APPROXIMATE CONTROLLABILITY FOR DIFFERENTIAL EQUATIONS WITH QUASI-AUTONOMOUS OPERATORS

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Abstract. The approximate controllability for the nonlinear control system with nonlinear monotone hemicontinuous and coercive operator is studied. The existence, uniqueness and a variation of solutions of the system are also given.

1. Introduction

Let $H$ and $V$ be two real separable Hilbert spaces such that $V$ is a dense subspace of $H$. We are interested in the approximate controllability for the following nonlinear functional control system on $H$:

\[
\begin{align*}
\frac{dx(t)}{dt} + Ax(t) &\ni (Bu)(t), \\
x(0) &= x_0.
\end{align*}
\]

Assume that $A$ is a monotone hemicontinuous operator from $V$ to $V^*$ and satisfies the coercive condition. Here $V^*$ stands for the dual space of $V$. Let $U$ be a Banach space and the controller operator $B$ be a bounded linear operator from the Banach space $L^2(0, T; U)$ to $L^2(0, T; H)$. If $Bu \in L^2(0, T; V^*)$, it is well known as the quasi-autonomous differential equation (see Theorem 2.6 of Chapter III in Barbu [5]). In [5], the existence and the norm estimate of a solution of the above equation on $L^2(0, T; V) \cap W^{1,2}(0, T; V^*)$ was given, and results similar to this case were obtained by many authors (see bibliographical notes of [5, 6, 7, 10, 11]), which is also applicable to optimal control problem.

The optimal control problems for a class of systems governed by a class of nonlinear evolution equations with nonlinear operator $A$ have been studied in references by Ahmed, Teo and Xiang [1, 2, 3]. The condition equivalent to the approximate controllability for semilinear control system have been obtained in by Naito [9] and Zhou [11]. As for the semilinear control system with the linear operator $A$ generated $C_0$-semigroup, Naito [9] proved the approximate
controllability under the range conditions of the controller $B$. The papers treating the controllability for systems with nonlinear principal operator $A$ are not many.

In the present article, we will prove the approximately controllable for (E) under a rather applicable assumption on the range of the control operator $B$, namely that $\{y : y(t) = Bu(t), u \in L^2(0,T;U)\}$ is dense subspace of $L^2(0,T;H)$, which is reasonable and widely used in case of the nonlinear system (refer to [11, 9, 8]).

2. Quasi-autonomous differential equations

If $H$ is identified with its dual space we may write $V \subset H \subset V^*$ densely and the corresponding injections are continuous. The norm on $V$, $H$ and $V^*$ will be denoted by $|| \cdot ||_1$, $|| \cdot ||$ and $|| \cdot ||_*$, respectively. Thus, in terms of the intermediate theory we may assume that $(V, V_1^2, V^*)_\lambda^2 = H$, where $(V, V_1^2, V^*)_\lambda^2$ denotes the real interpolation space between $V$ and $V^*$.

We note that a nonlinear operator $A$ is said to be hemicontinuous on $V$ if

$$\omega \lim_{t \to 0} A(x + ty) = Ax$$

for every $x, y \in V$ where “$\omega$-lim” indicates the weak convergence on $V$.

Let $A : V \to V^*$ be given as a monotone operator and hemicontinuous from $V$ to $V^*$ such that

$$A(0) = 0,$$

$$(Au - Av, u-v) \geq \omega_1 ||u-v||^2 - \omega_2 ||u-v||^2,$$

$$||Au||_* \leq \omega_2(||u|| + 1)$$

for every $u, v \in V$, where $\omega_2$ is a real number and $\omega_1, \omega_3$ are some positive constants.

Here, we note that if $0 \neq A(0)$ we need the following assumption

$$(Au, u) \geq \omega_1 ||u||^2 - \omega_2 ||u||^2$$

for every $u \in V$. It is also known that $A$ is maximal monotone and $R(A) = V^*$, where $R(A)$ denotes the range of $A$.

Let $h \in L^2(0,T;V^*)$ and $x$ be the solution of the following quasi-autonomous differential equation with $B = I$:

$$\begin{cases}
\frac{dx(t)}{dt} + Ax(t) \ni h(t), & 0 < t \leq T, \\
x(0) = x_0,
\end{cases}$$

(1)
where \( A \) is given satisfying the hypotheses mentioned above. The following result is from Theorem 2.6 of Chapter III in [5].

