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AN APPLICATION OF THE LERAY-SCHAUDER DEGREE THEORY TO THE VARIABLE COEFFICIENT SEMILINEAR BIHARMONIC PROBLEM

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Abstract. We obtain multiplicity results for the nonlinear biharmonic problem with variable coefficient. We prove by the Leray-Schauder degree theory that the nonlinear biharmonic problem has multiple solutions for the biharmonic problem with the variable coefficient semilinear term under some conditions.

1. Introduction

In this paper we consider the multiplicity result for the following biharmonic equation with the variable coefficient semilinear term and Dirichlet boundary condition

$$\Delta^2 u + c\Delta u = b(x)u^+ + s\psi_1(x), \qquad \text{in } \Omega, \qquad\qquad (1.1)$$
$$u = 0, \qquad \Delta u = 0, \qquad \text{on } \partial\Omega$$

where Δ is the Laplace operator is the positive eigenfunction of $\Delta + c\Delta - b(x)$ with Dirichlet boundary condition. Here Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$, $b(x)$ is *Hölder* continuous in Ω. We set $c \in R$, $u^+ = \max\{u, 0\}$ and $u^- = -\min\{u, 0\}$.

Let $\lambda_k (k = 1, 2, \cdots)$ denote the eigenvalues and $\phi_k (k = 1, 2, \cdots)$ the corresponding eigenfunctions, suitably normalized with respect to $L^2(\Omega)$ inner product, of the eigenvalue problem

$$\Delta u + \lambda u = 0 \qquad \text{in } \Omega,$$

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$$u = 0 \qquad \text{on} \ \ \partial\Omega,$$

where each eigenvalue λ_k is repeated as often as its multiplicity.

Choi and Jung [1] showed that the problem

$$\Delta^2 u + c\Delta u = bu^+ + s, \qquad \text{in } \Omega, \tag{1.2}$$
$$u = 0, \qquad \Delta u = 0, \qquad \text{on } \partial\Omega$$

has at least two solutions when $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$, $s < 0$ and when $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$, $s > 0$. They obtained these results by using the variational reduction method. They [2] also proved that when $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$ and $s < 0$, (1.2) has at least three nontrivial solutions by using degree theory. Tarantello [5] also studied the jumping problem

$$\Delta^2 u + c\Delta u = b((u+1)^+ - 1), \qquad \text{in } \Omega, \tag{1.3}$$
$$u = 0, \qquad \Delta u = 0, \qquad \text{on } \partial\Omega.$$

She show that if $c < \lambda_1$ and $b \geq \lambda_1(\lambda_1 - c)$, then (1.3) has at least two solutions, one of which is a negative solution. She obtained this result by degree theory. Micheletti and Pistoia [4] also proved that if $c < \lambda_1$ and $b \geq \lambda_2(\lambda_2 - c)$, then (1.3) has at least four solutions by the variational linking theorem and Leray-Schauder degree theory.

In section 2 we investigate a priori estimate of the solutions of (1.1) and the no solvability condition. In section 3 we prove the existence of multiple solutions of (1.1).

2. Preliminaries

Let $\lambda_k (k = 1, 2, \cdots)$ denote the eigenvalues and $\phi_k (k = 1, 2, \cdots)$ the corresponding eigenfunctions, suitably normalized with respect to $L^2(\Omega)$ inner product, of the eigenvalue problem

$$\Delta u + \lambda u = 0 \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on} \ \ \partial\Omega,$$

where each eigenvalue λ_k is repeated as often as its multiplicity. We recall that $\lambda_1 < \lambda_2 \leq \lambda_3 \ldots \to +\infty$, and that $\phi_1(x) > 0$ for $x \in \Omega$. The eigenvalue problem

$$\Delta^2 u + c\Delta u = \Gamma u \qquad \text{in } \Omega,$$
$$u = 0, \qquad \Delta u = 0 \qquad \text{on} \ \ \partial\Omega$$

has also infinitely many eigenvalues $\Gamma_k = \lambda_k(\lambda_k - c)$, $k \geq 1$ and corresponding eigenfunctions ϕ_k, $k \geq 1$. We note that

$$\lambda_1(\lambda_1 - c) < \lambda_2(\lambda_2 - c) \leq \lambda_3(\lambda_3 - c) < \cdots .$$

