Korean J. Math. **19** (2011), No. 2, pp. 171–180

ON THE SUPERSTABILITY OF THE GENERALIZED SINE FUNCTIONAL EQUATIONS

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Abstract. In this paper, we study the superstability problem bounded by two-variables of Th. M. Rassias type for the generalized sine functional equations

$$g(x+y)f(x-y) = f(x)^2 - f(y)^2$$
$$f(x+y)g(x-y) = f(x)^2 - f(y)^2$$
$$g(x+y)g(x-y) = f(x)^2 - f(y)^2,$$

which does not use his iteration method.

1. Introduction

The stability problem of the functional equation was conjectured by Ulam [12] during the conference in the university of Wisconsin in 1940. In the next year, it was solved by Hyers [8] in the case of additive mapping, which is called the Hyers-Ulam stability.

In 1979, J. Baker et al. in [4] introduced the following: if f satisfies the inequality $|E_1(f) - E_2(f)| \leq \varepsilon$, then either f is bounded or $E_1(f) = E_2(f)$. This is frequently referred to as superstability.

The superstability of the cosine functional equation (**C**) (also called the d'Alembert equation)

$$\text{(C)} \qquad f(x+y) + f(x-y) = 2f(x)f(y)$$

Received February 25, 2011. Revised April 12, 2011. Accepted April 15, 2011.

2000 Mathematics Subject Classification: 39B82, 39B52.

Key words and phrases: sine functional equation, exponential, Hyers-Ulam-Rassias stability, superstability.

The first author of this work was partially supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (Grant number: 2010-0010243).

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and the sine functional equation

$$f(x)f(y) = f(\frac{x+y}{2})^2 - f(\frac{x-y}{2})^2$$

were investigated by Baker [3] and Cholewa [5]. Their results were improved by Badora [1] and Badora and Ger [2], Forti [6] and Găvruta [7] as well as by Kim([9], [10], [11]). Since the above sine functional equation is equivalent to

$$f(x+y)f(x-y) = f(x)^2 - f(y)^2, \tag{S}$$

we will use the latter as the sine equation.

In this paper, we investigate the superstability of the generalized functional equations of the sine functional equation as follows:

$$g(x+y)f(x-y) = f(x)^2 - f(y)^2, \tag{GFFF}$$

$$f(x+y)g(x-y) = f(x)^2 - f(y)^2, \tag{FGFF}$$

$$g(x+y)g(x-y) = f(x)^2 - f(y)^2. \tag{GGFF}$$

2. Stability of the Equations

In this section, we investigate the superstability of the generalized functional equations (GFFF), (FGFF),(GGFF) of the sine functional equations, which are not used the traditional iteration method by Th. M. Rassias, even though they are considered in his form.

Theorem 2.1. *Let A be a commutative Banach algebra and $f, g : A \to \mathbb{C}$ be functions satisfying the inequality*

$$|g(x+y)f(x-y) - f(x)^2 + f(y)^2| \leq \delta(\|x\| + \|y\|) \tag{2.1}$$

for all $x, y \in A$. If g is unbounded, then f satisfies the Wilson functional equation

$$f(x+y) + f(x-y) = 2f(x)h(y) \tag{2.2}$$

with the function h defined by $h(x) := \frac{f(x+w)-f(x-w)}{2f(w)}$, where $f(w) \neq 0$. Moreover, if f is unbounded with $f(0) = 0$, then f satisfies the sine functional equation

$$f(x+y)f(x-y) = f(x)^2 - f(y)^2.$$

Proof. For every $x, y \in A$, let $u = x + y$, $v = x - y$. Then by putting $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$ in (2.1) we get

$$\begin{aligned}(2.3)\qquad & |g(u)f(v) - f(\frac{u+v}{2})^2 + f(\frac{u-v}{2})^2| \\ &\leq \delta(\|\frac{u+v}{2}\| + \|\frac{u-v}{2}\|) \\ &\leq \delta(\|u\| + \|v\|).\end{aligned}$$

Define the function h on A by $h(x) := \frac{f(x+w)-f(x-w)}{2f(w)}$ such that $f(w) \neq 0$. Then, we claim that $f(x+y) + f(x-y) = 2f(x)h(y)$. At first, we need to show that $|g(x+y)+g(x-y)-2g(x)h(y)| \leq \frac{4\delta}{|f(w)|}(\|x\|+\|y\|+\|w\|)$.