**Proposition 2.1.** Let \( x_0 \in H \) and \( h \in L^2(0,T;V^*) \). Then there exists a unique solution \( x \) of (2.1) belonging to

\[
C([0,T];H) \cap L^2(0,T;H) \cap W^{1,2}(0,T;V^*)
\]

and satisfying

\[
|x(t)|^2 + \int_0^t ||x(s)||^2 ds \leq C_1(|x_0|^2 + \int_0^t ||h(s)||^2 ds + 1),
\]

\[
\int_0^t ||x(s)||^2 ds \leq C_1(|x_0|^2 + \int_0^t ||h(s)||^2 ds + 1),
\]

where \( C_1 \) is a constant.

**Lemma 2.2.** Let \( x_h \) and \( x_k \) be the solutions of (1) corresponding to \( h \) and \( k \) in \( L^2(0,T;V^*) \). Then we have that

\[
\frac{1}{2} |x_h(t) - x_k(t)|^2 + \omega_1 \int_0^t ||x_h(s) - x_k(s)||^2 ds \leq \int_0^t e^{2\omega_2(t-s)} ||x_h(s) - x_k(s)|| ||h(s) - k(s)||_s ds,
\]

and

\[
\frac{1}{2} |x_h(t)|^2 + \omega_1 \int_0^t ||x_h(s)||^2 ds \leq \frac{e^{2\omega_2 t}}{2} |x_0|^2 + \int_0^t e^{2\omega_2(t-s)} ||x_h(s)|| ||h(s)||_s ds.
\]

**Proof.** In order to prove (5), taking scalar product on both sides of (1) by \( x(t) \),

\[
\frac{1}{2} \frac{d}{dt} |x_h(t)|^2 + \omega_1 |x_h(t)|^2 \leq \omega_2 |x_h(t)|^2 + ||x_h(t)|| ||h(t)||_s.
\]

Integrating on \([0,t]\), we get

\[
\frac{1}{2} |x_h(t)|^2 + \omega_1 \int_0^t ||x_h(s)||^2 ds \leq \frac{1}{2} |x_0|^2 + \omega_2 \int_0^t |x_h(s)|^2 ds + \int_0^t ||x_h(s)|| ||h(s)||_s ds.
\]

From (6) it follows that

\[
\frac{d}{dt} (e^{-2\omega_2 t} \int_0^t |x_h(s)|^2 ds) = 2e^{-2\omega_2 t} \left\{ \frac{1}{2} |x_h(t)|^2 - \omega_2 \int_0^t |x_h(s)|^2 ds \right\} \leq 2e^{-2\omega_2 t} \left\{ \frac{1}{2} |x_0|^2 + \int_0^t ||x_h(s)|| ||h(s)||_s ds \right\}.
\]
Integrating (7) over $(0,t)$ we have
\[
e^{-2\omega_2 t} \int_0^t |x_h(s)|^2 ds
\leq 2 \int_0^t e^{-2\omega_2 \tau} \int_0^\tau ||x_h(s)|| \|h(s)||_* ds d\tau + \frac{1 - e^{-2\omega_2 t}}{2\omega_2} |x_0|^2
\]
\[
= 2 \int_0^t \int_s^t e^{-2\omega_2 \tau} ||x_h(s)|| \|h(s)||_* ds + \frac{1 - e^{-2\omega_2 t}}{2\omega_2} |x_0|^2
\]
\[
= 2 \int_0^t \frac{e^{-2\omega_2 s} - e^{-2\omega_2 t}}{2\omega_2} ||x_h(s)|| \|h(s)||_* ds + \frac{1 - e^{-2\omega_2 t}}{2\omega_2} |x_0|^2
\]
\[
= \frac{1}{\omega_2} \int_0^t (e^{-2\omega_2 s} - e^{-2\omega_2 t}) ||x_h(s)|| \|h(s)||_* ds + \frac{1 - e^{-2\omega_2 t}}{2\omega_2} |x_0|^2,
\]
and hence,
\[
\omega_2 \int_0^t |x_h(s)|^2 ds \leq \int_0^t (e^{2\omega_2 (t-s)} - 1) ||x_h(s)|| \|h(s)||_* ds + \frac{e^{2\omega_2 t} - 1}{2} |x_0|^2.
\]
Combining (6) with (8) it follows that
\[
\frac{1}{2} |x(t)|^2 + \omega_1 \int_0^t ||x_h(s)||^2 ds \leq \frac{e^{2\omega_2 t}}{2} |x_0|^2 + \int_0^t e^{2\omega_2 (t-s)} ||x_h(s)|| \|h(s)||_* ds.
\]
We also obtain (4) by the similar argument in the proof of (5).

