{s}(-1)^k\frac{1}{(k-s+1)!(s-k)!}$$

$$= (-1)^{s-1}\frac{1}{0!(s-s+1)!} + (-1)^s\frac{1}{1!0!} = 0.$$

Let us show that both the right member and the left one of (2.11) become zero at $2\beta + 2n + 2s - 1 = 0$. Vanishing of the right-hand side is obvious, and the left member becomes the form

$$\sum_{k=0}^{s}C_n^k 2^{-k}\frac{\frac{1}{2}\frac{3}{2}...\frac{2s-2k-1}{2}(-2n)(-2n-2)...(-2n-2s+2k+2)}{(2s-2k)!}$$

$$\equiv 2^s P_{2s}(-s) \equiv 0.$$

We can prove analogously that the left-hand and right-hand sides of (2.11) become zero identically at $2n + 2\beta + 2s - 3, ..., 2\beta + 2n + 1 = 0$. (Case B.) Consider now the case of odd l. Let $l = 2s + 1$. Then it is necessary to prove the following identity

$$\sum_{k=0}^{s}\frac{\Gamma(2\delta+2)\Gamma^2(k+\delta+1)}{\Gamma^2(\delta+1)\Gamma(2k+2\delta+2)}N_k(\alpha,n,\delta)4^{-2k}\prod_{l=0}^{2k-1}(-2\beta-2s-1)$$

$$\times\frac{(k+\delta+1)_{2s-2k+1}(2\beta+2k)_{2s-2k+1}}{(2\beta+2n+2k)_{2s-2k+1}(2s-2k+1)!} \equiv \frac{\beta(\beta+1)...(\beta+2s)}{(2s+1)!}$$

or

$$(2.18) \qquad \sum_{k=0}^{n} C_n^k \frac{2^{-k}(n+\beta+s+1)_{s-2k+1}[2\beta+2k+1]_{2s-2k-2}}{(2s-2k+1)!}$$

$$= \frac{(\beta+s+1)_{s-1}(2\beta+2n+1)_{2s-2}}{(2s+1)!}.$$

(2.18) can be proved analogously to (2.11). Investigation of coefficients at the degrees of Z^l and $[l/2] > n$ in (2.3) is reduced to the proof of the identity

$$(2.19) \qquad \sum_{k=0}^{n} \frac{\Gamma(2\delta+2)\Gamma^2(k+\delta+1)}{\Gamma^2(\delta+1)\Gamma(2k+2\delta+1)} N_k(\alpha,n,\delta) 4^{-2k} \prod_{l=0}^{2k-1}(-2\beta-l)$$

$$\times \frac{(\delta+k+1)_{l-2k}(2\beta+2k)_{l-2k}}{(2\beta+2k+2n)_{l-2k}(l-2k)!} = \frac{(\beta)_l}{l!}.$$

Proof of (2.19) is divided for l to two cases: even ($l = 2s$) and odd ($l = 2s+1$) ones. For choosing m, one should change places of s and n, the further proof doesn't differ from the case of $[l/2] \leqslant n$. (2.7) is proved by the same method as (2.6). The role of (2.14) plays the formula[22]

$$\sum_{k=0}^{n}(-1)^k C_n^k C_{a+bk}^m = 0, \quad 0 < m < n.$$

Theorem is proved. Some special cases of the proved theorem are obtained in [17]. On the base of the theorem formulated above, let us write expressions $I_1(\xi,\eta,\zeta)$ and $I_2(\xi,\eta,\zeta)$ in the form

$$(2.20) \qquad I_1(\xi,\eta,\zeta) = \frac{\Gamma^2(\delta+1)}{\Gamma(2\delta+1)}(\eta-\xi)^{2\delta+1}(\eta-\zeta)^{-\beta}(\xi-\zeta)^{-\beta}$$

(2.21)
$$I_2(\xi,\eta,\zeta) = \frac{\Gamma(\delta+1)\Gamma(-2\delta-1)}{\Gamma(-\delta)}(-1)^n(\eta-\xi)^{2\delta+1}(\eta-\zeta)^{-\beta}(\zeta-\xi)^{-\beta}.$$

Substituting (2.20), (2.21) into (1.1) we obtain the representation of a generalized solution of the class R [24]:
(2.22)
$$U(\xi,\eta) = \int_{0}^{\xi}(\eta-\zeta)^{-\beta}(\xi-\zeta)^{-\beta}T(\zeta)d\zeta + \int_{\xi}^{\eta}(\eta-\zeta)^{-\beta}(\zeta-\xi)^{-\beta}N(\zeta)d\zeta$$

where

$$N(\zeta) = \frac{1}{2\cos\pi\beta}T(\zeta) - (-1)^n 2^{4\beta-2}\gamma_2\nu(\zeta).$$

