Korean J. Math. **19** (2011), No. 2, pp. 205–209

ON QUASI-CLASS $\mathcal{A}$ OPERATORS

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Abstract. Let $\mathcal{QA}$ denote the class of bounded linear Hilbert space operators T which satisfy the operator inequality

$$T^*|T^2|T \geq T^*|T|^2T.$$

Let f be an analytic function defined on an open neighbourhood $\mathcal{U}$ of $\sigma(T)$ such that f is non–constant on the connected components of $\mathcal{U}$. We generalize a theorem of Sheth [10] to $f(T) \in \mathcal{QA}$.

1. Introduction

Let $B(\mathcal{H})$ denote the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$. Recall ([3]) that $T \in B(\mathcal{H})$ is called *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$ for $p \in (0,1]$ and T is called *paranormal* if $||T^2x|| \geq ||Tx||^2$ for all unit vector $x \in \mathcal{H}$, and we say that $T \in B(\mathcal{H})$ belongs to *class* $\mathcal{A}$ if $|T^2| \geq |T|^2$. We shall denote classes of p-hyponormal operators, paranormal operators, and class A operators by $\mathcal{H}(p)$, $\mathcal{PN}$, and $\mathcal{A}$, respectively. It is well known that

$$\mathcal{H}(p) \subset \mathcal{A} \subset \mathcal{PN}. \tag{1}$$

In [6] Jeon and Kim considered an extension of the notion of class $\mathcal{A}$ operators; we say that $T \in B(\mathcal{H})$ is *quasi-class* $\mathcal{A}$ if

$$T^*|T^2|T \geq T^*|T|^2T.$$

Received May 9, 2011. Revised June 1, 2011. Accepted June 3, 2011.
2000 Mathematics Subject Classification: 47B20.
Key words and phrases: Quasi-class $\mathcal{A}$ operator, Sheth's theorem.
This work of the third author was supported by the Korea Research Foundation Grant funded by the Korean Government(KRF-2008-313-C00044).
This work of the first author was supported by the University of Incheon Research Grant in 2010.
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We shall denote the set of quasi-class $\mathcal{A}$ operators by $\mathcal{QA}$. Class $\mathcal{QA}$ properly contains class $\mathcal{A}$. Actually, as shown in [6, Lemma 1], an operator $T \in \mathcal{QA}$ has a matrix representation

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{T(\mathcal{H})} \\ T^{*-1}(0) \end{pmatrix}, \tag{2}$$

where $A \in \mathcal{A}$. It is well known that

$$\mathcal{H}(p) \subset \mathcal{A} \subset \mathcal{QA}. \tag{3}$$

In view of inclusions (1) and (3), it seems reasonable to expect that the operators in class $\mathcal{QA}$ are paranormal. But there exists an example of a class $\mathcal{QA}$ operator which is not paranormal ([6]).

We denote the spectrum and the closure of numerical range of an operator $T \in B(\mathcal{H})$ by $\sigma(T)$ and $\overline{W(T)}$, respectively. I. H. Sheth [10] showed that if T is a hyponormal operator and $S^{-1}TS = T^*$ for an operator S such that $0 \notin \overline{W(S)}$, then T is self-adjoint.

In this paper we extend this result to $f(T) \in \mathcal{QA}$, where f is an analytic function defined on an open neighbourhood $\mathcal{U}$ of $\sigma(T)$ such that f is non–constant on the connected components of $\mathcal{U}$: this also generalizes results proved in [8] and [7].

2. Sheth's theorem

For $T \in B(\mathcal{H})$, let $\mathcal{H}(T)$ denote the set of functions f which are analytic on an open neighbourhood $\mathcal{U}$ of $\sigma(T)$; let $\mathcal{H}_c(T)$ denote those $f \in \mathcal{H}(T)$ which are non–constant on the connected components of $\mathcal{U}$. Generalizing a result of Sheth [10], we show in the following that if $f(T)^*S = Sf(T)$, where $f(T) \in \mathcal{QA}$ and $0 \notin \overline{W(S)}$, then $T = T_0 \oplus T_1$ for some nilpotent operator T_0 and an operator T_1 quasisimilar to a self-adjoint operator. We start with some complementary results.

