Korean J. Math. **19** (2011), No. 2, pp. 129–147

THE ZETA-DETERMINANTS OF HARMONIC OSCILLATORS ON $\mathbb{R}^2$

Kyounghwa Kim

Abstract. In this paper we discuss the zeta-determinants of harmonic oscillators having general quadratic potentials defined on $\mathbb{R}^2$. By using change of variables we reduce the harmonic oscillators having general quadratic potentials to the standard harmonic oscillators and compute their spectra and eigenfunctions. We then discuss their zeta functions and zeta-determinants. In some special cases we compute the zeta-determinants of harmonic oscillators concretely by using the Riemann zeta function, Hurwitz zeta function and Gamma function.

1. Introduction

The zeta-determinant of a differential operator is a global spectral invariant, which is a natural extension of the usual determinant of a linear map or a matrix acting on a finite dimensional vector space. The zeta-determinant is defined by the spectral zeta function associated to a differential operator, which plays an important role in geometry, topology and mathematical physics (cf. [2] and [3]). In this paper we discuss the zeta-determinants of harmonic oscillators having general quadratic potentials defined on $\mathbb{R}^2$. The harmonic oscillators are Laplacians equipped with quadratic potentials, which are used quite often in mathematics and physics including functional analysis and quantum mechanics. It is a well known fact that the harmonic oscillators have discrete point spectra. By using change of variables we reduce the harmonic oscillators with general quadratic potentials to the standard harmonic oscillators and compute their spectra and eigenfunctions. We then discuss the spectral zeta function consisting of the eigenvalues of the harmonic oscillators

Received January 25, 2011. Revised May 30, 2011. Accepted June 3, 2011.
2000 Mathematics Subject Classification: 58J52, 58Z05, 58J50.
Key words and phrases: zeta-determinant, harmonic oscillator, quadratic potential.

and discuss their zeta determinants. Finally we compute, in some special cases, the zeta-determinants concretely by using the Riemann zeta function, Hurwitz zeta function and Gamma function.

We begin with the definition of the *Harmonic oscillators*. The harmonic oscillators of one variable $H : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is defined by

$$H = -\frac{d^2}{dx^2} + a^2x^2 \quad (a \in \mathbb{R}^+), \tag{1.1}$$

where $\mathcal{S}(\mathbb{R})$ is the Schwartz space on $\mathbb{R}$. In Section 2 we are going to review some basic spectral properties of the harmonic oscillator H including their spectra and eigenfunctions. In Section 3 we are going to consider the harmonic oscillators of the form

$$H_n = \left(-\frac{\partial^2}{\partial x_1^2} + {a_1}^2 x_1^2\right) + \left(-\frac{\partial^2}{\partial x_2^2} + {a_2}^2 x_2^2\right) + \cdots + \left(-\frac{\partial^2}{\partial x_n} + {a_n}^2 x_n^2\right),$$

where $a_1, \cdots, a_n \in \mathbb{R}^+$ and discuss their spectra and eigenfunctions. We then discuss harmonic oscillators with general quadratic forms on $\mathbb{R}^n$. Using change of variables we reduce them to the standard harmonic oscillators and compute their spectra in terms of the eigenvalues of the symmetric matrices associated to the quadratic forms. Finally, in Section 4 we review the definition of the zeta-determinant and show how it generalizes the usual determinant of a linear map acting on a finite dimensional vector space. We then discuss the zeta-determinants of the harmonic oscillators defined on $\mathbb{R}^2$ of the following type

$$H(a,b) = -(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + (a^2x^2 + b^2y^2) \quad (a,\ b > 0).$$

When $b = a$ and $a = \frac{c}{2}$, $b = \frac{3c}{2}$, we compute the zeta-determinant of $H(a,b)$ explicitly by using the Riemann zeta function, Hurwitz zeta function and Gamma function We finally discuss the zeta-determinant of $H(a,b)$ when $a, b \in \mathbb{R}^+ - \{n^2 \mid n \in \mathbb{N}\}$ but so far we didn't find an explicit description of the zeta-determinant of $H(a,b)$.

2. Harmonic oscillators of one variable

In this section we are going to introduce harmonic oscillators of one variable acting on the Schwartz space and investigate their spectral structures. Using the materials in this section we are going to compute the eigenvalues and eigenfunctions of harmonic oscillators of n variables in the next section.

We first define a few concepts. Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space on $\mathbb{R}$. The *creation operator* A and *annihilation operator* $A^\dagger : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ are defined, for $a > 0$, by

$$A = ax - \frac{d}{dx}, \qquad A^\dagger = ax + \frac{d}{dx}.$$

Let H be the Harmonic oscillator defined in (1.1). The function $\psi_0 \in \mathcal{S}(\mathbb{R})$ is called the *ground state* of H if

$$A^\dagger \psi_0 = 0, \qquad \|\psi_0\| = 1.$$

For $k \geq 1$, we define a function $\psi_k \in \mathcal{S}(\mathbb{R})$ by

$$\psi_k = \frac{1}{(2ka)^{\frac{1}{2}}} A\psi_{k-1} \tag{2.1}$$

and call ψ_k the *excited state* of H. Then we have the following result, which are well known.

Lemma 2.1. *Let H be a Harmonic oscillator defined in (1.1) and $A, A^\dagger$ be the creation operator and annihilation operator, respectively. Then we have the following equalities.*
(1) $AA^\dagger = H - a$, $A^\dagger A = H + a$,
(2) $[A^\dagger, A] = 2a$,
(3) $[H, A] = 2aA, [H, A^\dagger] = -2aA^\dagger$, where $[A, B] = AB - BA$.