The eigenvalue problem

$$\Delta^2 u + c\Delta u - b(x)u = \Lambda u \qquad \text{in } \Omega,$$

$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega$$

has also infinitely many eigenvalues Λ_k, $k \geq 1$, and ψ_k, $k \geq 1$ the corresponding eigenfunctions. We assume that the eigenfunctions are normalized with respect to H inner product. Standard eigenvalue theory gives that

$$\Lambda_1 < \Lambda_2 \leq \Lambda_3 \leq \cdots , \qquad \Lambda_k \to +\infty \qquad \text{as } k \to +\infty,$$

$$\psi_1(x) > 0 \qquad \text{in } \Omega.$$

Let $L^2(\Omega)$ be a square integrable function space defined on Ω. Any element u in $L^2(\Omega)$ can be written as

$$u = \sum h_k \psi_k \quad \text{with} \quad \sum h_k^2 < \infty.$$

We define a subspace H of $L^2(\Omega)$ as follows

$$H = \{u \in L^2(\Omega) | \sum |\Lambda_k| < \infty\}.$$

Then this is a complete normed space with a norm

$$\|u\| = [\sum |\Lambda_k| h_k^2]^{\frac{1}{2}}.$$

Since $\Lambda_k \to +\infty$ and c is fixed, we have

(i) $\Delta^2 u + c\Delta u - b(x)u \in H$ implies $u \in H$.
(ii) $\|u\| \geq C\|u\|_{L^2(\Omega)}$, for some $C > 0$.
(iii) $\|u\|_{L^2(\Omega)} = 0$ if and only if $\|u\| = 0$.

Now we investigate the no solvability condition for (1.1):

Lemma 2.1. *Assume that* $c < \lambda_1$ *and* $\lambda_n(\lambda_n - c) < b(x) < \lambda_{n+1}(\lambda_{n+1} - c)$, $n \geq 1$. *Then we have:*
(i) If $s > 0$, *then (1.1) has no solution.*
(ii) If $s = 0$, *then (1.1) has only the trivial solution* $u = 0$.

Proof. We rewrite (1.1) as

$$(2.1)\qquad \begin{aligned}(\Delta^2 + c\Delta - b(x) &- \Lambda_1)u \\ &= -\Lambda_1 u^+ + (b(x) + \Lambda_1)u^- + s\psi_1(x).\end{aligned}$$

Taking the inner product of both sides of (2.1), we have

$$(2.2)\qquad \begin{aligned}0 &= ((\Delta^2 + c\Delta - b(x) - \Lambda_1)u, \psi_1(x)) \\ &= (-\Lambda_1 u^+ + (b(x) + \Lambda_1)u^- + s\psi_1(x), \psi_1(x)).\end{aligned}$$

The conditions $c < \lambda_1$ and $\lambda_n(\lambda_n - c) < b(x) < \lambda_{n+1}(\lambda_{n+1} - c)$, $n \geq 1$ imply that $\Lambda_1 < 0$ and $b(x) + \Lambda_1 \geq 0$. Thus it follows that for $s > 0$ the left hand side of (2.2) is 0 and the right hand side of (2.2) is positive. Then (1.1) has no solution. If $s = 0$, then the only possibility to hold (2.2) is $u = 0$. □

LEMMA 2.2. *Assume that $\lambda_1 < c < \lambda_2$ and $b(x) < \lambda_1(\lambda_1 - c)$. Then we have:*
(i) If $s < 0$, then (1.1) has no solution.
(ii) If $s = 0$, then (1.1) has only trivial solution.

