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QUADRATIC MAPPINGS ASSOCIATED WITH INNER PRODUCT SPACES

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Abstract. In [7], Th.M. Rassias proved that the norm defined over a real vector space V is induced by an inner product if and only if for a fixed integer $n \geq 2$

$$\sum_{i=1}^{n} \left\| x_i - \frac{1}{n} \sum_{j=1}^{n} x_j \right\|^2 = \sum_{i=1}^{n} \|x_i\|^2 - n \left\| \frac{1}{n} \sum_{i=1}^{n} x_i \right\|^2$$

holds for all $x_1, \cdots, x_n \in V$.

Let V, W be real vector spaces. It is shown that if an even mapping $f : V \to W$ satisfies

$$\sum_{i=1}^{2n} f\left(x_i - \frac{1}{2n} \sum_{j=1}^{2n} x_j \right) = \sum_{i=1}^{2n} f(x_i) - 2nf\left(\frac{1}{2n} \sum_{i=1}^{2n} x_i \right) \tag{0.1}$$

for all $x_1, \cdots, x_{2n} \in V$, then the even mapping $f : V \to W$ is quadratic.

Furthermore, we prove the generalized Hyers-Ulam stability of the quadratic functional equation (0.1) in Banach spaces.

1. Introduction

The stability problem of functional equations was originated from a question of Ulam [15] concerning the stability of group homomorphisms. Hyers [5] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [6] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias

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[6] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [4] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

A square norm on an inner product space satisfies the important parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [14] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. In [3], Czerwik proved the generalized Hyers-Ulam stability of the quadratic functional equation. Several functional equations have been investigated in [8]–[13].

Throughout this paper, assume that n is a fixed positive integer. Let X be a real normed vector space with norm $||\cdot||$, and Y a real Banach space with norm $\|\cdot\|$.

In this paper, we investigate the quadratic functional equation (0.1), and prove the generalized Hyers-Ulam stability of the quadratic functional equation (0.1) in Banach spaces.

2. Quadratic mappings associated with inner product spaces

We investigate the quadratic functional equation (0.1).

LEMMA 2.1. *Let V and W be real vector spaces. If an even mapping $f : V \to W$ satisfies*

$$\sum_{i=1}^{2n} f\left(x_i - \frac{1}{2n}\sum_{j=1}^{2n} x_j\right) = \sum_{i=1}^{2n} f(x_i) - 2nf\left(\frac{1}{2n}\sum_{i=1}^{2n} x_i\right) \tag{2.1}$$

for all $x_1, \cdots, x_{2n} \in V$, then the mapping $f : V \to W$ is quadratic, i.e.,

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in V$.

Proof. Assume that $f : V \to W$ satisfies (2.1).

Letting $x_1 = \cdots = x_n = x$, $x_{n+1} = \cdots = x_{2n} = y$ in (2.1), we get

$$nf\left(x - \frac{x+y}{2}\right) + nf\left(y - \frac{x+y}{2}\right) = nf(x) + nf(y) - 2nf\left(\frac{x+y}{2}\right)$$

for all $x, y \in V$. Since $f : V \to W$ is even,

$$2nf\left(\frac{x-y}{2}\right) = nf(x) + nf(y) - 2nf\left(\frac{x+y}{2}\right)$$

for all $x, y \in V$. So

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) = f(x) + f(y) \tag{2.2}$$

for all $x, y \in V$. Letting $x = y = 0$ in (2.2), we get $f(0) = 0$. Letting $y = 0$ in (2.2), we get $f(\frac{x}{2}) = \frac{1}{4}f(x)$ for all $x \in V$. It follows from (2.2) that

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in V$. □

Corollary 2.2. *Let V and W be real vector spaces. An even mapping $f : V \to W$ satisfies*

$$\begin{aligned} f\left(x - \frac{x+y}{2}\right) &+ f\left(y - \frac{x+y}{2}\right) \\ &= f(x) + f(y) - 2f\left(\frac{x+y}{2}\right) \end{aligned} \tag{2.3}$$

for all $x, y \in V$ if and only if the mapping $f : V \to W$ is quadratic.

Proof. By Lemma 2.1, it is enough to show that if $f : V \to W$ is quadratic, then $f : V \to W$ satisfies (2.3).

