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THE EQUIVALENT CONDITIONS FOR THE HOMOMORPHISM OF MINIMAL SETS TO BE REGULAR

H. S. Song

Abstract. In this paper we study some properties on regular homomorphisms. In particular, we investigate the equivalent conditions for the homomorphism of minimal sets to be regular

1. Introduction

Regular minimal sets were first studied by Auslander in [1]. These minimal sets may be described as minimal subsets of enveloping semigroups. In [5], Shoenfeld introduced the regular homomorphisms which are defined by extending regular minimal sets to homomorphisms with minimal range and obtained several characterizations on regular homomorphisms.

The purpose of this paper is to study some properties on regular homomorphisms and investigate the equivalent conditions for the homomorphism of minimal sets to be regular.

2. Preliminaries

A *transformation group*, or *flow*, (X, T), will consist of a jointly continuous action of the topological group T on the compact Hausdorff space X. The group T, with identity e, is assumed to be topologically discrete

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and remain fixed throughout this paper, so we may write X instead of (X, T).

A *homomorphism* of flows is a continuous, equivariant map. A homomorphism from X into itself is called an *endomorphism* of X. Especially, a one-one homomorphism of X onto X is called an automorphism of X. We denote the group of automorphisms of X by $A(X)$.

A flow is said to be *minimal* if every point has dense orbit. Minimal flows are also referred to as minimal sets. M is said to be a *universal minimal set* if it is a minimal set such that every minimal set is a homomorphic image of M. A homomorphism whose range is minimal is always onto.

The compact Hausdorff space X carries a natural uniformity whose indices are the neighborhoods of the diagonal in $X \times X$. Two points $x, x' \in X$ are said to be *proximal* if, given any index α, there exists $t \in T$ such that $(xt, x't) \in \alpha$. The *proximal relation* in X, denoted by $P(X, T)$, is the set of all proximal pairs in X. X is said to be *distal* if $P(X, T) = \triangle$, the diagonal of $X \times X$ and is said to be *proximal* if $P(X, T) = X \times X$. Given $x \in X$, we define $P(x) = \{x' \in X \mid (x, x') \in P(X, T)\}$.

A homomorphism $\pi : X \to Y$ is said to be *proximal* if whenever $x, x' \in \pi^{-1}(y)$ then x and x' are proximal.

Given a flow (X, T), we may regard T as a set of self-homeomorphisms of X. We define $E(X)$, the *enveloping semigroup* of X to be the closure of T in X^X, taken with the product topology. $E(X)$ is at once a flow and a sub-semigroup of X^X. The minimal right ideals of $E(X)$, considered as a semigroup, coincide with the minimal sets of $E(X)$.

If E is some enveloping semigroup, and there exists a homomorphism $\theta : E \to E(X)$ we say that E is an *enveloping semigroup for* X. If such a homomorphism exists, it must be unique, and, given $x \in X$ and $p \in E$ we may write xp to mean $x\theta(p)$ unambiguously.

Given a minimal right ideal I in some enveloping semigroup, we denote the set of idempotent elements in I by $J(I)$.

Lemma 2.1. *([5]) Given $x \in X$ and minimal subset N of $\overline{xT}$, there exists a minimal right ideal I such that $N = xI$.*

Lemma 2.2. *([3]) Let $\pi : (X, T) \to (Y, T)$ be an epimorphism(or onto homomorphism). Then there exists a unique epimorphism $\psi : E(X) \to E(Y)$ such that $\pi(x)\psi(p) = \pi(xp)$ for all $p \in E(X)$.*

DEFINITION 2.3. *([1]) A minimal set X which satisfies any one (and therefore all) of the following properties is called regular.*

(a) *For any two points $x, x' \in X$ there exists an endomorphism $h : X \to X$ such that $h(x)$ and x' are proximal.*
(b) *For any two points $x, x' \in X$ with (x, x') almost periodic there exists an endomorphism $h : X \to X$ such that $h(x) = x'$.*
(c) *(X, T) and (I, T) are isomorphic, where I is a minimal right ideal in $E(X)$.*

REMARK 2.4. *Since X is distal iff $E(X)$ is minimal, Definition 2.3 shows that if X is distal and regular, (X, T) and $(E(X), T)$ are isomorphic(see [3, Proposition 5.3]).*

3. The equivalent conditions for the homomorphism to be regular

Let $\pi : X \to Y$ be a fixed homomorphism with Y minimal and suppose $y \in Y$. Then $X^{\pi^{-1}(y)}$ is a flow whose elements are functions from $\pi^{-1}(y)$ to X.