**Theorem 2.3.** If $(x_0, h) \in H \times L^2(0, T; V^*)$, then $x \in L^2(0, T; V) \cap C([0, T]; H)$ and the mapping
\[
H \times L^2(0, T; V^*) \ni (x_0, h) \mapsto x \in L^2(0, T; V) \cap C([0, T]; H)
\]
is continuous.

**Proof.** By virtue of Proposition 2.1 for any $(x_0, h) \in H \times L^2(0, T; V^*)$, the solution $x$ of (1) belongs to $L^2(0, T; V) \cap C([0, T]; H)$. Let $(x_{0i}, h_i) \in H \times L^2(0, T; V^*)$ and $x_i$ be the solution of (1) with $(x_{0i}, h_i)$ instead of $(x_0, h)$ for $i = 1, 2$. Multiplying on (1) by $x_1(t) - x_2(t)$, we have
\[
\frac{1}{2} \frac{d}{dt} |x_1(t) - x_2(t)|^2 + \omega_1 ||x_1(t) - x_2(t)||^2
\leq \omega_2 |x_1(t) - x_2(t)|^2 + ||x_1(t) - x_2(t)|| ||h_1(t) - h_2(t)||_*.
\]
By the similar process of the proof of (5) it holds
\[
\frac{1}{2} |x_1(t) - x_2(t)|^2 + \omega_1 \int_0^t ||x_1(s) - x_2(s)||^2 ds
\leq \frac{e^{2\omega_2 t}}{2} |x_0_1 - x_0_2|^2 + \int_0^t e^{2\omega_2 (t-s)} ||x_1(s) - x_2(s)|| ||h_1(s) - h_2(s)||_* ds.
\]
We can choose a constant $c > 0$ such that
\[
\omega_1 - e^{2\omega_2 T} \frac{c}{2} > 0
\]
and, hence

\[
\int_0^T e^{2c_2(t-s)} ||x_1(s) - x_2(s)|| ||h_1(s) - h_2(s)|| \, ds \\
\leq e^{2c_2T} \int_0^T \left\{ \frac{c}{2} ||x_1(s) - x_2(s)||^2 + \frac{1}{2c} ||h_1(s) - h_2(s)||_2^2 \right\} \, ds.
\]

Thus, there exists a constant \( C > 0 \) such that

\[
||x_1 - x_2||_{L^2(0,T;V) \cap C([0,T];H)} \leq C(||x_{01} - x_{02}|| + ||h_1 - h_2||_{L^2(0,T;V^*)}).
\]

Suppose \((x_{0n}, h_n) \to (x_0, h)\) in \( H \times L^2(0,T;V^*)\), and let \(x_n\) and \(x\) be the solutions \((E)\) with \((x_{0n}, h_n)\) and \((x_0, h)\), respectively. Then, by virtue of (9), we see that \(x_n \to x\) in \( L^2(0,T,V) \cap C([0,T];H)\).

\[
\square
\]

3. Approximate controllability

In what follows we assume that the embedding \( V \subset H \) is compact. Let \(x_h\) be the solution of \((1)\) corresponding to \(h\) in \( L^2(0,T;V^*) \). We define the solution mapping \(S\) from \( L^2(0,T;V^*) \) to \( L^2(0,T;V) \) by

\(\quad (Sh)(t) = x_h(t), \quad h \in L^2(0,T;V^*)\).

Let \(A\) be the Nemitsky operator corresponding to the map \(A\), which is defined by \(A(x) = Ax\). Then

\(\quad x_h(t) = \int_0^t ((I-A)S)h(s) \, ds,\)

and with the aid of Proposition 2.1

\[
||Sh||_{L^2(0,T;V) \cap W^{1,2}(0,T;V^*)} = ||x_h||_{L^2(0,T;V) \cap W^{1,2}(0,T;V^*)} \\
\leq C_1(||x_0|| + ||h||_{L^2(0,T;V^*)} + 1).
\]

Hence if \(h\) is bounded in \( L^2(0,T;V^*) \), then so is \(x_h\) in \( L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \). Since \( V \) is compactly embedded in \( H \) by assumption, the embedding \( L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset L^2(0,T;H) \) is compact in view of Theorem 2 of Aubin [4]. Hence, since the embedding \( L^2(0,T;H) \subset L^2(0,T;V^*) \) is continuous, the mapping \( h \mapsto Sh = x_h \) is compact from \( L^2(0,T;V^*) \) to itself.