Assume that $f : V \to W$ is quadratic. Since $f(2x) = 4f(x)$ for all $x \in V$, $f(\frac{x}{2}) = \frac{1}{4}f(x)$ for all $x \in V$. So

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) = f(x) + f(y)$$

for all $x, y \in V$. Thus

$$\begin{aligned} f\left(x-\frac{x+y}{2}\right) &+ f\left(y-\frac{x+y}{2}\right) \\ &= 2f\left(\frac{x-y}{2}\right) = f(x)+f(y)-2f\left(\frac{x+y}{2}\right) \end{aligned}$$

for all $x, y \in V$, as desired. □

For a given mapping $f : X \to Y$, we define

$$Df(x_1, \cdots, x_{2n}) := \sum_{i=1}^{2n} f\left(x_i - \frac{1}{2n}\sum_{j=1}^{2n} x_j\right) - \sum_{i=1}^{2n} f(x_i) + 2nf\left(\frac{1}{2n}\sum_{i=1}^{2n} x_i\right)$$

for all $x_1, \cdots, x_{2n} \in X$.

Now we prove the generalized Hyers-Ulam stability of the quadratic functional equation $Df(x_1, \cdots, x_{2n}) = 0$ in real Banach spaces.

THEOREM 2.3. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^{2n} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x_1, \cdots, x_{2n}) : = \sum_{j=0}^{\infty} 4^j \varphi\left(\frac{x_1}{2^j}, \cdots, \frac{x_{2n}}{2^j}\right) < \infty, \tag{2.4}$$

$$\|Df(x_1, \cdots, x_{2n})\| \leq \varphi(x_1, \cdots, x_{2n}) \tag{2.5}$$

for all $x_1, \cdots, x_{2n} \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (2.1) such that

$$\begin{aligned} &\|f(x)+f(-x)-Q(x)\| \\ &\leq \frac{1}{n}\widetilde{\varphi}(\underbrace{x, \cdots, x}_{n \text{ times}}, \underbrace{0, \cdots, 0}_{n \text{ times}}) + \frac{1}{n}\widetilde{\varphi}(\underbrace{-x, \cdots, -x}_{n \text{ times}}, \underbrace{0, \cdots, 0}_{n \text{ times}}) \end{aligned} \tag{2.6}$$

for all $x \in X$.

Proof. Letting $x_1 = \cdots = x_n = x$ and $x_{n+1} = \cdots = x_{2n} = 0$ in (2.5), we get

$$\begin{aligned} &\left\|3nf\left(\frac{x}{2}\right) + nf\left(\frac{-x}{2}\right) - nf(x)\right\| \\ &\leq \varphi(\underbrace{x, \cdots, x}_{n \text{ times}}, \underbrace{0, \cdots, 0}_{n \text{ times}}) \end{aligned} \tag{2.7}$$

for all $x \in X$. Replacing x by $-x$ in (2.7), we get

$$\left\|3nf\left(\frac{-x}{2}\right)+nf\left(\frac{x}{2}\right)-nf(-x)\right\| \leq \varphi(\underbrace{-x,\cdots,-x}_{n \text{ times}},\underbrace{0,\cdots,0}_{n \text{ times}}) \tag{2.8}$$

for all $x \in X$. Let $g(x) := f(x)+f(-x)$ for all $x \in X$. It follows from (2.7) and (2.8) that

$$\left\|4ng\left(\frac{x}{2}\right)-ng(x)\right\| \leq \varphi(\underbrace{x,\cdots,x}_{n \text{ times}},\underbrace{0,\cdots,0}_{n \text{ times}})+\varphi(\underbrace{-x,\cdots,-x}_{n \text{ times}},\underbrace{0,\cdots,0}_{n \text{ times}}) \tag{2.9}$$

for all $x \in X$. So

$$\left\|g(x)-4g\left(\frac{x}{2}\right)\right\| \leq \frac{1}{n}\varphi(\underbrace{x,\cdots,x}_{n \text{ times}},\underbrace{0,\cdots,0}_{n \text{ times}})+\frac{1}{n}\varphi(\underbrace{-x,\cdots,-x}_{n \text{ times}},\underbrace{0,\cdots,0}_{n \text{ times}})$$