Proof. Let ψ be a function in $\mathcal{S}(\mathbb{R})$. Then

$$\begin{aligned} AA^\dagger(\psi) &= \left(ax - \frac{d}{dx}\right)\left(ax + \frac{d}{dx}\right)(\psi) \\ &= a^2x^2 + ax\frac{d}{dx}\psi - \frac{d}{dx}ax\psi - \frac{d^2}{dx^2}\psi \\ &= a^2x^2 + ax\frac{d}{dx}\psi - a\psi - ax\frac{d}{dx}\psi - \frac{d^2}{dx^2}\psi \\ &= \left(a^2x^2 - a - \frac{d^2}{dx^2}\right)(\psi) = (H-a)(\psi), \end{aligned}$$

which shows $AA^\dagger = (H - a)$. Similarly, we can show $A^\dagger A = (H + a)$, which completes the proof of the statement (1). Since $[A^\dagger, A] = A^\dagger A -$

$AA^\dagger$, we have $[A^\dagger, A] = H + a - (H - a) = 2a$, which shows the equality (2). Let ϕ be a function in $\mathcal{S}(\mathbb{R})$. Then

$$
\begin{aligned}
&(HA - AH)(\phi)\\
=&\left(a^2x^2 - \frac{d^2}{dx^2}\right)\left(ax - \frac{d}{dx}\right)(\phi) - \left(ax - \frac{d}{dx}\right)\left(a^2x^2 - \frac{d^2}{dx^2}\right)(\phi)\\
=&\left(a^2x^2 - \frac{d^2}{dx^2}\right)\left(ax\phi - \frac{d}{dx}\phi\right) - \left(ax - \frac{d}{dx}\right)\left(a^2x^2\phi - \frac{d^2}{dx^2}\phi\right)\\
=&- 2a\frac{d}{dx}\phi + 2a^2x\phi = 2a\left(ax - \frac{d}{dx}\right)(\phi) = 2aA(\phi),
\end{aligned}
$$

which shows $[H, A] = 2aA$. Similarly,

$$(HA^\dagger - A^\dagger H)(\phi) = -2a\frac{d}{dx}\phi - 2a^2x\phi = -2aA^\dagger(\phi),$$

which completes the proof of the lemma. □

Lemma 2.2. *Let $\psi_k \in \mathcal{S}(\mathbb{R})$ be the excited state of H defined in (2.1). Then ψ_k is the normalized eigenfunction of H such that $H\psi_k = (2k + 1)a\psi_k$ and $\|\psi_k\| = 1$ for all $k \geq 0$.*

Proof. To prove this theorem, we apply the mathematical induction and the statement (3) in Lemma 2.1. First we show that $H\psi_k = (2k + 1)a\psi_k$. For $k = 0$, by the statement (1) in Lemma 2.1., $AA^\dagger = H - a$, $AA^\dagger\psi_0 = (H - a)\psi_0$. Since $A^\dagger\psi_0 = 0$, we get $H\psi_0 = a\psi_0$. Now for $k = 1$,

$$
\begin{aligned}
H\psi_1 &= H\frac{1}{(2a)^{\frac{1}{2}}}A\psi_0 = \frac{1}{(2a)^{\frac{1}{2}}}HA\psi_0 = \frac{1}{(2a)^{\frac{1}{2}}}(AH + 2aA)\psi_0\\
&= \frac{1}{(2a)^{\frac{1}{2}}}(AH\psi_0 + 2aA\psi_0) = \frac{1}{(2a)^{\frac{1}{2}}}(aA\psi_0 + 2aA\psi_0)\\
&= \frac{1}{(2a)^{\frac{1}{2}}}3aA\psi_0 = 3a\psi_1.
\end{aligned}
$$

We now assume that it is true for $k = m - 1$. Then

$$
\begin{aligned}
H\psi_m &= H\frac{1}{(2ma)^{\frac{1}{2}}}A\psi_{m-1} = \frac{1}{(2ma)^{\frac{1}{2}}}HA\psi_{m-1} \\
&= \frac{1}{(2ma)^{\frac{1}{2}}}(AH+2aA)\psi_{m-1} = \frac{1}{(2ma)^{\frac{1}{2}}}(AH\psi_{m-1}+2aA\psi_{m-1}) \\
&= \frac{1}{(2ma)^{\frac{1}{2}}}\{(2m-1)aA\psi_{m-1}+2aA\psi_{m-1}\} \\
&= \frac{1}{(2ma)^{\frac{1}{2}}}(2m+1)aA\psi_{m-1} = (2m+1)a\psi_m.
\end{aligned}
$$

The induction shows that $H\psi_k = (2k+1)a\psi_k$ holds for all nonnegative integers k, and hence ψ_k is an eigenfunction of H with eigenvalue $(2k+1)a$. By the same way, we show that $\|\psi_k\| = 1$ for all $k \geq 0$. Indeed, it is true for $k = 0$ because of the definition of the ground state. If we assume that it is true for $k = m-1$, then

$$
\begin{aligned}
\|\psi_m\|^2 &= \langle\psi_m,\psi_m\rangle = \langle\frac{1}{(2ma)^{\frac{1}{2}}}A\psi_{m-1}, \frac{1}{(2ma)^{\frac{1}{2}}}A\psi_{m-1}\rangle \\
&= \frac{1}{2ma}\langle A\psi_{m-1}, A\psi_{m-1}\rangle = \frac{1}{2ma}\langle A^\dagger A\psi_{m-1}, \psi_{m-1}\rangle \\
&= \frac{1}{2ma}\langle (H+a)\psi_{m-1}, \psi_{m-1}\rangle = \frac{1}{2ma}\langle H\psi_{m-1}+a\psi_{m-1}, \psi_{m-1}\rangle \\
&= \frac{1}{2ma}\langle (2m-1)a\psi_{m-1}+a\psi_{m-1}, \psi_{m-1}\rangle \\
&= \frac{1}{2ma}\langle 2ma\psi_{m-1}, \psi_{m-1}\rangle = \langle\psi_{m-1},\psi_{m-1}\rangle = \|\psi_{m-1}\|^2 = 1.
\end{aligned}
$$

Hence by induction $\|\psi_k\| = 1$ for all nonnegative integers k, which completes the proof of the lemma. □

Remark : If we denote by $\mathcal{P} := \text{span} < \psi_i \mid i = 1,2,3,\cdots >$, it can be shown that $\mathcal{P}$ is dense in $L^2(\mathbb{R})$ and hence there is no more eigenvalues than we obtained in the above theorem. We refer to [1] for the proof of this fact.

Now we consider the eigenvalues and eigenfunctions of an n-dimensional harmonic oscillator.