Lemma 2.1. *(i)* [2, Theorem 2.1] *If $T \in \mathcal{A}$, then $|T|T$ is semi–hyponormal and $\sigma(|T|T) = \{r^2e^{i\theta} : re^{i\theta} \in \sigma(T)\}$.*

(ii) [7, Lemma 3] *For every $T \in B(\mathcal{H})$, $(T^*|T|^2T)^{\frac{1}{2}} \geq |T|^2 \iff (|T^*||T|^2|T^*|)^{\frac{1}{2}} \geq |T^*|^2$.*

(iii) [5, Theorem 1(i)] *If $M, N \in B(\mathcal{H})$ are positive operators and $p, r \geq 0$, then*

$$(N^{\frac{r}{2}} M^p N^{\frac{r}{2}})^{\frac{r}{p+r}} \geq N^r \Longrightarrow (M^{\frac{p}{2}} N^r M^{\frac{p}{2}})^{\frac{r}{p+r}} \leq M^p.$$

We say that a pair of operators $X, Y \in B(\mathcal{H})$ are semi–similar if there exists a sequence of mutually orthogonal self–adjoint projections $\{P_i\}$ commuting with X and Y such that $\sum_i P_i = I$ and for each i there exists an invertible operator L_i on $P_i\mathcal{H}$ so that $L_i^{-1} X L_i = Y|_{P_i\mathcal{H}}$. Observe that semi–similarity implies quasi–similarity.

THEOREM 2.2. *Let $T \in B(\mathcal{H})$ be such that $f(T) \in \mathcal{QA}$. If*

$$Sf(T) = f(T)^* S$$

for some operator $S \in B(\mathcal{H})$ such that $0 \notin \overline{W(S)}$, then T is the direct sum of a nilpotent operator T_0 and an operator T_1 which is semi similar to a self–adjoint operator. In particular, if $f(z) = z^n$ for some odd integer $n \geq 1$, then T is the direct sum of an n-nilpotent operator and a self–adjoint operator.

Proof. Letting $\mathbf{R}$ denote the real line, the hypothesis $Sf(T) = f(T)^* S$ implies that $f(\sigma(T)) = \sigma(f(T)) \subseteq \mathbf{R}$ [11, Theorem 1]. Set $f(T) = E$; then $E \in \mathcal{QA}$, and so

$$E = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{E(\mathcal{H})} \\ E^{*-1}(0) \end{pmatrix},$$

where $A \in \mathcal{A}$ and $\sigma(E) \subseteq \sigma(A) \cup \{0\}$ [6, Lemma 1]. Let $S_1 = S|_{\overline{E(H)}}$. Then $S_1 \in B(\overline{E(\mathcal{H})})$, $0 \notin W(S_1)$ and $S_1 A = A^* S_1$. Hence $\sigma(A) \subseteq \mathbf{R}$, and this (by Lemma 2.1(i)) implies that the semi–hyponormal operator $|A|A$ has real spectrum. Since a semi–hyponormal operator with real spectrum is necessarily self–adjoint, $|A|A$ is self–adjoint. Consequently, $|A||A|^2|A| = A^*|A|^2 A = |A^2|^2$, and hence

$$|A^2| = (|A||A^*|^2|A|)^{\frac{1}{2}} = (A^*|A|^2 A)^{\frac{1}{2}}.$$

Observe that $A \in \mathcal{A}$ implies that $|A|^2 \leq |A^2| = (A^*|A|^2 A)^{\frac{1}{2}}$. Applying Lemma 2.1(ii), it follows that $(|A^*||A|^2|A^*|)^{\frac{1}{2}} \geq |A^*|^2$. Choose $p = r = 2$, $M = |A|$ and and $N = |A^*|$ in Lemma 2.1(iii); then $(|A||A^*|^2|A|)^{\frac{1}{2}} \leq |A|^2$. Hence $|A|^2 = |A^2|$, which implies that $A^*(A^*A - AA^*)A = 0$, i.e., A is quasihyponormal. Since a quasihyponormal operator A with real spectrum (hence, Lebesgue area measure $m(\sigma(A)) = 0$) is normal [1], it follows that A is self–adjoint.