Indeed, it holds from (2.3) that

$$\begin{aligned}(2.4)\qquad & |g(x+y) + g(x-y) - 2g(x)h(y)| \\ &= \frac{1}{|f(w)|}|g(x+y)f(w) + g(x-y)f(w) - 2g(x)f(w)h(y)| \\ &\leq \frac{1}{|f(w)|}|g(x+y)f(w) - f(\frac{x+y+w}{2})^2 + f(\frac{x+y-w}{2})^2| \\ &\quad + \frac{1}{|f(w)|}|g(x-y)f(w) - f(\frac{x-y+w}{2})^2 + f(\frac{x-y-w}{2})^2| \\ &\quad + \frac{1}{|f(w)|}|f(\frac{x+y+w}{2})^2 - f(\frac{x-y-w}{2})^2 - g(x)f(y+w)| \\ &\quad + \frac{1}{|f(w)|}|g(x)f(y-w) - f(\frac{x+y-w}{2})^2 + f(\frac{x-y+w}{2})^2| \\ &\quad + |2g(x)\frac{f(y+w)-f(y-w)}{2f(w)} - 2g(x)h(y)| \\ &\leq \frac{1}{|f(w)|}\delta(\|x+y\| + \|w\|) + \frac{1}{|f(w)|}\delta(\|x-y\| + \|w\|) \\ &\quad + \frac{1}{|f(w)|}\delta(\|x\| + \|y+w\|) + \frac{1}{|f(w)|}\delta(\|x\| + \|y-w\|) \\ &\leq \frac{4\delta}{|f(w)|}(\|x\| + \|y\| + \|w\|), \quad \forall x, y \in A.\end{aligned}$$

Using (2.3) and (2.4), we have for every $z \in A$

$$
\begin{aligned}
(2.5) \qquad & |g(z)||f(x+y)+f(x-y)-2f(x)h(y)| \\
&= |g(z)f(x+y)+g(z)f(x-y)-2g(z)f(x)h(y)| \\
&\leq |g(z)f(x+y)-f(\frac{z+x+y}{2})^2+f(\frac{z-x-y}{2})^2| \\
&\quad +|g(z)f(x-y)-f(\frac{z+x-y}{2})^2+f(\frac{z-x+y}{2})^2| \\
&\quad +|f(\frac{z+x+y}{2})^2-f(\frac{z-x+y}{2})^2-g(z+y)f(x)| \\
&\quad +|f(\frac{z+x-y}{2})^2-f(\frac{z-x-y}{2})^2-g(z-y)f(x)| \\
&\quad +|[g(z+y)+g(z-y)]f(x)-2g(z)f(x)h(y)| \\
&\leq 4\delta(\|x\|+\|y\|+\|z\|)+4\delta\frac{|f(x)|}{|f(w)|}(\|z\|+\|y\|+\|w\|).
\end{aligned}
$$

Therefore, since $\|x\| \leq \|w\|$ or $\|x\| > \|w\|$, the above inequality (2.5) states

$$
\begin{aligned}
(2.6) \qquad & |f(x+y)+f(x-y)-2f(x)h(y)| \\
& \qquad \leq 4\delta\frac{(\|x\|+\|y\|+\|z\|)}{|g(z)|}(\frac{|f(x)|}{|f(w)|}+1)
\end{aligned}
$$

or

$$
\begin{aligned}
(2.7) \qquad & |f(x+y)+f(x-y)-2f(x)h(y)| \\
& \qquad \leq 4\delta\frac{(\|w\|+\|y\|+\|z\|)}{|g(z)|}(\frac{|f(x)|}{|f(w)|}+1).
\end{aligned}
$$

Since g is assumed to be an unbounded function, then we can choose $(z_n) \in G$ so that $|g(z_n)| \to \infty$ as $n \to \infty$. So the right sides of above inequalities (2.6) and (2.7) vanish for fixed elements x, y. Hence f and h satisfy (2.2).