Theorem 2.1. *Let $H_n : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ be an n-dimensional harmonic oscillator defined by*

$$H_n = \left(-\frac{\partial^2}{\partial x_1^2} + a_1 x_1^2\right) + \left(-\frac{\partial^2}{\partial x_2^2} + a_2 x_2^2\right) + \cdots + \left(-\frac{\partial^2}{\partial x_n} + a_n x_n^2\right),$$

where $a_1, \cdots, a_n \in \mathbb{R}^+$ and let $\psi_{k_i}(x_i)$ be the eigenfunction of $-\frac{\partial^2}{\partial x_i^2} + a_i x_i^2$ with the eigenvalue λ_{k_i}. Then the eigenfunctions and eigenvalues of H_n are

$$\psi_{k_1}(x_1)\psi_{k_2}(x_2)\cdots\psi_{k_n}(x_n) \quad \text{and} \quad \lambda_{k_1} + \lambda_{k_2} + \cdots + \lambda_{k_n},$$

respectively, where $\lambda_{k_i} = (2k_i + 1)\sqrt{a_i} \quad (k_i = 0, 1, 2, \cdots)$.

Proof. For $k_i \in \{0\} \cup \mathbb{Z}^+$, we note that

$$\left(-\frac{\partial^2}{\partial x_i^2} + a_i x_i^2\right) \psi_{k_i}(x_i) = \lambda_{k_i} \psi_{k_i}(x_i).$$

Then

$$\begin{aligned}
&H_n(\psi_{k_1}(x_1)\psi_{k_2}(x_2)\cdots\psi_{k_n}(x_n)) \\
&= \sum_{i=1}^{n} \left\{ \left(-\frac{\partial^2}{\partial x_i^2} + a_i x_i^2\right) \psi_{k_1}(x_1)\psi_{k_2}(x_2)\cdots\psi_{k_n}(x_n) \right\} \\
&= \sum_{i=1}^{n} \{\lambda_{k_i} \psi_{k_1}(x_1)\psi_{k_2}(x_2)\cdots\psi_{k_n}(x_n)\} \\
&= (\lambda_{k_1} + \lambda_{k_2} + \cdots + \lambda_{k_n})\psi_{k_1}(x_1)\psi_{k_2}(x_2)\cdots\psi_{k_n}(x_n),
\end{aligned}$$

which shows that $\psi_{k_1}(x_1)\psi_{k_2}(x_2)\cdots\psi_{k_n}(x_n)$ is an eigenfunction of H_n with eigenvalue $\lambda_{k_1} + \lambda_{k_2} + \cdots + \lambda_{k_n}$. This completes the proof of the theorem. □

3. Harmonic oscillators of n variables

In this section we introduce harmonic oscillators with quadratic potentials and investigate their eigenvalues and eigenfunctions.

Theorem 3.1. *Let $q(x_1, x_2, \cdots, x_n)$ be a quadratic polynomial of n variables. Let*

$$H = -\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right) + q(x_1, x_2, \cdots, x_n)$$

be a harmonic oscillator with quadratic polynomial $q(x_1, x_2, \cdots, x_n)$. *Then there exist a harmonic oscillator*

$$\tilde{H} = \left(-\frac{\partial^2}{\partial X_1^2} + \lambda_1 X_1^2\right) + \left(-\frac{\partial^2}{\partial X_2^2} + \lambda_2 X_2^2\right) + \cdots + \left(-\frac{\partial^2}{\partial X_n^2} + \lambda_n X_n^2\right)$$

and a linear map ϕ *such that the following diagram*

$$\begin{array}{ccc} S(\mathbb{R}^n)_{x_i} & \xrightarrow{H} & S(\mathbb{R}^n)_{x_i} \\ \phi \uparrow & & \downarrow \phi^{-1} \\ S(\mathbb{R}^n)_{X_i} & \xrightarrow[\widetilde{H}]{} & S(\mathbb{R}^n)_{X_i} \end{array}$$

commutes.

Proof. Suppose $q(x_1, x_2, \cdots, x_n) = \sum_{i,j=1}^n a_{ij} x_i x_j$, where $A = (a_{ij})$ is an $n \times n$ real symmetric matrix. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of A with corresponding mutually orthogonal eigenvectors $v_1, v_2, \cdots, v_n$, where $\lambda_i \in \mathbb{R}$ and $||v_i|| = 1$. Put $U = [v_1, v_2, \cdots, v_n]$. Then

$$U^{-1}AU = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} = U^T AU.$$

Setting $D = U^{-1}AU$, we get

$$\begin{aligned} \sum_{i,j=1}^n a_{ij} x_i x_j = & \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} u_{ij} \end{pmatrix} D \begin{pmatrix} u_{ji} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} \\ = & \ \lambda_1 (u_{11} x_1 + u_{21} x_2 + \cdots + u_{n1} x_n)^2 \\ & + \lambda_2 (u_{12} x_1 + u_{22} x_2 + \cdots + u_{n2} x_n)^2 \\ & + \cdots + \lambda_n (u_{1n} x_1 + u_{2n} x_2 + \cdots + u_{nn} x_n)^2. \end{aligned}$$

Now we put $X_i = u_{1i} x_1 + u_{2i} x_2 + \cdots + u_{ni} x_n$ and define $\tilde{H}$ by

$$\tilde{H} = \left(-\frac{\partial^2}{\partial X_1^2} + \lambda_1 X_1^2\right) + \left(-\frac{\partial^2}{\partial X_2^2} + \lambda_2 X_2^2\right) + \cdots + \left(-\frac{\partial^2}{\partial X_n^2} + \lambda_n X_n^2\right).$$

We also define a linear map ϕ as follows.

$$\phi : \mathcal{S}(\mathbb{R}^n)_{X_1, X_2, \cdots, X_n} \longrightarrow \mathcal{S}(\mathbb{R}^n)_{x_1, x_2, \cdots, x_n} \quad \text{by}$$
$$f(X_1, X_2, \cdots, X_n) \longmapsto f\left(\sum_{k=1}^{n} u_{k1}x_k, \sum_{k=1}^{n} u_{k2}x_k, \cdots, \sum_{k=1}^{n} u_{kn}x_k\right).$$