Proof. The conditions $\lambda_1 < c < \lambda_2$ and $b(x) < \lambda_1(\lambda_1 - c)$ imply that $\Lambda_1 > 0$ and $b(x) + \Lambda_1 \leq 0$. It follows that from (2.2),

$$\begin{aligned}0 &= ((\Delta^2 + c\Delta - b(x) - \Lambda_1)u, \psi_1(x)) \\ &= (-\Lambda_1 u^+ + (b(x) + \Lambda_1)u^- + s\psi_1(x), \psi_1(x)) \leq s.\end{aligned}$$

If $s < 0$, the left hand side of (2.2) is 0 and the right hand side of (2.2) is negative. Thus (1.1) has no solution. If $s = 0$, then the only possibility to hold the above equation is $u = 0$. □

We have a *prior* bound for the solutions of (1.1).

LEMMA 2.3. *Assume that $\Lambda_1 < -\epsilon < 0$ and $b(x) + \Lambda_1 \geq \epsilon > 0$. Then there exist a constant $C > 0$ and $s_0 < 0$ such that if u is a solution of (1.1) with s, $s \geq s_0$, then $\|u\| \leq C$.*

Proof. From (2.2) we have

$$s = ((\Lambda_1 u^+ - (b(x) + \Lambda_1)u^-, \psi_1(x)).$$

Since $(\Lambda_1 u^+ - (b(x) + \Lambda_1)u^- \leq -\epsilon|u|$, we have

$$-s \geq \epsilon \int_\Omega |u|\psi_1(x) \geq \epsilon|\int_\Omega u\psi_1(x)|.$$

Thus if u is a solution of (1.1), we have

$$|(u, \psi_1(x))| \leq \frac{1}{\epsilon}(-s), \tag{2.3}$$

where $s \leq 0$. We argue by contradiction. Suppose that there exists a sequence (u_n, s_n) such that $s_n \leq 0$, s_n is bounded, $\|u_n\| \to \infty$ and u_n satisfy the equations

$$(\Delta^2 + c\Delta - b(x) - \Lambda_1)u_n = -\Lambda_1 u_n^+ + (b(x) + \Lambda_1)u_n^- + s_n\psi_1(x).$$

Let $v_n = \frac{u_n}{\|u_n\|}$. By the compactness of v_n, there exists v such that $v_n \to v$. v satisfies $\|v\| = 1$ and

$$(\Delta^2 + c\Delta - b(x) - \Lambda_1)v + \Lambda_1 v^+ - (b(x) + \Lambda_1)v^- = 0. \tag{2.4}$$

Since, from (1.1), we have

$$(\Delta^2 + c\Delta)v_n = b(x)v_n^+ + s_n\frac{\psi_1(x)}{\|u_n\|},$$

(2.3) with u_n instead of u and the boundedness of s_n implies that

$$|(v_n, \psi_1(x))| \leq \frac{1}{\epsilon\|u_n\|}(-s_n) \to 0 \qquad \text{as} \quad n \to \infty.$$

So we have that $|(v, \psi_1(x))| = 0$. By (2.4), we obtain

$$\int_\Omega (-\Lambda_1 v^+ + (b(x) + \Lambda_1)v^-)\psi_1(x) = 0. \tag{2.5}$$

Since $-\Lambda_1 v^+ + (b(x) + \Lambda_1)v^- \geq \epsilon|v|$ and $\psi_1(x) > 0$, the only possibility to hold (2.5) is that $v = 0$, which is impossible, since $\|v\| = 1$. Thus we prove the lemma. □

3. Main result

We have the main result of this paper.

THEOREM 3.1. *Let $c < \lambda_1$ and $\lambda_n(\lambda_n - c) < b(x) < \lambda_{n+1}(\lambda_{n+1} - c)$, $n \geq 1$. Then there exists $s_0 < 0$ such that for any s with $0 < s \leq s_0$ if n is even then (1.1) has at least three solutions, one of which is a positive solution, and if n is odd then (1.1) has at least two solutions, one of which is a positive solution.*

Throughout this section we assume that $c < \lambda_1$, $\lambda_n(\lambda_n - c) < b(x) < \lambda_{n+1}(\lambda_{n+1} - c)$, $n \geq 1$.