for all $x \in X$. Hence

$$\begin{aligned}\|4^l g(\frac{x}{2^l})-4^m g(\frac{x}{2^m})\| \leq & \sum_{j=l}^{m-1}\frac{4^j}{n}\varphi\left(\underbrace{\frac{x}{2^j},\cdots,\frac{x}{2^j}}_{n \text{ times}},\underbrace{0,\cdots,0}_{n \text{ times}}\right) \\ & + \sum_{j=l}^{m-1}\frac{4^j}{n}\varphi\left(\underbrace{-\frac{x}{2^j},\cdots,-\frac{x}{2^j}}_{n \text{ times}},\underbrace{0,\cdots,0}_{n \text{ times}}\right)\end{aligned} \tag{2.10}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.4) and (2.10) that the sequence $\{4^k g(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{4^k g(\frac{x}{2^k})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{k\to\infty} 4^k g\left(\frac{x}{2^k}\right)$$

for all $x \in X$.

By (2.4) and (2.5),

$$\begin{aligned}\|DQ(x_1,\cdots,x_{2n})\| &= \lim_{k\to\infty} 4^k \left\| Dg\left(\frac{x_1}{2^k},\cdots,\frac{x_{2n}}{2^k}\right)\right\| \\ &\le \lim_{k\to\infty} 4^k \left(\varphi\left(\frac{x_1}{2^k},\cdots,\frac{x_{2n}}{2^k}\right) + \varphi\left(-\frac{x_1}{2^k},\cdots,-\frac{x_{2n}}{2^k}\right)\right) = 0\end{aligned}$$

for all $x_1,\cdots,x_{2n}\in X$. So $DQ(x_1,\cdots,x_{2n})=0$. By Lemma 2.1, the mapping $Q:X\to Y$ is quadratic. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (2.10), we get (2.6). So there exists a quadratic mapping $Q:X\to Y$ satisfying (2.1) and (2.6).

Now, let $Q':X\to Y$ be another quadratic mapping satisfying (2.1) and (2.6). Then we have

$$\begin{aligned}&\|Q(x)-Q'(x)\| = 4^q\left\|Q\left(\frac{x}{2^q}\right)-Q'\left(\frac{x}{2^q}\right)\right\| \\ &\le 4^q\left\|Q\left(\frac{x}{2^q}\right)-f\left(\frac{x}{2^q}\right)-f\left(\frac{-x}{2^q}\right)\right\| \\ &\qquad\qquad +4^q\left\|Q'\left(\frac{x}{2^q}\right)-f\left(\frac{x}{2^q}\right)-f\left(\frac{-x}{2^q}\right)\right\| \\ &\le \frac{2\cdot 4^q}{n}\widetilde{\varphi}\left(\underbrace{\frac{x}{2^q},\cdots,\frac{x}{2^q}}_{n\text{ times}},\underbrace{0,\cdots,0}_{n\text{ times}}\right)+\frac{2\cdot 4^q}{n}\widetilde{\varphi}\left(\underbrace{\frac{-x}{2^q},\cdots,\frac{-x}{2^q}}_{n\text{ times}},\underbrace{0,\cdots,0,}_{n\text{ times}}\right),\end{aligned}$$

which tends to zero as $q\to\infty$ for all $x\in X$. So we can conclude that $Q(x)=Q'(x)$ for all $x\in X$. This proves the uniqueness of Q. □

Corollary 2.4. *Let $p>2$ and θ be positive real numbers, and let $f:X\to Y$ be a mapping such that*

$$\|Df(x_1,\cdots,x_{2n})\| \le \theta\sum_{j=1}^{2n}||x_j||^p \tag{2.11}$$

for all $x_1,\cdots,x_{2n}\in X$. Then there exists a unique quadratic mapping $Q:X\to Y$ satisfying (2.1) such that

$$\|f(x)+f(-x)-Q(x)\| \le \frac{2^{p+1}\theta}{2^p-4}||x||^p$$

for all $x\in X$.

Proof. Define $\varphi(x_1,\cdots,x_{2n})=\theta\sum_{j=1}^{2n}||x_j||^p$, and apply Theorem 2.3 to get the desired result. □

Corollary 2.5. *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^{2n} \to [0,\infty)$ satisfying* (2.4) *and* (2.5)*. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying* (2.1) *such that*

$$\|f(x) - Q(x)\| \le \frac{1}{n}\widetilde{\varphi}(\underbrace{x,\cdots,x}_{n\ \text{times}},\underbrace{0,\cdots,0}_{n\ \text{times}})$$

for all $x \in X$, where $\widetilde{\varphi}$ is defined in (2.4).