3. The modified Tricomi problem for the elliptic-hyperbolic equation of the second kind

PROBLEM. *Find a function $U(x,y)$, defined in the domain D which is a generalized solution in the following sense:*
a) $U(x,y) \in C(\overline{D})$.
b) $U(x,y)$ *is a generalized solution in the class R in D_- and twice continuously differentiable. $U(x,y)$ satisfies the following equation in the domain D^+*

(3.1) $$L(U) \equiv yU_{yy} + U_{xx} + \alpha U_y = 0$$

c) *Along the line of degeneration AB of the equation (3.1) the function $U(x,y)$ satisfies the following gluing condition:*

$$(-1)^p \lim(-y)^\alpha[U - A_n^-(\tau)]_y' = \lim y^\alpha[U_y' + A_n^+(U)] = \nu(x),$$

where $U(x,-0) = U(x,+0) = \tau(x)$ follows from condition a), $p = [\frac{3}{2}-\alpha]$,

$$A_n^-(\tau) = \gamma_1 \sum_{k=0}^{p} N_k(p,\alpha,\delta)(-y)^k \int_0^1 \tau^{(2k)}(\lambda)[t(1-t)]^{k+\delta}dt,$$

$$\lambda = x - 2\sqrt{-y}(1-2t),$$
$$\gamma_1 = \frac{\Gamma(2+2\delta)}{\Gamma^2(1+\delta)},$$

$$\delta = \begin{cases} \alpha_0 - 3/2, & for\ 1/2 < \alpha_0 < 1, \\ \alpha_0 - 1/2, & for\ \ 0 < \alpha_0 < 1/2 \end{cases}$$

$$N_k(p,\alpha,\delta) = \frac{2^{2k}C_p^k\Gamma(1+\delta)}{\Gamma(1+\delta+k)\prod_{s=0}^{k-1}(\alpha+s)}$$

$$A(U)_n^+ = \sum_{i=1}^{n} \delta_i y^{i-1}\frac{\partial^{2i}U}{\partial x^{2i}}, \quad \delta_i - const;$$

d) $U(x, y)$ satisfies the boundary conditions

(3.2) $$U\mid_\sigma = f(s), \quad 0 \le s \le l,$$

(3.3) $$U\mid_{AC} = \varphi(x), \quad 0 \le x \le 1/2,$$

where f and φ are given continuous functions.

Reducing of the problem T_n to singular integral equation.
The equation conjugate to equation (3.1), is

(3.4) $$L^*(v) \equiv yv_{yy} + v_{xx} + (2 - \alpha)v_y = 0$$

Let σ concede with normal contour $\sigma_0 : x(1 - x) = 4y$.

The properties of Grinn function $G(\xi, t; x, y)$ of the problem T_n was given in work [18], namely,

(3.5) $$G(\xi, t; x, y) = q(\xi, t; x, y) - (4R^2)^{-\beta}q(\xi - 1/2, t; \overline{x}, \overline{y}),$$

where

$$\beta = \alpha - 1/2, \quad 4R^2 = (2x - 1)^2 + 16y$$

$$q(\xi, t; x, y) = kt^{\alpha-1}r_1^{-2\beta}F(\beta, \beta, 2\beta; \frac{16\sqrt{yt}}{r_1^2})$$

$$r_1^2 = (\xi - x)^2 + 4(\sqrt{t} - \sqrt{y})^2, \quad \overline{x} = \frac{x - 0.5}{4R^2}, \quad \overline{y} = \frac{y}{(4R^2)^2}.$$

Properties of Grin function $G(\xi, t; x, y)$ are following:
1) $L_{(x,y)}(G) = 0$ and $L^*_{\xi,t}(G) = 0$ for $(x, y) \ne (\xi, t)$;
2) $G(\xi, t; x, y) = 0$ if $(x, y) \in \sigma_0$ or $(\xi, t) \in \sigma_0$;
3) $\lim_{t \to +0} \left[t\overline{G}_t + (1 - \alpha)\overline{G} + \sum_{i=1}^{n} \delta_i t^i \frac{\partial^{2i}\overline{G}}{\partial\xi^{2i}} \right] = 0,$

where

$$\overline{G}(\xi, t; x) = G(\xi, t; x, 0).$$

Let D_h^+ denote a part of domain D^+ for $y \ge h > 0$

$$D_h^+ = D^+ \cap \{y \ge h > 0\}.$$

The following identity is valid

(3.6) $$\iint\limits_{D_h^+} [vL(U) - UL^*(v)]d\xi dt$$

$$= \int\limits_{A'B'U\sigma_0} [vU_\xi - Uv_\xi]dt - [tvU_t - tUv_t - (1 - \alpha)Uv]d\xi.$$

Let suppose that in (3.6) U — a solution of the equation (3.1) in D_h^+, in the capacity of v we take Grin function for $y = 0$, i. e.

$$v = \overline{G}(\xi, t; x) = kt^{\alpha-1}\left[(\xi - x)^2 + 4t\right]^{-\beta}$$

$$- k(2x-1)^{-2\beta}t^{\alpha-1}\left[\left(\xi - \frac{1}{2} - \frac{1}{2(2x-1)^2}\right)^2 + 4t\right]^{-\beta}.$$

