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TOPOLOGICAL APPROACH FOR THE MULTIPLE SOLUTIONS OF THE NONLINEAR PARABOLIC PROBLEM WITH VARIABLE COEFFICIENT JUMPING NONLINEARITY

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Abstract. We get a theorem which shows that there exist at least two or three nontrivial weak solutions for the nonlinear parabolic boundary value problem with the variable coefficient jumping nonlinearity. We prove this theorem by restricting ourselves to the real Hilbert space. We obtain this result by approaching the topological method. We use the Leray-Schauder degree theory on the real Hilbert space.

1. Introduction

Let Ω be a bounded domain in $\mathbf{R^n}$ with smooth boundary $\partial\Omega$. Let $a(x)$ and $b(x)$ be *Hölder* continuous in Ω and let λ_k, ϕ_k $(k = 1, 1, 3, \ldots)$ be the eigenvalues and the corresponding eigenfunctions of the eigenvalue problem

$$\Delta u + \lambda u = 0 \qquad \text{in} \ \ \Omega,$$
$$u = 0 \qquad \text{on} \ \ \partial\Omega.$$

We assume that the eigenfunctions ϕ_i are an orthonormal basis for $L_2(\Omega)$. We note that

$$\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots, \qquad \lambda_k \to +\infty \qquad \text{as} \ \ k \to +\infty,$$

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$$\phi_1(x) > 0 \qquad \text{in } \ \Omega.$$

Let ν_k, ψ_k $(k = 1, 1, 3, \ldots)$ be the eigenvalues and the corresponding eigenfunctions of the eigenvalue problem

$$(\Delta + b(x))u + \nu u = 0 \qquad \text{in } \ \Omega,$$

$$u = 0 \qquad \text{on } \ \partial\Omega.$$

Standard eigenvalue theory gives that

$$\nu_1 < \nu_2 \le \nu_3 \le \cdots, \qquad \nu_k \to +\infty \qquad \text{as } \ k \to +\infty,$$

$$\psi_1(x) > 0 \qquad \text{in } \ \Omega.$$

In this paper we investigate the number of the weak solutions for the following parabolic equation with the variable coefficient jumping nonlinearity and Dirichlet boundary condition

$$u_t - \Delta u \ = \ b(x)u^+ - a(x)u^- - s\psi_1 - h(x,t) \qquad \text{in } \Omega \times (0, 2\pi), \tag{1.1}$$

$$u = 0 \quad \text{ on } \quad \partial\Omega \times (0, 2\pi).$$

We assume that the function $h(x, t)$ is 2π periodic in t and in the space $L_2^*(\Omega \times (0, 2\pi))$. The physical model for this kind of the jumping nonlinearity problem can be furnished by travelling waves in suspension bridges. The nonlinear equations with jumping nonlinearity have been extensively studied by McKenna and Walter [5], Tarantello [9], Micheletti and Pistoia [7,8] and many the other authors[1,2]. Tarantello, Micheletti and Pistoia dealt with the biharmonic equations with jumping nonlinearity and proved the existence of nontrivial solutions by degree theory and critical points theory. Lazer and McKenna [4] dealt with the one dimensional elliptic equation with jumping nonlinearity for the existence of nontrivial solutions by the global bifurcation method. For the multiplicity results of the solutions for the nonlinear parabolic problem we refer to [3, 6].

The steady-state case of (1.1) is the elliptic problem

$$-\Delta u = b(x)u^+ - a(x)u^- \ - \ s\psi_1 - h(x,t) \qquad \text{in } \Omega \times (0, 2\pi), \tag{1.2}$$

$$u = 0 \quad \text{ on } \quad \partial\Omega \times (0, 2\pi).$$

Our main result is as follows:

THEOREM 1.1. *Assume that $a(x) < \lambda_1 < \cdots < \lambda_n < b(x) < \lambda_{n+1}$, $s > 0$ and $h(x,t)$ is 2π periodic in t and in the space $H^* = L^2(\Omega\times(0,2\pi))$. Then (1.1) has at least two weak solutions if n is odd and at least three solutions if n is even.*

For the proof of Theorem 1.1 we restrict ourselves to the real Hilbert space and approach the topological method. We use the Leray-Schauder degree theory on the real Hilbert space. The outline of this paper is the following: In section 2 we introduce the Hilbert space H whose elements are expressed by the square integrable Fourier series expansions on $\Omega\times(0,2\pi)$ and consider the parabolic problem (1.1) on H and obtain *a priori* bound for the weak solutions of (1.1). In section 3 we prove Theorem 1.1.