DEFINITION 3.1. *([5]) Define $z_y \in X^{\pi^{-1}(y)}$ by $z_y(x) = x$ for all $x \in \pi^{-1}(y)$. Let $E(\pi, y)$ be the orbit closure of z_y, i.e., $E(\pi, y) = \overline{z_y T} \subset X^{\pi^{-1}(y)}$.*

REMARK 3.2. (1) *If Y is a singleton $\{y\}$, then $E(\pi, y) = E(X)$.*
(2) *For each $y \in Y$, $E(X)$ is an enveloping semigroup for $E(\pi, y)$.*

DEFINITION 3.3. *([5]) Let $\overline{\pi}_y : E(\pi, y) \to Y$ be the unique homomorphism with $\overline{\pi}_y(z_y) = y$.*

THEOREM 3.4. *([5]) Suppose N and N' are minimal subsets of $E(\pi, y)$ and $E(\pi, y')$ respectively. Then there exists an isomorphism $\psi : N \to N'$ such that $(\overline{\pi}_{y'}|_{N'}) \circ \psi = \overline{\pi}_y|_N$.*

REMARK 3.5. *Theorem 3.4 shows that the minimal sets of $E(\pi, y)$ are isomorphic and independent of the choice of y and hence it defines a minimal set N and an essentially (up to isomorphism) unique homomorphism which we call $\overline{\pi} : N \to Y$.*

DEFINITION 3.6. *([5]) Given a homomorphism $\pi : X \to Y$ with Y minimal, we call the homomorphism $\overline{\pi} : N \to Y$ the regularizer of π and we say that $\overline{\pi}$ is a regular homomorphism.*

Thus a homomorphism is regular if and only if it is isomorphic to the regularizer of some homomorphism.

Given a homomorphism $\psi : Z \to W$, we define $\mathrm{Aut}\psi = \{\theta \in A(Z) \mid \psi\circ\theta = \psi\}$. We say that ψ is a *group extension* if whenever $z, z' \in Z$ and $\psi(z) = \psi(z')$, there exists $\theta \in \mathrm{Aut}\psi$ such that $\theta(z) = z'$.

Now let M be a fixed universal minimal set and let $G = A(M)$. Given a homomorphism $\gamma : M \to X$ we define the subgroups $G(X, \gamma)$ and $S(X, \gamma)$ of G as follows(see [2], [7]);

$G(X, \gamma) = \mathrm{Aut}\gamma$

$S(X, \gamma) = \{\alpha \in G \mid h\circ\gamma\circ\alpha = \gamma \text{ for some } h \in A(X)\}$.

REMARK 3.7. (1) *$G(X, \gamma)$ is a normal subgroup of $S(X, \gamma)$.*

(2) *Suppose we have homomorphisms of minimal sets $\gamma : M \to X$ and $\pi : X \to Y$. Then $G(X, \gamma) \subset G(Y, \pi\circ\gamma)$.*

DEFINITION 3.8. *([5]) We say that a homomorphism $\psi : Z \to Y$, Z and Y minimal, is* regular with respect to *$\pi : X \to Y$ if, given any pair of homomorphisms $\gamma : M \to X$ and $\delta : M \longrightarrow Z$ with $\pi\circ\gamma = \psi\circ\delta$, there exists a homomorphism $\theta : Z \to X$ with $\theta\circ\delta = \gamma$ and $\pi\circ\theta = \psi$.*

THEOREM 3.9. *([5]) $\overline{\pi}$ is regular with respect to π.*

THEOREM 3.10. *([7]) Suppose we have homomorphisms of minimal sets $\gamma : M \to X$ and $\pi : X \to Y$. Then the following conditions are equivalent :*

(1) *π is regular with respect to itself.*
(2) *$G(Y, \pi\circ\gamma) \subset S(X, \gamma)$.*

THEOREM 3.11. *Suppose we have homomorphisms of minimal sets $\gamma : M \to X$ and $\pi : X \to Y$. Then the following conditions are equivalent :*