Then (3.6) can be rewrited as

(3.7)
$$\int\limits_{A'B'} U\left[t\overline{G}_t + (1-\alpha)\overline{G} + \sum_{i=1}^{n}\delta_i t^i\frac{\partial^{2i}\overline{G}}{\partial\xi^{2i}}\right]d\xi$$

$$- \int\limits_{A'B'} t^{1-\alpha}\overline{G}t^\alpha[U_t + A_n^+(U)]d\xi$$

$$- \sum_{i=1}^{n}\delta_i t^i\sum_{k=0}^{2i-1}(-1)^k\frac{\partial^k U}{\partial\xi^k}\frac{\partial^{2i-1-k}\overline{G}}{\partial\xi^{2i-1-k}}\Bigg|_{A'}^{B'}$$

$$+ \int\limits_{\sigma_0} U\left\{[t\overline{G}_t + (1-\alpha)\overline{G}]d\xi - \overline{G}_\xi dt\right\} = 0.$$

Suppose that

$$\frac{\partial^k U}{\partial\xi^k}\Big|_{(0,0)} = \frac{\partial^k U}{\partial\xi^k}\Big|_{(1,0)} = 0, \quad k = \overline{0, 2n-1},$$

and

(3.8) $f(s) = f(\xi) = [\xi(1-\xi)]^{2n-1}f_1(\xi), \quad f_1(\xi) \in C[0,1]$

pass to the limit for $h \to 0$, we get the principal relation between $\tau(x)$ and $\nu(x)$ from D^+

(3.9) $$\tau(x) = k\int\limits_0^1 \nu(t)[|t-x|^{-2\beta} - (x+t-2xt)^{-2\beta}]dt + F(x)$$

here

$$F(x) = 4^{1-\alpha}k\beta x(1-x)\int\limits_0^1 f_1(\xi)[\xi(1-\xi)]^{n+\alpha_0-2}[x^2 + \xi - 2\xi x]^{-\beta-1}d\xi.$$

In the domain D_- we have to use the representation of the generalized solution in the class R (2.22).

DEFINITION. The function $U_\alpha(x, y)$, defined by the formula (1.9), is called a generalized solution of the equation (3.1) in the class R in the domain D_-, if $\nu(x) \in C[0, 1]$ and function $\tau(x)$ ia represented as

$$(3.10) \qquad \tau(x) = \int_0^x (x - t)^{-2\beta} T(t)dt,$$

where $T(t)$ is some continuous function on $[0; 1]$.

The generalized solution of the equation (3.1) in the class R in the D_- as it was proved above, is represented in the form
(3.11)

$$U(\xi, \eta) = \int_0^\xi (\eta - \zeta)^{-\beta}(\xi - \zeta)^{-\beta} T(\zeta)d\zeta + \int_\xi^\eta (\eta - \zeta)^{-\beta}(\zeta - \xi)^{-\beta} N(\zeta)d\zeta,$$

where

$$N(\zeta) = \frac{1}{2\cos\pi\beta} T(\zeta) - (-1)^n 2^{4\beta - 2}\gamma_2\nu(\zeta).$$

Let $U(x, y) \in R(D_-)$. Using the boundary conditions (3.3), we find another relation between $\tau(x)$ and $\nu(x)$, from D_-

$$(3.12) \qquad \tau(x) = \gamma_3 \int_0^x \nu(t)(x - t)^{-2\beta}dt + \Phi(x, \varphi),$$

where

$$\gamma_3 = (-1)^p 2\cos\beta\pi\gamma_2.$$

Except from (3.9) and (3.12) the function $\tau(x)$ we get for $\nu(x) \in H(\delta)$ the following singular integral equation, which is equivalent to the researched problem.

$$(3.13) \quad \nu(x) - \lambda \int_0^1 (\frac{t}{x})^{1-2\beta} \left[\frac{1}{t - x} - \frac{1}{x + t - 2xt} \right] \nu(t)dt = \Psi(x, f, \varphi);$$

here λ – const.

Performing conditions (3.8) and $\varphi(x) = x^{2n+\varepsilon}\varphi_1(x)$, $\varepsilon > 0$, $\varphi_1 \in C[0, \frac{1}{2}]$ it can be shown that $\Psi(x, f, \varphi) \in H(\theta)$, where

$$\theta = \begin{cases} 1 - 2\alpha_0, & 0 < \alpha_0 < \dfrac{1}{2} \\[2mm] 2 - 2\alpha_0, & \dfrac{1}{2} < \alpha < 1. \end{cases}$$

As it was shown in [15], the singular integral equation (3.13), has unique solution for $\nu(x) \in H(\theta)$, which reduce to infinity of order less then one on the ends of the section AB.

The uniquenessof the solution of the problem T_n is proved analogously as in work [18].

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