2. A priori bound

We shall work with the complex Hilbert space $H^* = L^2(\Omega\times(0,2\pi))$, equipped with the usual inner product

$$\langle v,\omega\rangle^* = \int_0^{2\pi}\int_\Omega v(x,t)\overline{\omega}(x,t)dxdt$$

and norm $\|v\| = \langle v,v\rangle^{*\frac{1}{2}}$. Later we shall switch to the real subspace H. The functions $\psi_{mn} = \frac{\psi_n(x)e^{imt}}{\sqrt{2\pi}}, n \geq 1, m = 0, \pm1, \pm2, \ldots$ are a complete orthonormal basis for H^*. Let Σ^* denote sums over the indices m,n. Every $v \in H^*$ has a Fourier expansion

$$v = \Sigma^* v_{mn}\psi_{mn},$$

with $\Sigma|v_{mn}|^2 = \|v\|^2, v_{mn} = \langle v,\psi_{mn}\rangle^*$. A weak solution to the boundary value problem (1.1) is, by definition, a function $u \in H$ satisfying $\Sigma^*|u_{mn}|^2(m^2+\lambda_n^2) < \infty$. For real $b(x) \neq \lambda_n$, the operator $R = (D_t - \Delta - b(x))^{-1}$ denoted by

$$u = Rh \leftrightarrow u_{mn} = \frac{h_{mn}}{\lambda_n - b(x) + im}$$

is a compact linear operator on H^* and the operator norm of R is $\|R\| = \frac{1}{|b(x)-\lambda_n|}$.

From now on, we restrict ourselves to the real subspace H and observe that it is invariant under R.

We have the *a priori* bound for the weak solutions of (1.1).

LEMMA 2.1. *Assume that $a(x) < \lambda_1 < \cdots < \lambda_n < b(x) < \lambda_{n+1}$, s is bounded and $h(x,t)$ is 2π periodic in t and that $h \in H$ satisfies $\|h\| \le r$. Then there exist C and s^*such that*

$$D_t u = \Delta u + b(x)u^+ - a(x)u^- - s\phi_1 - h$$
$$u(x, t+2\pi) = u(x,t)$$

satisfies $\|u\| \le C$ for any s with $s \le s^*$

Proof. Suppose not. Then there exist (u_n, s_n, h_n) with $\|u_n\| \to \infty$, $s_n \to s^*$, s^* is bounded and $\|u_n\| \le r$ which satisfy the equation (1.1). Now let $v_n = \frac{u_n}{\|u_n\|}$, and v_n satisfies

$$D_t v_n = \Delta v_n + (b(x)v_n^+ - a(x)v_n^-) - \frac{s\psi_1(x)}{\|u_n\|} - \frac{h_n(x,t)}{\|u_n\|} \tag{2.1}$$

Taking the inner product of both sides of (2.1) with ψ_1, we have

$$\begin{aligned} \langle D_t v_n - \Delta v_n - \nu_1 v_n, \psi_1\rangle &= \langle -\nu_1 v_n^+ + (\nu_1 + b(x) - a(x))v_n^-, \psi_1\rangle \\ &\quad -\langle \frac{s_n\psi_1(x) + h_n}{\|u_n\|}, \psi_1\rangle. \end{aligned} \tag{2.2}$$

The condition $a(x) < \lambda_1 < \cdots < \lambda_n < b(x) < \lambda_{n+1}$ implies that $-\nu_1 \ge \epsilon > 0$ and $\nu_1 + b(x) - a(x) \ge \epsilon > 0$. Now we observe that

$$-\nu_1 v_n^+ + (\nu_1 + b(x) - a(x))v_n^- \ge \epsilon |v_n|$$

and thus

$$| < \frac{s_n\psi_1(x) + h_n}{\|u_n\|}, \psi_1 > | \ge \epsilon \int |v_n|\psi_1(x) \ge \epsilon | \int v_n\psi_1(x)|.$$

Thus if v_n is a solution of (2.1), then

$$| < v_n, \psi_1(x) > | \le \frac{1}{\epsilon} | < \frac{s_n\psi_1(x) + h_n}{\|u_n\|}, \psi_1(x) > |. \tag{2.3}$$