(1) *π is regular.*
(2) *π is regular with respect to itself.*
(3) *π is its own regularizer.*
(4) *For any two points $x, x' \in X$ with $\pi(x) = \pi(x')$ there exists an endomorphism $\theta : X \to X$ such that $\theta(x)$ and x' are proximal and $\pi\circ\theta = \pi$.*
(5) *For any two points $x, x' \in X$ with (x, x') almost periodic and $\pi(x) = \pi(x')$ there exists an endomorphism $\theta : X \to X$ such that $\theta(x) = x'$ and $\pi\circ\theta = \pi$.*
(6) *$G(Y, \pi\circ\gamma) \subset S(X, \gamma)$.*

(7) *Suppose $y \in Y$. Then the fiber $\pi^{-1}(y)$ of π is partitioned by the collection of sets* $\{\pi^{-1}(y)u \mid u \in J(M)$ and $yu = y\}$ *and* $\pi\ |_{\pi^{-1}(y)u}$ *is a group extension.*

Proof. That (1), (2), (3), (4), and (5) are equivalent follows from [5, Proposition 2.2.8]. That (2) and (6) are equivalent by Theorem 3.10. Finally we show that (5) and (7) are equivalent.

That $\{\pi^{-1}(y)u \mid u \in J(M)$ and $yu = y\}$ partitions $\pi^{-1}(y)$ follows from [5, Corollary 2.2.9]. Let $x, x' \in \pi^{-1}(y)u$. Then there exist $x_1, x_2 \in \pi^{-1}(y)$ with $x = x_1u$ and $x' = x_2u$. This implies that $(x, x')u = (x, x')$ so (x, x') is almost periodic. Applying (5) then yields the required group extension.

Now let $x, x' \in X$ with (x, x') almost periodic and $\pi(x) = \pi(x') = y$. Since (x, x') is almost periodic, it follows that there exists $u \in J$ such that $(x, x')u = (x, x')$. Hence $x, x' \in \pi^{-1}(y)u$. By hypothesis there exists $\theta \in$Autπ such that $\theta(x) = x'$. □

Remark 3.12. (1) *If X is regular, the homomorphism of minimal sets $\pi : X \to Y$ is regular.*

(2) *We may use Theorem 3.11 (5) and the following statement interchangeably by [5, Lemma 2.2.2].*

For any two points $x, x' \in X$ with (x, x') almost periodic and $\pi(x) = \pi(x')$ there exists $\theta \in$Autπ such that $\theta(x) = x'$.

(3) *Theorem 3.11 shows that a proximal homomorphism of minimal sets is always regular.*

(4) *If X is proximal and minimal, it is regular.*

(5) *If π is regular, then $G(X, \gamma)$ is a normal subgroup of $G(Y, \pi \circ \gamma)$ (see [5, Proposition 2.3.3]).*

Theorem 3.13. *Suppose we have homomorphisms of minimal sets $\gamma : M \to X$ and $\pi : X \to Y$. Then the following conditions are equivalent :*

(1) *π is proximal.*

(2) *Given a homomorphism $\delta : M \to X$ with $\pi\delta = \pi\gamma$, we have $\delta = \gamma$.*

(3) *$G(X, \gamma) = G(Y, \pi\circ\gamma)$.*

(4) *For any two points $x, x' \in X$ with (x, x') almost periodic and $\pi(x) = \pi(x')$, we have $x = x'$.*

(5) *Suppose that $y \in Y$ and that $u \in J(I)$ with $yu = y$. Then $\pi^{-1}(y)u$ is a singleton.*

(6) *Suppose $y \in Y$. Then $\pi^{-1}(y) \subset xJ(M)$ for any $x \in \pi^{-1}(y)$.*

Proof. That (1) and (5) are equivalent follows from [5, Lemma 2.5.8]. That (1) and (6) are equivalent follows from immediately from [4, Proposition 4.1] and the fact that if $x, x' \in \pi^{-1}(y)$, then there exists $u \in J$ such that $x' = xu$.

(1) implies (4). This follows from the fact that if a pair of points is both proximal and almost periodic, the two points are identical.

(4) implies (1). Suppose $x, x' \in X$ and $\pi(x) = \pi(x')$. Since X is minimal, there exists $v \in J$ with $xv = x$. Let $x'' = x'v$. Then x'' and x' are proximal and $(x, x'')v = (x, x'')$ implies that (x, x'') is almost periodic. But $\pi(x) = \pi(x'')$. Thus $x'' = x$ and therefore $x = x'v$ which implies that x and x' are proximal. This proves that π is proximal.