The solution of \((E)\) is denoted by \(x(T;u)\) associated with the control \(u\) at time \(T\). The system \((E)\) is said to be \emph{approximately controllable} at time \(T\) if \(Cl\{x(T;u) : u \in L^2(0,T;U)\} = H\), where \(Cl\) denotes the closure in \( H\).

We assume

\(\quad (B) \quad Cl\{y : y(t) = (Bu)(t), \quad \text{a.e.} \quad u \in L^2(0,T;U)\} = L^2(0,T;H),\)

where \(Cl\) denotes also the closure in \( L^2(0,T;H)\).

The main results of this paper is the following:
Theorem 3.1. Let the assumption (B) be satisfied. If our constants condition in (F) contains the following inequality: $\omega_3 < \omega_1$, then

\begin{equation}
C(I - AS)h : h \in L^2(0, T; V^*) = L^2(0, T; V^*).
\end{equation}

Therefore, the nonlinear differential control system (E) is approximately controllable at time $T$.

Proof. Let us fix $T_0 > 0$ so that

\begin{equation}
N = \omega_1^{-1} \omega_2 e^{\omega_2 T_0} < 1.
\end{equation}

Let $z \in L^2(0, T_0; V^*)$ and $r$ be a constant such that

$z \in U_r = \{x \in L^2(0, T_0; V^*) : \|x\|_{L^2(0, T_0; V^*)} < r\}.$

Take a constant $d > 0$ such that

\begin{equation}
(r + \omega_3 + \omega_3 \omega_1^{-1/2} e^{\omega_2 T_0}|x_0|)(1 - N)^{-1} < d.
\end{equation}

Let the assumption (B) holds. Since the assumption (B), there exists a sequence $\{u_n\} \subset L^2(0, T_0; U)$ such that $B_{U_n} \rightarrow h$ in $L^2(0, T_0; V^*)$. Then by Theorem 2.3 we have that $x(t; u_n) \rightarrow x_h$ in $L^2(0, T_0; V) \cap C([0, T_0]; H)$. Let $g \in H$. We can choose $g \in W^{1, 2}(0, T_0; V^*)$ such that $g(0) = x_0$ and $g(T_0) = y$ and from the equation
(15) there is \( h \in L^2(0, T_0; V^*) \) such that \( g' = (I - AS)h \). By the assumption (B) there exists \( u \in L^2(0, T_0; U) \) such that
\[
\| h - Bu \|_{L^2(0, T_0; V^*)} \leq \frac{\sqrt{2} \omega_1}{e^{\omega_2 T_0}} \epsilon
\]
for every \( \epsilon > 0 \). From (4)
\[
\frac{1}{2} \| x_h(t) - x_{Bu}(t) \|^2 + \omega_1 \int_0^t ||x_h(s) - x_{Bu}(s)||^2 \, ds \\
\leq \int_0^t e^{2\omega_2(t-s)} ||x_h(s) - x_{Bu}(s)|| \cdot ||h(s) - (Bu)(s)|| \, ds \\
\leq \omega_1 \int_0^t ||x_h(s) - x_{Bu}(s)||^2 \, ds + \frac{e^{2\omega_2 t}}{4\omega_1} \int_0^t ||h(s) - (Bu)(s)||^2 \, ds,
\]
it holds
\[
\| x_h - x_{Bu} \|_{C([0, T_0]; H)} \leq \frac{e^{\omega_2 T_0}}{\sqrt{2} \omega_1} \| h - Bu \|_{L^2(0, T_0; V^*)},
\]
thus, we have
\[
\| y - x_h(T) \| = \bigg\| \int_0^T ((I - AS)h)(s) \, ds - \int_0^T ((I - AS)Bu)(s) \, ds \bigg\| \\
\leq \frac{e^{\omega_2 T_0}}{\sqrt{2} \omega_1} \| h - Bu \|_{L^2(0, T_0; V^*)} \leq \epsilon.
\]
Therefore, the system (E) is approximately controllable at time \( T_0 \). Since the condition (12) is independent of initial values, we can solve the equation in \([T_0, 2T_0]\) with the initial value \( x(T_0) \). By repeating this process, the approximate controllability for (E) can be extended the interval \([0, nT_0]\) for natural number \( n \), i.e., for the initial value \( x(nT_0) \) in the interval \([nT_0, (n+1)T_0] \).

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