Next, let f be an unbounded function with $f(0) = 0$ satisfying (2.1). Applying $x = 0$ in (2.2), then f is odd, i.e., $f(-y) = -f(y)$. Substituting $(x, y) = (\frac{u+v}{2}, \frac{u-v}{2})$ in (2.2), it implies

$$
(2.8) \qquad f(u)+f(v) = 2f(\frac{u+v}{2})h(\frac{u-v}{2}) \quad \forall u, v \in A.
$$

Then, from (2.8) and $f(0) = 0$, we infer that

$$\begin{aligned} f(x+y) &= f(x+y) + f(0) \\ &= 2f(\frac{x+y}{2})h(\frac{x+y}{2}) \quad \forall x, y \in A, \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} f(x-y) &= f(x-y) + f(0) \\ &= 2f(\frac{x-y}{2})h(\frac{x-y}{2}) \quad \forall x, y \in A. \end{aligned} \tag{2.10}$$

The oddness of f and (2.8) implies

$$\begin{aligned} f(x) - f(y) &= f(x) + f(-y) \\ &= 2f(\frac{x-y}{2})h(\frac{x+y}{2}) \quad \forall x, y \in A. \end{aligned} \tag{2.11}$$

From (2.8), (2.9), (2.10), and (2.11), we obtain

$$\begin{aligned} f(x+y)f(x-y) &= [2f(\frac{x+y}{2})h(\frac{x+y}{2})][2f(\frac{x-y}{2})h(\frac{x-y}{2})] \\ &= [2f(\frac{x+y}{2})h(\frac{x-y}{2})][2f(\frac{x-y}{2})h(\frac{x+y}{2})] \\ &= [f(x) + f(y)][f(x) - f(y)] \\ &= f(x)^2 - f(y)^2 \end{aligned}$$

for all $x, y \in A$. □

THEOREM 2.2. *Let A be a commutative Banach algebra and $f, g : A \to \mathbb{C}$ be functions satisfying the inequality*

$$|f(x+y)g(x-y) - f(x)^2 + f(y)^2| \leq \delta(\|x\| + \|y\|) \tag{2.12}$$

for all $x, y \in A$. If g is unbounded, then f satisfies the Wilson functional equation

$$f(x+y) + f(x-y) = 2f(x)h(y) \tag{2.13}$$

with the function h defined by $h(x) := \frac{f(w+x)+f(w-x)}{2f(w)}$, where $f(w) \neq 0$. Moreover, if f is unbounded with $f(0) = 0$, then f satisfies the sine functional equation (S).

Proof. An obvious slight change in the steps of the proof applied in Theorem 2.2 gives us the required result. Indeed, let g be unbounded.

the inequality

$$|f(u)g(v) - f(\frac{u+v}{2})^2 + f(\frac{u-v}{2})^2| \leq \delta(\|u\| + \|v\|).$$

Define the functional h on A by $h(x) := \frac{f(w+x)+f(w-x)}{2f(w)}$ such that $f(w) \neq 0$. Then, we claim that $f(x+y) + f(x-y) = 2f(x)h(y)$. Due to the similar calculation with (2.3) and (2.4) in Theorem 2.1, we show that

$$|g(x+y) + g(x-y) - 2g(x)h(y)| \leq \frac{4\delta}{|f(w)|}(\|x\| + \|y\| + \|w\|)$$

and also

$$\begin{aligned} &|g(z)||f(x+y) + f(x-y) - 2f(x)h(y)| \\ &\quad \leq 4\delta(\|x\| + \|y\| + \|z\|) + 4\delta\frac{|f(x)|}{|f(w)|}(\|z\| + \|y\| + \|w\|). \end{aligned}$$

Since g is an unbounded, for $(z_n) \in G$ so that $|g(z_n)| \to \infty$, and x, y are fixed elements, Hence the same reason with Theorem 2.1 give to us that f and h satisfy (2.2).