Let

$$|E| = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} \begin{pmatrix} \overline{E(\mathcal{H})} \\ E^{*-1}(0) \end{pmatrix}.$$

Then $A_{11} = |A|$ and

$$|E|^2 = \begin{pmatrix} |A|^2 + |A_{12}^*|^2 & * \\ * & |A_{12}|^2 + |A_{22}|^2 \end{pmatrix} = \begin{pmatrix} |A|^2 & AB \\ B^*A & |B|^2 \end{pmatrix},$$

which implies that $A_{12} = 0$ and $|E| = |A| \oplus |B|$ with $AB = 0$. Since $E^2 = A^2 \oplus 0$, $|E^2| = |A^2| \oplus 0$ and it follows that

$$E^*\{(|A^2| \oplus 0) - (|A|^2 \oplus |B|^2)\}E = E^*\{0 \oplus -|B|^2\}E \geq 0.$$

Hence $B = 0$, which implies that $E = f(T)$ is self–adjoint. Applying [4, Theorem 3.1] to the equation $f(T) = E$, E self-adjoint, it follows that T is the direct sum of a nilpotent operator T_0 and an operator T_1 semi–similar to a self-adjoint operator.

Now let $f(z) = z^{2m+1}$ for some non–negative integer m. Then

$$T = T_0 \oplus T_1 = T_0 \oplus H \oplus \begin{pmatrix} B & C \\ 0 & -B \end{pmatrix},$$

where T_0 is (now) $2m+1$-nilpotent, H is self-adjoint and B, C are commuting one–one positive operators [9, Theorem 3]. Evidently,

$$F = \begin{pmatrix} B & C \\ 0 & -B \end{pmatrix} \in \mathcal{QA}.$$

Hence

$$\begin{aligned} & F^*\{|F^2| - |F|^2\}F \\ = \ & F^*\left[\begin{pmatrix} B^{4m+2} & 0 \\ 0 & B^{4m+2} \end{pmatrix} - \begin{pmatrix} B^{4m+2} & CB^{4m+1} \\ CB^{4m+1} & B^{4m+2} + C^2B^{4m} \end{pmatrix}\right]F \\ = \ & F^*\left[\begin{pmatrix} 0 & -CB^{4m+1} \\ -CB^{4m+1} & -C^2B^{4m} \end{pmatrix}\right]F \\ = \ & \begin{pmatrix} 0 & CB^{8m+3} \\ CB^{8m+3} & C^2B^{8m+2} \end{pmatrix} \geq 0. \end{aligned}$$

This, however, is impossible. Hence the part F is absent, i.e., $T = T_0 \oplus H$. □

References

[1] S. L. Campbell and B. C. Gupta, *On k-quasihyponormal operators*, Math. Japon. **23**(1978-79), 185–189.

[2] M. Cho and T. Yamazaki, *An operator transform from class A to the class of hyponormal operators and its application*, Integral Equations Operator Theory, **53**(2005), 497–508.

[3] T. Furuta, Invitation to Linear Operators, Taylor and Francis, London (2001).

[4] F. Gilfeather, *Operator valued roots of abelian analytic functions*, Pacific J. Math. **55**(1974), 127–148.

[5] M. Ito and T. Yamazaki, *Relations between two inequalities $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ and $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{r}{p+r}}$ and their applications*, Integral Equations Operator Theory, **44**(2002), 442–450.

[6] I. H. Jeon and I. H. Kim, *On operators satisfying $T^*|T^2|T \geq T^*|T|^2T$*, Linear Algebra Appl. **418**(2006), 854–862.

[7] I. H. Jeon, I. H. Kim, K. Tanahashi and A. Uchiyama, *Conditions implying self–adjointness of operators*, Integral Equations Operator Theory, **61**(2008), 549–557.

[8] I. H. Kim, *The Fuglede-Putnam theorem for (p,k)-quasihyponormal operators*, J. Inequal. Appl. (2006), 397–403.

[9] H. Radjavi and P. Rosenthal, *On roots of normal operators*, J. Math. Anal. Appl. **34**(1971), 653-664.

[10] I. H. Sheth, *On hyponormal operators*, Proc. Amer. Math. Soc., **17**(1966), 998–1000.

[11] J. P. Williams, *Operators similar to their adjoints*, Proc. Amer. Math. Soc. **20**(1969), 121–123.

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