Then we claim that $\tilde{H} = \phi^{-1} \circ H \circ \phi$. By definition of ϕ and H, we have

$$\begin{aligned}
& H\phi\left(f(X_1, X_2, \cdots, X_n)\right) \\
= & H\left(f(\sum_{k=1}^{n} u_{k1}x_k, \sum_{k=1}^{n} u_{k2}x_k, \cdots, \sum_{k=1}^{n} u_{kn}x_k)\right) \\
= & -\frac{\partial^2}{\partial x_1^2} f(\sum_{k=1}^{n} u_{k1}x_k, \sum_{k=1}^{n} u_{k2}x_k, \cdots, \sum_{k=1}^{n} u_{kn}x_k) \\
= & -\frac{\partial^2}{\partial x_2^2} f(\sum_{k=1}^{n} u_{k1}x_k, \sum_{k=1}^{n} u_{k2}x_k, \cdots, \sum_{k=1}^{n} u_{kn}x_k) \\
& - \cdots - \frac{\partial^2}{\partial x_n^2} f(\sum_{k=1}^{n} u_{k1}x_k, \sum_{k=1}^{n} u_{k2}x_k, \cdots, \sum_{k=1}^{n} u_{kn}x_k) \\
& +(\sum_{i,j=1}^{n} a_{ij}x_i x_j) f(\sum_{k=1}^{n} u_{k1}x_k, \sum_{k=1}^{n} u_{k2}x_k, \cdots, \sum_{k=1}^{n} u_{kn}x_k).
\end{aligned}$$

First of all, for x_1, we get

$$
\begin{aligned}
&\frac{\partial^2}{\partial x_1^2}\left[f(\sum_{k=1}^n u_{k1}x_k, \sum_{k=1}^n u_{k2}x_k, \cdots, \sum_{k=1}^n u_{kn}x_k)\right] \\
= &\frac{\partial}{\partial x}\left[u_{11}\frac{\partial f}{\partial X_1}(\sum_{k=1}^n u_{k1}x_k, \sum_{k=1}^n u_{k2}x_k, \cdots, \sum_{k=1}^n u_{kn}x_k)\right. \\
&\quad + u_{12}\frac{\partial f}{\partial X_2}(\sum_{k=1}^n u_{k1}x_k, \sum_{k=1}^n u_{k2}x_k, \cdots, \sum_{k=1}^n u_{kn}x_k) \\
&\quad \left. + \cdots + u_{1n}\frac{\partial f}{\partial X_1}(\sum_{k=1}^n u_{k1}x_k, \sum_{k=1}^n u_{k2}x_k, \cdots, \sum_{k=1}^n u_{kn}x_k)\right] \\
= &\sum_{r=1}^n u_{1r}^2 \frac{\partial^2 f}{\partial X_r^2}(\sum_{k=1}^n u_{k1}x_k, \sum_{k=1}^n u_{k2}x_k, \cdots, \sum_{k=1}^n u_{kn}x_k) \\
&+ \sum_{1 \le p \ne q \le \frac{n(n-1)}{2}} 2u_{1p}u_{1q}\frac{\partial^2 f}{\partial X_p \partial X_q}(\sum_{k=1}^n u_{k1}x_k, \sum_{k=1}^n u_{k2}x_k, \cdots, \sum_{k=1}^n u_{kn}x_k).
\end{aligned}
$$

We note that $u_{1k}^2 + u_{2k}^2 + \cdots + u_{nk}^2 = 1$ for each k. Since $\langle v_i, v_j \rangle = 0$ for each $i \ne j$, we have $u_{1i}u_{1j} + u_{2i}u_{2j} + \cdots + u_{ni}u_{nj} = 0$, which leads to the following equalities

$$
\begin{aligned}
&\sum_{m=1}^n \frac{\partial^2}{\partial x_m^2}\left[f(\sum_{k=1}^n u_{k1}x_k, \sum_{k=1}^n u_{k2}x_k, \cdots, \sum_{k=1}^n u_{kn}x_k)\right] \\
= &\sum_{m=1}^n \left[\sum_{r=1}^n u_{mr}^2 \frac{\partial^2 f}{\partial X_r^2}(\sum_{k=1}^n u_{k1}x_k, \sum_{k=1}^n u_{k2}x_k, \cdots, \sum_{k=1}^n u_{kn}x_k)\right. \\
&\left. + \sum_{1 \le p \ne q \le \frac{n(n-1)}{2}} 2u_{mp}u_{mq}\frac{\partial^2 f}{\partial X_p \partial X_q}(\sum_{k=1}^n u_{k1}x_k, \sum_{k=1}^n u_{k2}x_k, \cdots, \sum_{k=1}^n u_{kn}x_k)\right] \\
= &\sum_{m=1}^n \left[\frac{\partial^2 f}{\partial X_m^2}(\sum_{k=1}^n u_{k1}x_k, \sum_{k=1}^n u_{k2}x_k, \cdots, \sum_{k=1}^n u_{kn}x_k)\right].
\end{aligned}
$$

Finally, we get the following result:

$$
\begin{aligned}
&\psi\left[(H\phi)f(X_1, X_2, \cdots, X_n)\right] \\
= &\ \psi\left[\sum_{m=1}^{n}\left\{-\frac{\partial^2}{\partial X_m^2} f(\sum_{k=1}^{n} u_{k1}x_k, \sum_{k=1}^{n} u_{k2}x_k, \cdots, \sum_{k=1}^{n} u_{kn}x_k)\right\}\right. \\
&\qquad \left. + \sum_{i,j=1}^{n}(a_{ij}x_ix_j) f(\sum_{k=1}^{n} u_{k1}x_k, \sum_{k=1}^{n} u_{k2}x_k, \cdots, \sum_{k=1}^{n} u_{kn}x_k)\right] \\
= &\ \sum_{m=1}^{n}\left[\frac{\partial^2}{\partial X_m^2} f(\sum_{k=1}^{n} u_{k1}x_k, \sum_{k=1}^{n} u_{k2}x_k, \cdots, \sum_{k=1}^{n} u_{kn}x_k)\right. \\
&\qquad \left. + \lambda_m x_m^2 f(\sum_{k=1}^{n} u_{k1}x_k, \sum_{k=1}^{n} u_{k2}x_k, \cdots, \sum_{k=1}^{n} u_{kn}x_k)\right] \\
= &\ \tilde{H} f(\sum_{k=1}^{n} u_{k1}x_k, \sum_{k=1}^{n} u_{k2}x_k, \cdots, \sum_{k=1}^{n} u_{kn}x_k),
\end{aligned}
$$

which shows that the given diagram commutes and completes the proof of the theorem. □

Finally we investigate eigenvalues and eigenfunctions of harmonic oscillators with quadratic potentials. Let $A = \begin{pmatrix} a_{ij} \end{pmatrix}$ be a positive definite, symmetric matrix and

$$H = -\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right) + q(x_1, x_2, \cdots, x_n)$$

be a harmonic oscillator with a quadratic polynomial of n variables acting on $\mathcal{S}(\mathbb{R}^n)$ where $q(x_1, x_2, \cdots, x_n) = \sum_{i,j=1}^{n} a_{ij}x_ix_j$. By Theorem 3.1, we can find the harmonic oscillator

$$\tilde{H} = \left(-\frac{\partial^2}{\partial X_1^2} + \lambda_1 X_1^2\right) + \left(-\frac{\partial^2}{\partial X_2^2} + \lambda_2 X_2^2\right) + \cdots + \left(-\frac{\partial^2}{\partial X_n^2} + \lambda_n X_n^2\right)$$

such that the diagram in Theorem 3.1 commutes.