Theorem 2.6. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^{2n} \to [0,\infty)$ satisfying* (2.5) *such that*

$$\widetilde{\varphi}(x_1,\cdots,x_{2n}) := \sum_{j=1}^{\infty} 4^{-j}\varphi(2^j x_1,\cdots,2^j x_{2n}) < \infty \tag{2.12}$$

for all $x_1,\cdots,x_{2n} \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (2.1) *such that*

$$\begin{aligned}(2.13)\qquad &\|f(x) + f(-x) - Q(x)\| \\ &\le \frac{1}{n}\widetilde{\varphi}(\underbrace{x,\cdots,x}_{n\ \text{times}},\underbrace{0,\cdots,0}_{n\ \text{times}}) + \frac{1}{n}\widetilde{\varphi}(\underbrace{-x,\cdots,-x}_{n\ \text{times}},\underbrace{0,\cdots,0}_{n\ \text{times}})\end{aligned}$$

for all $x \in X$.

Proof. It follows from (2.9) that

$$\begin{aligned}\left\|g(x) - \frac{1}{4}g(2x)\right\| \;\le\; &\frac{1}{4n}\varphi(\underbrace{2x,\cdots,2x}_{n\ \text{times}},\underbrace{0,\cdots,0}_{n\ \text{times}}) \\ &+\frac{1}{4n}\varphi(\underbrace{-2x,\cdots,-2x}_{n\ \text{times}},\underbrace{0,\cdots,0}_{n\ \text{times}})\end{aligned}$$

for all $x \in X$. So

$$\begin{aligned}\left\|\frac{1}{4^l}g(2^l x) - \frac{1}{4^m}g(2^m x)\right\| \;\le\; &\sum_{j=l+1}^{m}\frac{1}{4^j n}\varphi(\underbrace{2^j x,\cdots,2^j x}_{n\ \text{times}},\underbrace{0,\cdots,0}_{n\ \text{times}}) \\ (2.14)\qquad + &\sum_{j=l+1}^{m}\frac{1}{4^j n}\varphi(\underbrace{-2^j x,\cdots,-2^j x}_{n\ \text{times}},\underbrace{0,\cdots,0}_{n\ \text{times}})\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.12) and (2.14) that the sequence $\{\frac{1}{4^k}g(2^k x)\}$ is Cauchy for all

$x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^k} g(2^k x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{k\to\infty} \frac{1}{4^k} g(2^k x)$$

for all $x \in X$.

By (2.5) and (2.12),

$$\begin{aligned}\|DQ(x_1, \cdots, x_{2n})\| &= \lim_{k\to\infty} \frac{1}{4^k} \|Dg(2^k x_1, \cdots, 2^k x_{2n})\| \\ &\le \lim_{k\to\infty} \frac{1}{4^k} (\varphi(2^k x_1, \cdots, 2^k x_{2n}) + \varphi(-2^k x_1, \cdots, -2^k x_{2n})) = 0\end{aligned}$$

for all $x_1, \cdots, x_{2n} \in X$. So $DQ(x_1, \cdots, x_{2n}) = 0$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.14), we get (2.13). So there exists a quadratic mapping $Q : X \to Y$ satisfying (2.1) and (2.13).

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 2.7. *Let $p < 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.11). *Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying* (2.1) *such that*

$$\|f(x) + f(-x) - Q(x)\| \le \frac{2^{p+1}\theta}{4 - 2^p} ||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x_1, \cdots, x_{2n}) = \theta \sum_{j=1}^{2n} ||x_j||^p$, and apply Theorem 2.6 to get the desired result. □

Corollary 2.8. *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^{2n} \to [0, \infty)$ satisfying* (2.5) *and* (2.12). *Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying* (2.1) *such that*

$$\|f(x) - Q(x)\| \le \frac{1}{n} \widetilde{\varphi}(\underbrace{x, \cdots, x}_{n \text{ times}}, \underbrace{0, \cdots, 0}_{n \text{ times}})$$

for all $x \in X$, where $\widetilde{\varphi}$ is defined in (2.12).

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