(2) implies (3). Let $\theta \in G(X, \gamma)$. Then $\gamma \circ \theta = \gamma$ whence $(\pi \circ \gamma) \circ \theta = \pi \circ \gamma$. This implies that $\theta \in G(Y, \pi \circ \gamma)$. Now let $\theta \in G(Y, \pi \circ \gamma)$. Then $\pi \circ (\gamma \circ \theta) = (\pi \circ \gamma) \circ \theta = \pi \circ \gamma$ and therefore $\gamma \circ \theta = \gamma$ which proves that $\theta \in G(X, \gamma)$.

(3) implies (4). Suppose $x, x' \in X$ with (x, x') almost periodic and $\pi(x) = \pi(x')$. Then there exists an almost periodic point $(p, p') \in M \times M$ such that $\gamma \times \gamma(p, p') = (x, x')$. Since M is regular, we have from Remark 3.12 (2) that there exists $\theta \in A(M)$ with $\theta(p) = p'$. Hence $\pi \circ \gamma \circ \theta(p) = \pi \circ \gamma(p') = \pi(x') = \pi(x) = \pi \circ \gamma(p)$. Since M is minimal, it follows that $\pi \circ \gamma \circ \theta = \pi \circ \gamma$. This implies that $\theta \in G(Y, \pi \circ \gamma) = G(X, \gamma)$. Thus $x = \gamma(p) = \gamma\theta(p) = \gamma(p') = x'$.

(4) implies (2). Given homomorphisms $\gamma : M \to X$ and $\delta : M \to X$ with $\pi\delta = \pi\gamma$ we pick $u \in J(M)$, and let $\gamma(u) = x'$ and $\delta(u) = x$. Then $(x', x)u = (x', x)$ so (x', x) is almost periodic. Now $\pi(x') = \pi\gamma(u) = \pi\delta(u) = \pi(x)$. Thus we get $x' = x$, so that $\gamma(u) = \delta(u)$. Since M is minimal, it follows that $\gamma = \delta$. □

THEOREM 3.14. *Suppose we have homomorphisms of minimal sets $\gamma : M \to X$ and $\pi : X \to Y$. Suppose $u \in J(M)$, $x_0 \in X$ with $\gamma(u) = x_0$ and $y \in Y$, $p \in M$ with $\pi \circ \gamma(p) = y$. If $x_0\theta(p) \in x_0 J(M)$ for all $\theta \in G(X, \gamma)$, then $P(\pi(x_0)) = Y$.*

Proof. Suppose that $y \in Y$ and that $p \in M$ with $\pi \circ \gamma(p) = y$. Then by hypothesis there exists $v \in J(M)$ such that $x_0\theta(p) = x_0 v$ for all $\theta \in G(X, \gamma)$. Since $\gamma \circ \theta = \gamma$, it follows that $\pi(x_0)v = \pi(\gamma(u)\theta(p)) = \pi \circ \gamma(\theta(p)) = \pi \circ \gamma \circ \theta(p) = \pi \circ \gamma(p) = y$. This means that $\pi(x_0)$ and y are proximal. □

The proof of the following result are similar to that of Theorem 3.14.

THEOREM 3.15. *Suppose we have homomorphisms of minimal sets* $\gamma : M \to X$ *and* $\pi : X \to Y$. *Suppose* $u \in J(M)$, $x_0 \in X$ *with* $\gamma(u) = x_0$ *and* $x \in X$, $p \in M$ *with* $\gamma(p) = x$. *If* $x_0\theta(p) \in x_0J(M)$ *for all* $\theta \in G(X, \gamma)$, *then* $P(x_0) = X$.

THEOREM 3.16. *Suppose that* $\pi : X \to Y$ *is a homomorphism of minimal sets and that* X *is distal. Then the following statements are equivalent :*

(1) π *is regular.*
(2) π *is a group extension.*

Proof. Suppose $x, x' \in X$ and $\pi(x) = \pi(x')$. Since X is distal, it follows from [3, Proposition 5.8] that $X \times X$ is distal whence $X \times X$ is pointwise almost periodic. Since (x, x') almost periodic and π is regular, there exists $\theta \in$Autπ such that $\theta(x) = x'$.

The converse is clear.

□

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Department of Mathematics
Kwangwoon University
Seoul 139–701, Korea
E-mail: songhs@kw.ac.kr