Theorem 3.2. *The eigenvalues of H are equal to the eigenvalues of $\tilde{H}$. In fact, if ψ is an eigenfunction of H with eigenvalue λ, then $\phi(\psi)$ is an eigenfunction of $\tilde{H}$ with the same eigenvalue λ.*

Proof. Let $\tilde{\psi}(X_1, X_2, \cdots, X_n)$ be an eigenfunction of $\tilde{H}$ with eigenvalue λ. Putting $\psi(x_1, x_2, \cdots, x_n) = \phi(\tilde{\psi}(X_1, X_2, \cdots, X_n))$, then we have

$$H\psi = \phi \circ \tilde{H} \circ \phi^{-1} \circ \phi \circ \tilde{\psi} = \phi \circ \tilde{H} \circ \tilde{\psi} = \phi(\lambda\tilde{\psi}) = \lambda\phi\tilde{\psi} = \lambda\psi.$$

This shows that $\tilde{\psi}$ is an eigenfunction of $\tilde{H}$ with eigenvalue λ if and only if $\phi(\tilde{\psi})$ is a eigenfunction of H with the same eigenvalue λ, which completes the proof of the theorem. □

4. The zeta-determinant of $H(a, b)$

In this section we are going to compute the zeta-determinants of 2-dimensional harmonic oscillators by using the Riemann zeta function, Hurwitz zeta function and Gamma function. First of all, let us review the zeta-determinants of linear operators acting on infinite dimensional vector spaces by using zeta functions. To give a motivation we first consider a linear operator T acting on an n-dimensional vector space. Suppose that $\lambda_1, \cdots, \lambda_2$ are positive eigenvalues of T. Then the determinant of T is given by

$$det(T) = \lambda_1\lambda_2 \cdots \lambda_n.$$

For $z \in \mathbb{C}$, we define the zeta function associated to T by

$$\zeta_T(z) = \lambda_1^{-z} + \lambda_2^{-z} + \cdots + \lambda_n^{-z}.$$

Then $\zeta_T(z)$ is an entire function and the derivative is given by

$$\zeta_T'(z) = -(\log\lambda_1)e^{-z\log\lambda_1} - (\log\lambda_2)e^{-z\log\lambda_2} - \cdots - (\log\lambda_n)e^{-z\log\lambda_n}.$$

Now we get $\zeta_T'(0) = -\log det(T)$. Finally, we can rewrite the determinant of T by

$$det(T) = e^{-\zeta_T'(0)}.$$

This fact gives a way of generalizing the determinant of an operator acting on a finite dimensional vector space to the determinant of an operator acting on infinite dimensional vector space.

Definition 4.1. *Let T_∞ be a linear operator acting on an infinite dimensional vector space with a discrete spectrum $\{\lambda_i \mid i = 1, 2, 3, \cdots\}$. Suppose that*

$$\zeta_{T_\infty}(z) = \sum_{i=1}^{\infty} \lambda_i^{-z}, \qquad Rez \gg 0$$

has an analytic continuation to the whole complex plane having a regular value at $z = 0$. *Then we define the* zeta-determinant $Det(T_\infty)$ *of* T_∞ *by*

$$Det(T_\infty) = e^{-\zeta'_{T_\infty}(0)}.$$

It is known that the zeta-determinants of 1-dimensional harmonic oscillator H is always $\sqrt{2}$, where $H = -\frac{d^2}{dx^2} + a^2x^2 \quad (a > 0)$ (cf. [6]). In other words, zeta-determinant does not depend on a. This is not the case for harmonic oscillators defined on $\mathbb{R}^n$, $n \geq 2$. We will see the reason in Theorem 4.1. Before that, we test this fact with the following example. Let us put

$$H(a,b) = \left(-\frac{\partial^2}{\partial x^2} + a^2x^2\right) + \left(-\frac{\partial^2}{\partial y^2} + b^2y^2\right),$$

where a, b are positive real numbers. We define the *Riemann* ζ *function* $\zeta_R(s)$ and *Hurwitz* ζ *function* $\zeta_R(s,a)$ as follows.

$$\zeta_R(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \zeta_R(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \quad a \neq 0, -1, -2, \cdots.$$

It is a well-known fact ([4]) that

$$\zeta_R(-1) = \frac{1}{12}, \quad \zeta_R(0,\alpha) = \frac{1}{2} - \alpha, \quad \zeta'_R(0,\alpha) = \log \frac{\Gamma(\alpha)}{\log\sqrt{2\pi}}. \tag{4.1}$$

Theorem 4.1. *The eigenvalues of* $H(a,a)$ *are given in the Table 1 and the zeta-determinants of* $H(a,a)$ *is* $e^{\frac{1}{12}\log(2a) + \zeta'_R(-1)}$.

$+$	a	$3a$	$5a$	$\cdots$	$(2k-1)a$	$\cdots$
a	$2a$	$4a$	$6a$		$2ka$	
$3a$	$4a$	$6a$	$8a$		$2(k+1)a$	
$\vdots$				$\ddots$	$\vdots$	
$(2k-1)a$	$2ka$	$2(k+1)a$	$2(k+2)a$	$\cdots$	$2(2k-1)a$	$\cdots$
$\vdots$					$\vdots$	$\ddots$

Table 1

Proof. For $n \in \mathbb{N}$, the multiplicities of eigenvalue $2(2n-1)a$ of $H(a,a)$ are equal to n. Indeed, the spectrum of $H(a,a)$ is given by

$$\{2a, 4a, 4a, \cdots, \underbrace{2an, \cdots, 2an}_{n \ times} \cdots\},$$

which implies

$$\zeta_{H(a,a)}(s) \quad = \sum_{n=1}^{\infty} n(2an)^{-s} = \frac{1}{(2a)^s} \sum_{n=1}^{\infty} n^{1-s} = \frac{1}{(2a)^s} \zeta_R(s-1).$$

Differentiating with respect to s, we obtain

$$\zeta'_{H(a,a)}(s) = -(2a)^{-s} \log 2a \zeta_R(s-1) + (2a)^{-s} \zeta'_R(s-1).$$

Putting $s = 0$ and using (4.1),

$$\zeta'_{H(a,a)}(0) = -\log 2a \zeta_R(-1) + \zeta'_R(-1) = \frac{1}{12} \log 2a + \zeta'_R(-1),$$

which completes the proof of the theorem. □

According to the above theorem, the zeta-determinant of $H(a,a)$ does depend on the parameter a. Moreover, the zeta-determinant of $H(a,b)$ does depend on the parameters a and b (see Theorem 4.2).