LEMMA 3.1. *Assume that* $c < \lambda_1$ *and* $\lambda_n(\lambda_n - c) < b(x) < \lambda_{n+1}(\lambda_{n+1} - c)$, $n \geq 1$. *Then there exist a constant* $R > 0$ *(depending on* C *which is introduced in Lemma 2.3) and* $s_{0<0}$ *such that the Leray-Schauder degree*

$$d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(b(x)u^+ + s\psi_1(x)), B_R(0), 0) = 0$$

for $R > C$ *and* $s \geq s_0$.

Proof. By Lemma 2.3, there exist a constant C and $s_0 < 0$ such that if u is a solution of (1.1) with s, $s \geq s_0$, then $\|u\| \leq C$. Let us choose R such that $R > C$. By Lemma 2.1, (1.1) has no solution when $s > 0$. Let us choose $s^* > 0$ such that (1.1) has no solution. Then the Leray-Schauder degree $d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(b(x)u^+ + s^*\psi_1(x)), B_R(0), 0) = 0$. Since the Leray-Schauder degree is invariant under a homotopy, we have that the Leray-Schauder degree

$$\begin{aligned} &d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(b(x)u^+ + s\psi_1(x)), B_R(0), 0) \\ &= d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(b(x)u^+ + s\psi_1(x) \\ &\qquad\qquad + \lambda(s^* - s)\psi_1(x)), B_R(0), 0) \\ &= d_{LS}(u - (u - (\Delta^2 + c\Delta)^{-1}(b(x)u^+ + s^*\psi_1(x)), B_R(0), 0) \\ &= 0, \end{aligned}$$

where $0 \leq \lambda \leq 1$ and $0 \geq s \geq s_0$. Thus we prove the lemma. □

We note that $u_1 = \frac{s}{\Lambda_1}\psi_1(x)$ is a solution of (1.1) and positive under the condition ($c < \lambda_1$, $\lambda_n(\lambda_n - c) < b(x) < \lambda_{n+1}(\lambda_{n+1} - c)$) or the condition ($\lambda_1 < c < \lambda_2$, $b(x) < \lambda_1(\lambda_1 - c)$). We consider the linear problem

$$\begin{aligned} \Delta^2 u + c\Delta u &= s\psi_1(x), \qquad \text{in } \Omega, \\ u = 0, \qquad \Delta u &= 0, \qquad \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

This linear problem has a unique solution under the above each condition, respectively. We now calculate the Leray-Schauder degree of the operator $u - (\Delta^2 + c\Delta)^{-1}(b(x)u^+ + s\psi_1(x))$ on the small neighborhood of the positive solution y_1 of (1.1).

LEMMA 3.2. *Assume that* $c < \lambda_1$ *and* $\lambda_n(\lambda_n - c) < b(x) < \lambda_{n+1}(\lambda_{n+1} - c)$, $n \geq 1$. *Then there exist* $s_0 < 0$ *and a small number* $\eta > 0$ *such that for any* s *with* $0 \geq s \geq s_0$, *the Leray-Schauder degree*

$$d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(b(x)u^+ + s\psi_1(x)), B_\eta(y_1), 0) = (-1)^n,$$