For the remainder of the proof, running along an obvious slight change in the step by step of that of Theorem 2.1, then we arrive the required result. □

THEOREM 2.3. *Let A be a commutative Banach algebra and $f, g : A \to \mathbb{C}$ be functions satisfying the inequality*

$$|g(x+y)g(x-y) - f(x)^2 + f(y)^2| \leq \delta(\|x\| + \|y\|) \tag{2.14}$$

for all $x, y \in A$. Then either g is bounded or g satisfies the Wilson functional equation $g(x+y) + g(x-y) = 2g(x)h(y)$ with the function h defined by $h(x) := \frac{g(w+x)+g(w-x)}{2g(w)}$, where $g(w) \neq 0$ and the sine functional equation

$$g(x+y)g(x-y) = g(x)^2 - g(y)^2.$$

Proof. Let g be unbounded. An obvious slight change in the steps of the proof applied in Theorem 2.1 gives us the inequality

$$|g(u)g(v) - f(\frac{u+v}{2})^2 + f(\frac{u-v}{2})^2| \leq \delta(\|u\| + \|v\|). \tag{2.15}$$

Define the functional h on A by $h(x) := \frac{g(w+x)+g(w-x)}{2g(w)}$. Since the defined function h and (2.8) are the same roles as h of Theorem 2.2 respectively, we obtain the equation $g(x+y)+g(x-y)=2g(x)h(y)$. The rest of the proof runs along the same line as in Theorem 2.2. $\square$

Corollary 2.4. *Let A be a commutative Banach algebra and $f : A \to \mathbb{C}$ be an unbounded function satisfying the inequality*

$$|f(x+y)f(x-y)-f(x)^2+f(y)^2| \le \delta(\|x\|+\|y\|)$$

for all $x, y \in A$. Then f satisfies the sine functional equation (S).

Remark 2.5. In all results in this section, letting $\delta(\|u\|+\|v\|)=\varepsilon$: constant, then we obtain the same numbers of corollaries, which are found in papers ([5], [10]).

3. Extension to Banach Algebra

All results in the Section 2 can be extended to the superstability on the Banach space.

In this section, let A be a commutative Banach algebra, and $(E, \|\cdot\|)$ be a semisimple commutative Banach space.

Theorem 3.1. *Assume that $f, g : A \to E$ satisfy the inequality*

$$\big\|g(x+y)f(x-y)-f(x)^2+f(y)^2\big\| \le \delta(\|x\|+\|y\|) \tag{3.1}$$

for all $x, y \in A$.

Then, for an arbitrary linear multiplicative functional $x^* \in E^*$,

if the superposition $x^* \circ g$ is unbounded, then f satisfies the Wilson functional equation

$$f(x+y)+f(x-y)=2f(x)h(y) \tag{3.2}$$

with the function h defined by $h(x) := \frac{f(x+w)-f(x-w)}{2f(w)}$, where $f(w) \neq 0$.

Moreover, if the superposition $x^* \circ f$ is unbounded with $(x^* \circ f)(0)=0$, then f satisfies the sine functional equation

$$f(x+y)f(x-y)=f(x)^2-f(y)^2. \tag{3.3}$$

Proof. Assume that (3.1) holds, and fix arbitrarily a linear multiplicative functional $x^* \in E$. As it is well known, we have from (3.1) that for every $x, y \in A$

$$\begin{aligned} \delta(\|x\|+\|y\|) &\geq \left\|g(x+y)f(x-y) - f(x)^2 + f(y)^2\right\| \\ &= \sup_{\|y^*\|=1} \left|y^*\left(g(x+y)f(x-y) - f(x)^2 + f(y)^2\right)\right| \\ &\geq \left|x^*\left(g(x+y)\right) \cdot x^*\left(f(x-y)\right) - x^*(f(x))^2 + x^*(f(y))^2\right|, \end{aligned}$$

which states that the superpositions $x^* \circ f$ and $x^* \circ g$ yield a solution of stability inequality (2.1) of Theorem 2.1. By assumption, since the superposition $x^* \circ g$ is unbounded, an appeal to Theorem 2.1 shows that the superpositions $x^* \circ f$ and $x^* \circ h$ solve (2.2), namely

$$(x^* \circ f)(x+y) + (x^* \circ f)(x-y) = 2(x^* \circ f)(x)(x^* \circ h)(y), \tag{3.4}$$

in which h is defined by $h(y) := \frac{f(y+w)-f(y-w)}{2f(w)}$ with $f(w) \neq 0$.