Remark : The exact value of $\zeta'_R(-1)$ was not known yet. However, in [5] we can see an approximation of $\zeta'_R(-1)$.

We next consider $H(\frac{a}{2}, \frac{3a}{2})$. The eigenvalues of $H(\frac{a}{2}, \frac{3a}{2})$ are given in Table 2.

$+$	$\frac{1}{2}a$	$\frac{3}{2}a$	$\frac{5}{2}a$	$\cdots$	$(2k-1)\frac{1}{2}a$	$\cdots$
$\frac{3}{2}a$	$2a$	$3a$	$4a$		$(k+1)a$	
$\frac{9}{2}a$	$5a$	$6a$	$7a$		$(k+4)a$	
$\vdots$				$\ddots$	$\vdots$	
$(2k-1)\frac{3}{2}a$	$(3k-1)a$	$3ka$	$(3k+1)a$	$\cdots$	$2(2k-1)a$	$\cdots$
$\vdots$					$\vdots$	$\ddots$

Table 2

From Table 2, for $n \in \mathbb{N}$, the multiplicities of eigenvalues $(3n-1)a, 3na, (3n+1)a$ of $H(\frac{a}{2}, \frac{3a}{2})$ are equal to n. Indeed, the spectrum of $H(\frac{a}{2}, \frac{3a}{2})$ is given by

$$\{\underbrace{2a, 3a, 4a}_{1 \ time}, \underbrace{5a, 6a, 7a}_{2 \ times}, \cdots, \underbrace{(3n-1)a, 3na, (3n+1)a}_{n \ times}, \cdots\}.$$

Therefore

$$
\begin{aligned}
\zeta_{H(\frac{a}{2},\frac{3a}{2})}(s) &= \sum_{\lambda_i \in Spec(H(\frac{a}{2},\frac{3a}{2}))} \lambda_i^{-s} \\
&= \sum_{n=1}^{\infty} n(3an)^{-s} + \sum_{n=1}^{\infty} n(3an+a)^{-s} + \sum_{n=0}^{\infty}(n+1)(3an+2a)^{-s} \\
&= \frac{1}{3a}\left(\sum_{n=1}^{\infty}\frac{3an}{(3an)^s} + \sum_{n=0}^{\infty}\frac{3an}{(3an+a)^s} + \sum_{n=0}^{\infty}\frac{3an+3a}{(3an+2a)^s}\right) \\
&= \frac{1}{3a}\left(\sum_{n=1}^{\infty}\frac{1}{(3an)^{s-1}} + \sum_{n=0}^{\infty}\frac{1}{(3an+a)^{s-1}} + \sum_{n=0}^{\infty}\frac{1}{(3an+2a)^{s-1}}\right. \\
&\qquad \left. - \sum_{n=0}^{\infty}\frac{a}{(3an+a)^s} - \sum_{n=0}^{\infty}\frac{a}{(3an+2a)^s}\right) \\
&= \frac{1}{3a}\left(\frac{1}{a^{s-1}}\sum_{n=1}^{\infty}\frac{1}{n^{s-1}} - \frac{a}{(3a)^s}\sum_{n=0}^{\infty}\frac{1}{(n+\frac{1}{3})^s} + \frac{a}{(3a)^s}\sum_{n=0}^{\infty}\frac{1}{(n+\frac{2}{3})^s}\right) \\
&= \frac{1}{3}\left(\frac{1}{a^s}\zeta_R(s-1) - \frac{1}{(3a)^s}\zeta_R(s,\frac{1}{3}) + \frac{1}{(3a)^s}\zeta_R(s,\frac{2}{3})\right).
\end{aligned}
$$

Putting $s = 0$ and using (4.1), we have

$$
\zeta_{H(\frac{a}{2},\frac{3a}{2})}(0) = \frac{1}{3}\left(\zeta_R(-1) + \zeta_R(0,\frac{2}{3}) + \zeta_R(0,\frac{1}{3})\right) = -\frac{5}{36},
$$

from which we can also see that $\zeta_{H(\frac{a}{2},\frac{3a}{2})}(0)$ does not depend on a. Differentiating $\zeta_{H(\frac{a}{2},\frac{3a}{2})}(s)$ with respect to s, we have

$$
\begin{aligned}
\zeta'_{H(\frac{a}{2},\frac{3a}{2})}(s) &= \tfrac{1}{3}\{-a^{-s}\log a\zeta_R(s-1) + a^{-s}\zeta'_R(s-1) \\
&\quad +(3a)^{-s}\log 3a\zeta_R(s,\tfrac{1}{3}) - (3a)^{-s}\zeta'_R(s,\tfrac{1}{3}) \\
&\quad -(3a)^{-s}\log 3a\zeta_R(s,\tfrac{2}{3}) + (3a)^{-s}\zeta'_R(s,\tfrac{2}{3})\}.
\end{aligned}
$$

Putting $s = 0$ and using (4.1), we get

$$\begin{aligned}
&\zeta'_{H(\frac{a}{2},\frac{3a}{2})}(0) \\
&= \frac{1}{3}\,\{-\log a\zeta_R(-1) + \zeta'_R(-1) + \log 3a\zeta_R(0,\frac{1}{3}) - \zeta'_R(0,\frac{1}{3}) \\
&\qquad\qquad - \log 3a\zeta_R(0,\frac{2}{3}) + \zeta'_R(0,\frac{2}{3})\} \\
&= \frac{1}{3}\{\zeta'_R(-1) - \log a(\zeta_R(-1) + \zeta'_R(0,\frac{2}{3}) - \zeta'_R(0,\frac{1}{3})) \\
&\qquad\qquad - \log 3(\zeta_R(0,\frac{2}{3}) - \zeta_R(0,\frac{1}{3})) + \zeta'_R(0,\frac{2}{3}) - \zeta'_R(0,\frac{1}{3})\} \\
&= \frac{1}{3}\{\frac{1}{3}\log 3 + \frac{5}{12}\log a + \zeta'_R(-1) + \log\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}\}.
\end{aligned}$$