where y_1 *is the unique negative solution of (3.1).*

Proof. The linear problem (3.1) has a unique solution y_1. For $s < 0$, $s\psi_1(x) < 0$. Since $c < \lambda_1$, by the standard strong the maximum principle to $w = \Delta u$ and consequently to u, the unique solution y_1 is negative. If u is a nontrivial solution of (1.1) with $u \leq 0$ on $\bar{\Omega}$, then $b(x)u^+ + s\psi_1(x) < 0$, so by the maximum principle as stated above, we derive that $u < 0$ in Ω. Moreover since $b(x)u^+ + s\psi_1(x)) \geq s\psi_1(x))$, by the maximum principle, $u \geq y_1$. Let K be the closure of $(\Delta^2 + c\Delta - b(x))^{-}(\bar{B})$, where $\bar{B}$ is the closed unit ball in $L_2(\Omega)$. Let u be a solution of (1.1) which is different from the positive solution $u_1 = \frac{s}{\Lambda_1}\psi_1(x)$ of (1.1). Since y_1 is negative, we can take $\eta < \max|u_1(x) - y_1(x)|$ such that the ball $B_\eta(y_1)$ with center y_1 and radius η does not contain u_1. Let us write $u = y_1 + v$ and $\|v\| = \eta$. Then v satisfies the equation

$$
\begin{aligned}
(\Delta^2 + c\Delta - b(x))v &= b(x)(y_1 + v)^- + b(x)y_1 \\
&= b(x)(y_1 + v)^+ - b(x)v
\end{aligned}
\tag{3.2}
$$

or

$$
v = (\Delta^2 + c\Delta - b(x))^{-1}(b(x)(y_1 + v)^+ - b(x)v). \tag{3.3}
$$

Let us set $\beta = \max b(x)$. We can easily check that $(y_1 + v)^+ < v^+ \leq \|v\|$. It follows that

$$
v \in 2\beta\eta K. \tag{3.4}
$$

It follows from (3.3) that

$$
\begin{aligned}
v &+ (\Delta^2 + c\Delta - b(x))^{-1}b(x)v \\
&= (\Delta^2 + c\Delta - b(x))^{-1}b(x)(y_1 + v)^+.
\end{aligned}
\tag{3.5}
$$

The function $w = \frac{v}{\eta}$ has the properties $\|w\| = 1$ and $w \in 2\beta K$. Since w is in compact set and different from zero and since $b(x)$ is not eigenvalue, $\inf_w \|w + (\Delta^2 + c\Delta - b(x))^{-1}b(x)w\| = a > 0$. Thus we get the estimate of the norm of the left hand side of (3.5)

$$
\|v + (\Delta^2 + c\Delta - b(x))^{-1}b(x)v\| \geq a\eta.
$$

By Lemma 1 of [3], there exists a modulus of continuity $\delta(t)$ with $\delta(t) \to 0$ as $t \to 0$ such that $v \in K$ and $y_1 < 0$ satisfies $\|(tv + y_1)^+\| \leq t\delta(t)$. It follows from (3.4) that

$$
\|(v + y_1)^+\| \leq 2\beta\eta\delta(2\beta\eta).
$$

keeping in mind that $\|(\Delta^2 + c\Delta - b(x))^{-1}\| = \frac{1}{|\Lambda_1|}$, we get the estimate of the norm of the right hand side of (3.5)

$$(\Delta^2 + c\Delta - b(x))^{-1} b(x)(y_1 + v)^+ \leq \frac{\beta}{|\Lambda_1|} 2\beta\eta\delta(2\beta\eta).$$

We can choose $\eta > 0$ so small that the right hand side is $< a\eta$ and $B_\eta(y_1) \cap \{u_1\} = \emptyset$. Thus for this value of η, there is no solution of (1.1) of the form $u = y_1 + v$ with $\|v\| = \eta$. That is,

$$u - (\Delta^2 + c\Delta)^{-1}(b(x)u^+ + s\psi_1(x)) \neq 0 \qquad \text{on } \partial B_\eta(y_1).$$

We apply the similar argument to the equation

$$\begin{aligned}(\Delta^2 &+ c\Delta - b(x))u \\ &= \lambda b(x)u^- + (\lambda - 1)b(x)y_1 + s\psi_1(x) \qquad \text{in } H,\end{aligned} \tag{3.6}$$

where $0 \leq \lambda \leq 1$ and u is of the form $u = y_1 + v$. Let u be a solution of the form $u = y_1 + v$ with $\|v\| = \eta$. When $\lambda = 1$, (3.6) is equal to (1.1), while for any λ the function v satisfies the equation

$$v = (\Delta^2 + c\Delta - b(x))^{-1}(\lambda b(x)(y_1 + v)^+ - \lambda b(x)v) \tag{3.7}$$

or

$$\begin{aligned}v + (\Delta^2 &+ c\Delta - b(x))^{-1}\lambda b(x)v \\ &= (\Delta^2 + c\Delta - b(x))^{-1}\lambda b(x)(y_1 + v)^+.\end{aligned} \tag{3.8}$$