Since v_n's are precompact in H, there exists v with $\|v\| = 1$ such that $v_n \to v$. Taking the limit of both sides of (2.2), we have

$$| < v, \psi_1(x) > | \le 0.$$

Thus we have

$$| < v, \psi_1(x) > | = 0.$$

From (2.2), we have

$$\int (-\nu_1 v^+ + (\nu_1 + b(x) - a(x))v^-)\psi_1 = 0.$$

Since $\psi_1 > 0$ in Ω and $-\nu_1 v^+ + (\nu_1 + b(x) - a(x))v^- \geq \epsilon|v|$, this implies that $v = 0$. This is impossible, since $\|v\| = 1$. $\square$

3. Proof of Theorem 1.1

LEMMA 3.1. *Assume that $a(x) < \lambda_1 < \cdots < \lambda_n < b(x) < \lambda_{n+1}$ and that $h \in H$ is 2π periodic in t and that $h \in H$. Then there exists $s_1 > 0$, $\epsilon > 0$ such that the Leray-Schauder degree*

$$deg(u - (D_t - \Delta)^{-1}(b(x)u^+ - a(x)u^- - s\psi_1 - h), \\ B^*_{s\epsilon}(s\theta), 0) = (-1)^n \tag{3.1}$$

for $s \geq s_1$. Here B^*_r denotes a ball of radius r in H and

$$\theta = -(D_t - \Delta - b(x))^{-1}\psi_1 = \frac{\psi_1}{-\nu_1} > 0.$$

Proof. Let $A = (D_t - \Delta)^{-1}$ and $R = (D_t - \Delta - b(x))^{-1}$. Then (1.1) is equivalent to

$$u = s\theta - Rh + R((b(x) - a(x))u^-) \equiv Su. \tag{3.2}$$

Let B^* be the open unit ball in H, let $K = \overline{R(B^*)}$. It follows that any solution $u \in s\theta + s \in \overline{B^*}$, of (3.2) belongs to $s\theta + \frac{3}{4}s\epsilon\overline{B^*}$ and this holds when $-h + (b(x) - a(x))u^-$ is replaced by $\lambda(-h + (b(x) - a(x))u^-)$, $0 \leq \lambda \leq 1$. We consider the problem

$$(D_t - \Delta - b(x))u = -s\psi_1 + \lambda(-h + (b(x) - a(x))u^-)$$

or

$$u = R(-s\psi_1 + \lambda(-h + (b(x) - a(x)u^-))).$$

Let $G = B^*_{s\epsilon}(s\theta)$. Since the degree is invariant under the homotopy, we have

$$\begin{aligned}
&\deg(u - R(-s\psi_1 - h + (b(x) - a(x))u^-, G, 0) \\
&= \deg(u - R(-s\psi_1 + \lambda(-h + (b(x) - a(x))u^-), G, 0) \\
&= \deg(u - R(-s\psi_1), G, 0) \\
&= \deg(u - s\theta - (D_t - \Delta)(b(x)u), G, 0) \\
&= \deg(u - A(b(x)u), B^*_{s\epsilon}(0), 0), \qquad 0 \leq \lambda \leq 1
\end{aligned}$$

Thus, to prove the lemma, we have to show that this degree is $(-1)^n$. To do this, we calculate the degree on finite dimensional subspaces which we now choose. The functions

$$\phi_{on} = \frac{1}{\sqrt{2\pi}}\phi_n(x),$$

$$\phi^c_{mn} = \frac{1}{\sqrt{\pi}}\phi_n(x)\cos mt \qquad m = 1, 2, 3\ldots,$$

$$\phi^s_{mn} = \frac{1}{\sqrt{\pi}}\phi_n(x)\sin mt$$

form a real orthonormal basis for H. If $h \in H$, then $h = \Sigma h_{mn}\phi_{mn}$ in H^* and h can be expanded in terms of $\phi_{on}, h^c_{mn}, h^s_{mn}$, with the identities

$$\|A - PA\|^2 = \Sigma\frac{1}{\lambda_n^2 + m^2}(|h_{mn}|^2 + |h_{-m,n}|^2).$$

It follows that

$$\|A - PA\|^2 \le \min_{m,n>b}\frac{1}{\lambda_n^2 + m^2} \le \max\left(\frac{1}{p+1}, \frac{1}{\lambda_{p+1}}\right)$$