In other words, bearing the linear multiplicativity of x^* in mind, for all $x, y \in A$, the difference of (3.2)

$$DS_{fh}(x,y) := f(x+y) + f(x-y) - 2f(x)h(y)$$

falls into the kernel of x^*. Therefore, in view of the unrestricted choice of x^*, we infer that

$$DS_{fh}(x,y) \in \bigcap\{\ker x^* : x^* \text{ is a multiplicative member of } E^*\}$$

for all $x, y \in A$. Since the algebra E has been assumed to be semisimple, the last term of the above formula coincides with the singleton $\{0\}$, i.e., which states our claimed (3.2).

In particular, if the superposition $x^* \circ f$ is unbounded with $(x^* \circ f)(0) = 0$, then, from Theorem 2.1, $x^* \circ f$ satisfies the sine functional equation (S), i.e.,

$$(x^* \circ f)(x+y)(x^* \circ f)(x-y) = (x^* \circ f)(x)^2 - (x^* \circ f)(y)^2. \tag{3.5}$$

In same above logic, bearing the linear multiplicativity of x^* in mind, for all $x, y \in A$, the difference of (3.3)

$$\begin{aligned} DS_f(x,y) &:= f(x+y)f(x-y) - f(x)^2 + f(y)^2 \\ &\in \bigcap\{\ker x^* : x^* \text{ is a multiplicative member of } E^*\} \end{aligned}$$

The semisimplity of E implies us the required result (3.3). $\square$

Theorem 3.2. *Assume that $f, g : A \to E$ satisfy the inequality*

$$\left\|f(x+y)g(x-y) - f(x)^2 + f(y)^2\right\| \leq \delta(\|x\|+\|y\|)$$

for all $x, y \in A$.

Then, for an arbitrary linear multiplicative functional $x^* \in E^*$,

if the superposition $x^* \circ g$ *is unbounded, then* f *satisfies the Wilson functional equation* (3.2) *with the function* h *defined by* $h(x) := \frac{f(w+x)+f(w-x)}{2f(w)}$, *where* $f(w) \neq 0$.

Moreover, if the superposition $x^* \circ f$ *is unbounded with* $(x^* \circ f)(0) = 0$, *then* f *satisfies the sine functional equation* (S).

Theorem 3.3. *Assume that* $f, g : A \to E$ *satisfy the inequality*

$$\left\| g(x+y)g(x-y) - f(x)^2 + f(y)^2 \right\| \leq \delta(\|x\| + \|y\|)$$

for all $x, y \in A$.

Then, for an arbitrary linear multiplicative functional $x^* \in E^*$,

if the superposition $x^* \circ g$ *is unbounded, then* f *satisfies the Wilson functional equation* (3.2) *with the function* h *defined by* $h(x) := \frac{g(w+x)+g(w-x)}{2g(w)}$, *where* $f(w) \neq 0$.

Moreover, if the superposition $x^* \circ f$ *is unbounded with* $(x^* \circ f)(0) = 0$, *then* f *satisfies the sine functional equation* (S).

Corollary 3.4. *Assume that* $f : A \to E$ *satisfy the inequality*

$$\left\| f(x+y)f(x-y) - f(x)^2 + f(y)^2 \right\| \leq \delta(\|x\| + \|y\|)$$

for all $x, y \in A$.

Then, for an arbitrary linear multiplicative functional $x^* \in E^*$,

if the superposition $x^* \circ f$ *is unbounded, then* f *satisfies the Wilson functional equation* (3.2) *with the function* h *defined by* $h(x) := \frac{f(x+w)-f(x-w)}{2f(w)}$, *where* $f(w) \neq 0$.

Moreover, if the superposition $x^* \circ f$ *is unbounded with* $(x^* \circ f)(0) = 0$, *then* f *satisfies the sine functional equation* (S).

Remark 3.5. As in Remark 2.5, for all results in this section, letting $\delta(\|u\| + \|v\|) = \varepsilon$: constant, then we obtain the same numbers of corollaries, which are found in papers ([5], [10]).

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