If w is the function $w = \frac{v}{\eta}$, then $\inf_w \|w + (\Delta^2 + c\Delta - b(x))^{-1}\lambda b(x)w\| = b > 0$. Thus we get the estimate of the norm of the left hand side of (3.8)

$$\|v + (\Delta^2 + c\Delta - b(x))^{-1}\lambda b(x)v\| \geq b\eta.$$

On the other hand, from (3.8) we have

$$v \in 2\beta\lambda\eta K.$$

By a modulus of continuity $\delta(t)$, we get the estimate of the norm of right hand side of (3.8)

$$\begin{aligned}\|(\Delta^2 &+ c\Delta - b(x))^{-1}\lambda b(x)(y_1 + v)^+\| \\ &\leq \frac{\beta}{|\Lambda_1|} 2\beta\lambda\eta\delta(2\beta\lambda\eta) \leq \frac{\beta}{|\Lambda_1|} 2\beta\lambda\eta\delta(2\beta\lambda\eta).\end{aligned}$$

We can choose η so small that the right hand side of (3.8) is $< b\eta$ and $B_\eta(y_1) \cap \{u_1\} = \emptyset$. Thus for this value of η there is no solution of (1.1) of the form $u = y_1 + v$ with $\|v\| = \eta$. That is,

$$u-(\Delta^2+c\Delta-b(x))^{-1}(\lambda b(x)u^-+(\lambda-1)b(x)y_1+s\psi_1(x)) \neq 0 \qquad \text{on } \partial B_\eta(y_1).$$

Since the Leray-Schauder degree is invariant under a homotopy, we have

$$\begin{aligned}&(u - (\Delta^2 + c\Delta - b(x))^{-1}(\lambda b(x)u^- + (\lambda - 1)b(x)y_1 + s\psi_1(x)), B_\eta(y_1), 0)\\ &= d_{LS}(u - (\Delta^2 + c\Delta - b(x))^{-1}(-b(x)y_1 + s\psi_1(x)), B_\eta(y_1), 0)\\ &= d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(b(x)u), B_\eta(0), 0)\end{aligned}$$

Now we are trying to find the number of the negative eigenvalues of the equation

$$u - (\Delta^2 + c\Delta)^{-1}(b(x)u) = \sigma u. \tag{3.9}$$

We note that $u - (\Delta^2 + c\Delta)^{-1}(b(x)u) = \sigma u$ is equivalent to the equation

$$(\Delta^2 + c\Delta)u - rb(x)u = 0, \text{where} \ \ r = \frac{1}{1-\sigma}$$

and $\sigma < 0$ corresponds to $0 < r < 1$. We first consider the eigenvalue problem

$$(\Delta^2 + c\Delta)u - r\lambda_n(\lambda_n - c)\frac{b(x)}{\lambda_n(\lambda_n - c)}u = 0.$$

Since $\frac{b(x)}{\lambda_n(\lambda_n-c)} > 1$, $r_k(\lambda_n(\lambda_n - c)) < \lambda_k(\lambda_k - c)$. Thus

$$r_k < \frac{\lambda_k(\lambda_k - c)}{\lambda_n(\lambda_n - c)}. \tag{3.10}$$

We next consider the eigenvalue problem

$$(\Delta^2 + c\Delta)u - r\lambda_{n+1}(\lambda_{n+1} - c)\frac{b(x)}{\lambda_{n+1}(\lambda_{n+1} - c)}u = 0.$$