and by the definition of degree

$$\deg(u - PAb(x)u, s\epsilon B^*(0), 0) = \deg(u - Ab(x)u, s\epsilon B^*(0), 0)$$

for large p, since the operator PA is of finite rank, with its range contained in PH. Taking the functions $\phi_{on}, \phi^c_{mn}, \phi^s_{mn}, 1 \le m, n \le p$, as a basis H_p, the equation $u + PAb(x)u$ becomes a matrix equation on the space H_p, of the form

$$(I_q + b(x)C)x = 0 \qquad \text{for} \qquad x \in R^q, \qquad q = p(2p + 1)$$

where I_q is the identity matrix of rank q, C is a $q \times q$ block diagonal matrix $C = diag(C_1, \cdots, C_p)$ and each C_n is a $2p + 1$ by $2p + 1$ block diagonal matrix given by

$$C_n = diag\ \left(-\frac{1}{\lambda_n}, A_{1n}, \cdots, A_{pn}\right)$$

Now let $D = I_q + b(x)C = diag(D_1, \cdots, D_n)$, where

$$D_n = diag\left(1 - \frac{b(x)}{\lambda_n}, I_2 - b(x)A_{1n}, \cdots, I_2 - b(x)A_{pn}\right).$$

Since $\det D_n = \left(1 - \frac{b(x)}{\lambda_n}\right) a_{1n}(x), \cdots, a_{pn}(x)$ where $\det(I_2 - b(x)A_{mn}) = a_{mn}(x) > 0$, we finally get for large p that

$$sign \det D = sign\left(1 - \frac{b(x)}{\lambda_1}\right) \cdots \left(1 - \frac{b(x)}{\lambda_p}\right) = (-1)^n.$$

Recall that $\lambda_n < b(x) < \lambda n + 1$. Since $sign \det D$ is equal to $\deg(u + PAb(x)u, s\epsilon B^*(0), 0)$ for large p, the theorem is proved by letting $p \to +\infty$. □

Lemma 3.2. *If $a(x) < \lambda_1 < \cdots < \lambda_n < b(x) < \lambda_{n+1}$ and that $h \in H$ is 2π periodic in t, then there exist positive constants s_2, ϵ such that*

$$\deg(u - (D_t - \Delta)^{-1}(b(x)u^+ - a(x)u^- - s\phi_1 - h), B^*_{s\epsilon}(s\overline{\theta}), 0) = 1$$

for $s \geq s_2$, where $\overline{\theta} = \frac{\phi_1}{a(x) - \lambda_1} < 0$.

The proof of this lemma is similar to the proof of Lemma 3.1.

We have the following no solvability condition for (1.1).

Lemma 3.3. *Assume that $a(x) < \lambda_1 < \cdots < \lambda_n < b(x) < \lambda_{n+1}$, $s > 0$ and $h(x,t)$ is 2π periodic in t and that $h \in H$ satisfies $\|h\| \leq r$. Then there exists a constant $s_0 < 0$ so small enough that if $s \leq s_0$, then the problem*

$$u_t - \Delta u = b(x)u^+ - a(x)u^- - s\psi_1 - h(x,t) \quad \text{in } \Omega \times (0, 2\pi),$$
$$u = 0 \quad \text{on} \quad \partial\Omega \times (0, 2\pi).$$

has no solution.

Proof. We can rewrite (1.1) as

$$(D_t - \Delta u - b(x) - \nu_1)u = -\nu_1 u^+ + (\nu_1 + b(x) - a(x))u^- - s\psi_1 - h, \tag{3.3}$$
$$u = 0 \qquad \text{on } \partial\Omega.$$

Taking the inner product of both sides of (3.3) with $\psi_1(x)$, we have

$$0 =< -\nu_1 u^+ + (\nu_1 + b(x) - a(x))u^- - s\psi_1 - h, \psi_1 >\geq -s - r$$

since $-\nu_1$ and $\nu_1 + b(x) - a(x)$ are positive. If $s < -r$, (1.1) has no solution. Thus we complete the proof. □

Lemma 3.4. *Let $a(x) < \lambda_1 < \cdots < \lambda_n < b(x) < \lambda_{n+1}$ and that $h \in H$ is 2π periodic in t, there exists $\beta > 0$, depending on C and s^* such that*

$$deg(u - (D_t - \Delta u)^{-1}(b(x)u^+ - a(x)u^- - (h(x,t) + s\psi_1)), B^*_\beta(0), 0) = 0$$

for $s \leq s^*$ and $\beta > C$.