Finally, we obtain zeta-determinant of $H(\frac{a}{2},\frac{3a}{2})$ as follows.

$$\begin{aligned}
Det\ \ H(\frac{a}{2},\frac{3a}{2}) &= e^{-\frac{1}{3}\{\frac{1}{3}\log 3+\frac{5}{12}\log a+\zeta'_{\mathbb{R}}(-1)+\log\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}\}} \\
&= 3^{-\frac{1}{9}}a^{-\frac{5}{36}}e^{-\frac{1}{3}\zeta'(-1)}\left(\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}\right)^{-\frac{1}{3}}.
\end{aligned}$$

We next consider the harmonic oscillator $H(a,b)$, where $a,\ b \in \mathbb{R} - \{n^2 | n \in \mathbb{N}\}$. Then simple computation shows that the multiplicities of eigenvalues of $H(a,b)$ are equal to 1 and Theorem 2.1 implies that the spectrum of $H(a,b)$ is given by

$$\{(2k-1)a + (2l-1)b \mid k, l \in \mathbb{N}\}.$$

Using the Mellin transform (cf. [4]), the zeta function $\zeta_{H(a,b)}(s)$ associated to $H(a,b)$ is given as follows.

$$\begin{aligned}
\zeta_{H(a,b)}(s) &= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\{(2k-1)a+(2l-1)b\}^{-s} \\
&= \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}\sum_{k=1}^{\infty} e^{-t(2k-1)a}\sum_{l=1}^{\infty} e^{-t(2l-1)b}dt \\
&= \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}\frac{e^{-at}}{1-e^{-2at}}\frac{e^{-bt}}{1-e^{-2bt}}dt \\
&= \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}\frac{e^{(a+b)t}}{(e^{2at}-1)(e^{2bt}-1)}dt \\
&= \frac{s}{\Gamma(s+1)}\int_0^{1} t^{s-1}\frac{e^{(a+b)t}}{(e^{2at}-1)(e^{2bt}-1)}dt \\
&\quad +\frac{s}{\Gamma(s+1)}\int_1^{\infty} t^{s-1}\frac{e^{(a+b)t}}{(e^{2at}-1)(e^{2bt}-1)}dt.
\end{aligned}$$

We split $\zeta_{H(a,b)}(s)$ into $\zeta^{\star}_{H(a,b)}(s)$ and $\zeta^{\star\star}_{H(a,b)}(s)$ as follows.

$$\zeta^{\star}_{H(a,b)}(s) = \tfrac{s}{\Gamma(s+1)}\int_0^1 t^{s-1}\tfrac{e^{(a+b)t}}{(e^{2at}-1)(e^{2bt}-1)}dt,$$
$$\zeta^{\star\star}_{H(a,b)}(s) = \tfrac{s}{\Gamma(s+1)}\int_1^{\infty} t^{s-1}\tfrac{e^{(a+b)t}}{(e^{2at}-1)(e^{2bt}-1)}dt.$$

Differentiating $\zeta^{\star\star}_{H(a,b)}(s)$ at $s=0$, we have

$$\begin{aligned}
\zeta^{\star\star\prime}_{H(a,b)}(0) &= \tfrac{d}{ds}\Big|_{s=0}\tfrac{s}{\Gamma(s+1)}\int_1^{\infty} t^{s-1}\tfrac{e^{(a+b)t}}{(e^{2at}-1)(e^{2bt}-1)}dt \\
&= \int_1^{\infty} t^{-1}\tfrac{e^{(a+b)t}}{(e^{2at}-1)(e^{2bt}-1)}dt,
\end{aligned}$$

which shows that $\zeta^{\star\star\prime}_{H(a,b)}(0)$ is well defined. We next investigate $\zeta^{\star}_{H(a,b)}(s)$. Using Taylor series and the fact $\sum_{n=1}^{\infty} e^{-nt} = \frac{1}{t}-\frac{1}{2}+O(e^{-\frac{c}{t}})$, we get

$$\begin{aligned}
\sum_{n=1}^{\infty} e^{-t(2n-1)a} &= e^{at}\sum_{n=1}^{\infty} e^{-2ant} \\
&\sim \left(1+at+\frac{1}{2}a^2t^2+\frac{1}{6}a^3t^3+\cdots\right)\left(\frac{1}{2at}-\frac{1}{2}\right) \\
&= \frac{1}{2at}-\frac{a}{4}t-\frac{a^2}{6}t^2+O(t^3),
\end{aligned}$$

which leads to

$$\begin{aligned}
&\sum_{k=1}^{\infty} e^{-t(2k-1)a} \sum_{l=1}^{\infty} e^{-t(2l-1)b} \\
&\sim \left(\frac{1}{2at} - \frac{a}{4}t - \frac{a^2}{6}t^2 + O(t^3)\right)\left(\frac{1}{2bt} - \frac{b}{4}t - \frac{b^2}{6}t^2 + O(t^3)\right) \\
&= \frac{1}{4ab}\frac{1}{t^2} - \frac{b}{8a} - \frac{a}{8b} + O(t).
\end{aligned}$$