Since $\frac{b(x)}{\lambda_{n+1}(\lambda_{n+1}-c)} < 1$, $r_k(\lambda_{n+1}(\lambda_{n+1} - c)) > \lambda_k(\lambda_k - c)$. Thus

$$\frac{\lambda_k(\lambda_k - c)}{\lambda_{n+1}(\lambda_{n+1} - c)} < r_k \tag{3.11}$$

By (3.10) and (3.11),

$$\frac{\lambda_k(\lambda_k - c)}{\lambda_{n+1}(\lambda_{n+1} - c)} < r_k < \frac{\lambda_k(\lambda_k - c)}{\lambda_n(\lambda_n - c)}.$$

Thus there exist n number of r_k, $k = 1, 2, \cdots, n$ in the area of $0 < r_k < 1$, so there exist n number of negative eigenvalues σ of (3.9). Thus we have

$$d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(b(x)u), B_\eta(0), 0) = (-1)^n,$$

so we prove the lemma. □

LEMMA 3.3. *Assume that* $c < \lambda_1$ *and* $\lambda_n(\lambda_n - c) < b(x) < \lambda_{n+1}(\lambda_{n+1} - c)$, $n \geq 1$. *Then there exists a small number* τ *such that the Leray-Schauder degree*

$$d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(b(x)u^+ + s\psi_1(x)), B_\tau(u_1), 0) = 1,$$

where u_1 *is the positive solution of (1.1).*

Proof. The function $u_1 = \frac{s}{\Lambda_1}\psi_1(x)$ is a positive solution. Since the solutions of (1.1) is discrete, we can choose a small number $\tau > 0$ such that $B_\tau(u_1)$ does not contain the other solutions of (1.1) except u_1. Let $u \in B_\tau(u_1)$. Then u can be written as $u = u_1 + w$, $\|w\| < \tau$. Then the Leray-Schauder degree

$$\begin{aligned} &d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(b(x)u^+ + s\psi_1(x)), B_\tau(u_1), 0) \\ &= d_{LS}(u - u_1, B_\tau(u_1), 0) \\ &= d_{LS}(u, B_\tau(0), 0) \\ &= 1 \end{aligned}$$

because $(\Delta^2 + c\Delta)u = b(x)u^+ + s\psi_1(x)$ has only one solution $u = u_1$ in $B_\tau(u_1)$. □

Proof of Theorem 3.1. By Lemma 3.1, there exists a large number $R > 0$ (depending on C) and $s_0 < 0$ such that the Leray-Schauder degree

$$d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(b(x)u^+ + s\psi_1(x)), B_R(0), 0) = 0$$

for $R > C$ and $s \geq s_0$. By Lemma 3.2, there exist $s_0 < 0$ and a small number $\eta > 0$ such that for $0 > s \geq s_0$, the Leray-Schauder degree

$$d_{LS}(u - (\Delta^2 + c\Delta - b(x))^{-1}(b(x)u^- + s\psi_1(x)), B_\eta(y_1), 0) = (-1)^n,$$

where y_1 is a solution of (3.1). By Lemma 3.3, there exists a small number τ such that the Leray-Schauder degree

$$d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(b(x)u^+ + s\psi_1(x)), B_\tau(u_1), 0) = 1,$$

where u_1 is the positive solution of (1.1). If n is even, then the Leray-Schauder degree in the region $B_R(0) \backslash \{B_\eta(y_1) \cup B_\tau(u_1)\}$ is -2, so there exists the third solution in the region $B_R(0)\setminus\{B_\eta(y_1)\cup B_\tau(u_1)\}$ of (1.1). Therefore there exist at least three solutions of (1.1), one of which is a positive solution. If n is odd, then the Leray-Schauder degree in the region $B_R(0)\setminus\{B_\eta(y_1)\cup B_\tau(u_1)\}$ is 0, so there is no solution in the region $B_R(0)\setminus\{B_\eta(y_1)\cup B_\tau(u_1)\}$ of (1.1). Therefore there exist at least two solutions of (1.1), one of which is a positive solution. Thus we complete the proof. □

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