Proof. By Lemma 3.3, there exists a constant $s_0 < 0$ such that if $s \leq s_0$, (1.1) has no solution. By Lemma 2.1, there exist a constant C and $s^* > 0$ such that if u is a solution of (1.1) with $s \leq s^*$, then $\|u\| \leq C$. Let us choose β so large that $\beta > C$. We note that

$$u - (D_t - \Delta u)^{-1}(b(x)u^+ - a(x)u^- - (h(x,t) + s\psi_1)) \neq 0 \qquad \text{on } \partial B_\beta(0)$$

and

$$u - (D_t - \Delta u)^{-1}(b(x)u^+ - a(x)u^- - (h(x,t) + s_0\psi_1)) + \lambda)(s^* - s_0)\psi_1(x)) \neq 0 \qquad \text{on } \partial B_\beta(0)$$

for $0 \leq \lambda \leq 1$. By the homotopy invariance property, we have that the Leray-Schauder degree

$$\begin{aligned}
&d_{LS}(u - (D_t - \Delta u)^{-1}(b(x)u^+ - a(x)u^- - (h(x,t) + s\psi_1)), B_\beta(0), 0) \\
&= d_{LS}(u - (D_t - \Delta u)^{-1}(b(x)u^+ - a(x)u^- \\
&\qquad\qquad -(h(x,t) + s_0\psi_1) + \lambda(s - s_0)\psi_1(x)), B_\beta(0), 0) \\
&= d_{LS}(u - (D_t - \Delta u)^{-1}(b(x)u^+ - a(x)u^- \\
&\qquad\qquad -(h(x,t) + s_0\psi_1)), B_\beta(0), 0) \\
&= 0,
\end{aligned}$$

where $s \leq s^*$ and $0 \leq \lambda \leq 1$. Thus we prove the lemma. □

Proof of Theorem 1.1. By Lemma 3.4, there exists a large number $\beta > 0$ (depending on C and s^*) such that all solutions of (1.1) are contained in the ball $B_\beta(0)$ and the Leray-Schauder degree

$$\deg(u - (D_t - \Delta u)^{-1}(b(x)u^+ - a(x)u^- - (h(x,t) + s\psi_1)), B^*_\beta(0), 0) = 0$$

for $\beta > C$. By Lemma 3.1, there exists $s_1 > 0$, $\epsilon > 0$ such that

$$\deg(u - (-\Delta + D_t)^{-1}(b(x)u^+ - a(x)u^- - s\psi_1 - h), B^*_{s\epsilon}(s\theta), 0) = (-1)^n$$

for $s \geq s_1$. Here B^*_r denotes a ball of radius r in H and

$$\theta = -(D_t - \Delta - b(x))^{-1}\psi_1 = \frac{\psi_1}{-\nu_1} > 0.$$

By Lemma 3.2, there exist positive constants s_2, ϵ such that

$$\deg(u - (D_t - \Delta)^{-1}(b(x)u^+ - a(x)u^- - s\phi_1 - h), B^*_{s\epsilon}(s\overline{\theta}), 0) = 1$$

for $s \geq s_2$, where $\overline{\theta} = \frac{\phi_1}{a(x) - \lambda_1} < 0$. If n is odd, by the homotopy excision property of the degree, the Leray-Schauder degree in the region $B_\beta(0) \setminus (B^*_{s\epsilon}(s\theta) \cup B^*_{s\epsilon}(s\overline{\theta}))$ is 0, so we have no conviction of the existence of the third solution of (1.1) in the region $B_\beta(0)\backslash(B^*_{s\epsilon}(s\theta) \cup B^*_{s\epsilon}(s\bar{\theta}))$. Thus (1.1) has at least two solutions if n is odd. If n is even, the Leray-Schauder degree in the region $B_\beta(0)\backslash(B^*_{s\epsilon}(s\theta) \cup B^*_{s\epsilon}(s\bar{\theta}))$ is -2, so there exists the third solution of (1.1) in the region $B_\beta(0)\backslash(B^*_{s\epsilon}(s\theta) \cup B^*_{s\epsilon}(s\bar{\theta}))$. Thus (1.1) has at least three solutions if n is even. Thus we complete the proof. $\square$

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