Now, we apply the above result to $\zeta^{\star}_{H(a,b)}(s)$. Then

$$\begin{aligned}
&\zeta^{\star}_{H(a,b)}(s) \\
&= \frac{s}{\Gamma(s+1)} \int_0^1 t^{s-1} \frac{e^{(a+b)t}}{(e^{2at}-1)(e^{2bt}-1)} dt \\
&= \frac{s}{\Gamma(s+1)} \int_0^1 t^{s-1} \left(\frac{1}{4ab}\frac{1}{t^2} - \frac{a^2+b^2}{8ab}\right) dt \\
&\quad + \frac{s}{\Gamma(s+1)} \int_0^1 t^{s-1} \left\{\frac{e^{(a+b)t}}{(e^{2at}-1)(e^{2bt}-1)} - \left(\frac{1}{4ab}\frac{1}{t^2} - \frac{a^2+b^2}{8ab}\right)\right\} dt \\
&= \frac{s}{\Gamma(s+1)} \left(\frac{1}{4ab}\left[\frac{1}{s-2}t^{s-2}\right]_0^1 - \frac{a^2+b^2}{8ab}\left[\frac{1}{s}t^s\right]_0^1\right) \\
&\quad + \frac{s}{\Gamma(s+1)} \int_0^1 t^{s-1} \left\{\frac{e^{(a+b)t}}{(e^{2at}-1)(e^{2bt}-1)} - \left(\frac{1}{4ab}\frac{1}{t^2} - \frac{a^2+b^2}{8ab}\right)\right\} dt \\
&= \frac{1}{4ab}\frac{s}{s-2}\frac{1}{\Gamma(s+1)} - \frac{a^2+b^2}{8ab}\frac{1}{\Gamma(s+1)} \\
&\quad + \frac{s}{\Gamma(s+1)} \int_0^1 t^{s-1} \left\{\frac{e^{(a+b)t}}{(e^{2at}-1)(e^{2bt}-1)} - \left(\frac{1}{4ab}\frac{1}{t^2} - \frac{a^2+b^2}{8ab}\right)\right\} dt.
\end{aligned}$$

Differentiating at $s=0$, we have

$$
\begin{aligned}
&\zeta^{\star\prime}_{H(a,b)}(0)\\
&= \frac{1}{4ab}\frac{(s-2)\Gamma(s+1)-s\{(s-2)\Gamma'(s+1)+\Gamma(s+1)\}}{(s-2)^2\Gamma^2(s+1)}\Big|_{s=0}\\
&\quad -\frac{a^2+b^2}{8ab}\frac{-\Gamma'(s+1)}{\Gamma^2(s+1)}\Big|_{s=0}\\
&\quad +\frac{d}{ds}\Big|_{s=0}\frac{s}{\Gamma(s+1)}\int_0^1 t^{s-1}\left\{\frac{e^{(a+b)t}}{(e^{2at}-1)(e^{2bt}-1)}\right.\\
&\qquad\qquad\left.-\left(\frac{1}{4ab}\frac{1}{t^2}-\frac{a^2+b^2}{8ab}\right)\right\}dt\\
&=\frac{1}{4ab}\frac{-2\Gamma(1)}{(-2)^2\Gamma^2(1)}-\frac{a^2+b^2}{8ab}\frac{-\Gamma'(1)}{\Gamma^2(1)}\\
&\quad +\int_0^1\frac{1}{t}\left\{\frac{e^{(a+b)t}}{(e^{2at}-1)(e^{2bt}-1)}-\left(\frac{1}{4ab}\frac{1}{t^2}-\frac{a^2+b^2}{8ab}\right)\right\}dt\\
&=-\frac{1}{8ab}-\frac{a^2+b^2}{8ab}\gamma\\
&\quad +\int_0^1\frac{1}{t}\left\{\frac{e^{(a+b)t}}{(e^{2at}-1)(e^{2bt}-1)}-\left(\frac{1}{4ab}\frac{1}{t^2}-\frac{a^2+b^2}{8ab}\right)\right\}dt\\
&=-\frac{\{(a^2+b^2)\gamma+1\}}{8ab}\\
&\quad +\int_0^1\frac{1}{t}\left\{\frac{e^{(a+b)t}}{(e^{2at}-1)(e^{2bt}-1)}-\left(\frac{1}{4ab}\frac{1}{t^2}-\frac{a^2+b^2}{8ab}\right)\right\}dt,
\end{aligned}
$$

where $\gamma=-\Gamma'(1)$ is the Euler constant (cf. [4]).

Finally, we obtain

$$
\begin{aligned}
\zeta'_{H(a,b)}(0) &= \zeta^{\star\prime}_{H(a,b)}(0)+\zeta^{\star\star\prime}_{H(a,b)}(0)\\
&= -\tfrac{\{(a^2+b^2)\gamma+1\}}{8ab}\\
&\quad +\int_0^1\tfrac{1}{t}\left\{\tfrac{e^{(a+b)t}}{(e^{2at}-1)(e^{2bt}-1)}-\left(\tfrac{1}{4ab}\tfrac{1}{t^2}-\tfrac{a^2+b^2}{8ab}\right)\right\}dt\\
&\quad +\int_1^\infty\tfrac{1}{t}\tfrac{e^{(a+b)t}}{(e^{2at}-1)(e^{2bt}-1)}dt,
\end{aligned}
$$

which yields the following theorem.

THEOREM 4.2. *For the harmonic oscillator $H(a,b)$ with a, $b \in \mathbb{R} - \{n^2 | n \in \mathbb{N}\}$, we have*

$$Det\ \ H(a,b) = e^{-\zeta'_{H(a,b)}(0)}$$

where

$$\begin{aligned}\zeta'_{H(a,b)}(0) \qquad &= -\tfrac{\{(a^2+b^2)\gamma+1\}}{8ab} \\ &+ \int_0^1 \tfrac{1}{t}\left\{\tfrac{e^{(a+b)t}}{(e^{2at}-1)(e^{2bt}-1)} - \left(\tfrac{1}{4ab}\tfrac{1}{t^2} - \tfrac{a^2+b^2}{8ab}\right)\right\} dt \\ &+ \int_1^\infty \tfrac{1}{t}\tfrac{e^{(a+b)t}}{(e^{2at}-1)(e^{2bt}-1)} dt.\end{aligned}$$

References

[1] John Roe, *Elliptic Operators, Topology, and Asymptotic Methods*, Chapman and Hall, 126–135, 1999.

[2] C. Soule, D. Abramovich, J.-F.Burnol, J. Kramer, *Lectures on Arakelov Geometry*, Cambridge University Press, 94–101, 1992.

[3] M. Dimassi, J. Sjostrand, *Spectral Asymptotics in the Semi-Classical Limit*, Cambridge University Press, 24–25, 1999.

[4] Dr. Wilhelm Magnus, Dr. Fritz Oberhettinger, Dr. Raj Pal Soni, *The gamma function and related functions"*, Chapter 1, Formulas and Theorems for the Special Functions of Mathematical Physics, Springer-Verlag, Berlin, 1966.

[5] Bejoy K. Choudhury , *"The Riemann zeta-function and its derivatives"*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **450** (1995), 477–499.

[6] Seung-Won Shin, *"The zeta-determinants of harmonic oscillators"*, Inha University, 2008.

Department of Mathematics
Inha University
Incheon 402-751, Korea
E-mail: `kyounghwa@inha.edu`