ON THE 2k-TH POWER MEAN VALUE OF THE GENERALIZED QUADRATIC GAUSS SUMS

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Abstract. The main purpose of this paper is using the elementary and analytic methods to study the properties of the 2k-th power mean value of the generalized quadratic Gauss sums, and give two exact mean value formulae for k = 3 and 4.

1. Introduction

Let q ≥ 2 be an integer, χ denotes a Dirichlet character modulo q. For any integer n, we define the generalized quadratic Gauss sums G(n, χ; q) as follows:

\[ G(n, \chi; q) = \sum_{a=1}^{q} \chi(a)e \left( \frac{na^2}{q} \right), \]

where e(y) = e^{2\pi iy}. This sum is important, because it is a generalization of the classical quadratic Gauss sums G(n; q), which is defined by

\[ G(n; q) = \sum_{a=1}^{q} e \left( \frac{na^2}{q} \right). \]

About the properties of G(n, χ; q), some authors had studied it, and obtained many interesting results. For example, for any integer n with (n, q) = 1, from the general result of Cochrane and Zheng [2] we can deduce that

\[ |G(n, \chi; q)| \leq 2^{\omega(q)}q^{\frac{1}{2}}, \]

where ω(q) denotes the number of all distinct prime divisors of q. The case where q is a prime is due to Weil [4]. Zhang [5] proved that for any odd prime p and integer n with (n, p) = 1, we have

\[
\sum_{\chi \mod p} |G(n, \chi; p)|^4 = \begin{cases} 
(p - 1)[3p^2 - 6p - 1 + 4(\frac{p}{2})\sqrt{p}], & \text{if } p \equiv 1 \mod 4; \\
(p - 1)(3p^2 - 6p - 1), & \text{if } p \equiv 3 \mod 4.
\end{cases}
\]
and
\[ \sum_{\chi \text{ mod } q} |G(n, \chi; q)|^6 = (p - 1)(10p^3 - 25p^2 - 4p - 1), \text{ if } p \equiv 3 \text{ mod } 4, \]
where \( \left( \frac{a}{p} \right) \) is the Legendre symbol.

Besides, W. Zhang and H. Liu [6] also proved the following conclusion:
Let \( q \) be a square-full number. Then for any integers \( n, k \) with \( (nk, q) = 1 \) and \( k \geq 1 \), we have the identity
\[ \sum_{\chi \text{ mod } q} \frac{|G(n, \chi; q)|^4}{(k, p - 1)^2} \prod_{p|q} \frac{\phi(p - 1)}{p - 1}, \]
where \( \prod_{p|q} \) denotes the product over all prime divisors of \( q \).

In this paper, we use the elementary and analytic methods to study the calculating problem of the 2\( k \)-th power mean value of the generalized quadratic Gauss sums, and give two exact calculating formulae for \( k = 3 \) and 4. That is, we shall prove the following:

**Theorem 1.** Let odd number \( q > 1 \) be a square-full number (i.e., for any prime \( p \), \( p \mid q \) if and only if \( p^2 \mid q \)). Then for any integer \( n \) with \( (n, q) = 1 \), we have the identity
\[ \sum_{\chi \text{ mod } q} |G(n, \chi; q)|^6 = 16^{\omega(q)} \cdot q^2 \cdot \phi^2(q), \]
where \( \omega(q) \) denotes the number of all distinct prime divisors of \( q \).

**Theorem 2.** Let odd number \( q > 1 \) be a square-full number. Then for any integer \( n \) with \( (n, q) = 1 \), we have
\[ \sum_{\chi \text{ mod } q} |G(n, \chi; q)|^8 = 64^{\omega(q)} \cdot q^3 \cdot \phi^3(q). \]

From our theorems we know that the estimates in reference [2] is the best one. In fact from Theorem 2 we know that there exists at least one Dirichlet character modulo \( q \) such that the inequality:
\[ |G(n, \chi; q)| \geq 2^{\frac{1}{2} \omega(q)} q^{\frac{3}{2}} \phi^{\frac{1}{2}}(q). \]
For general integer \( k \geq 5 \), we believe that the following conclusion is correct:

**Conjecture.** Let odd number \( q > 1 \) be a square-full number, \( k \geq 2 \) be an integer. Then for any integer \( n \) with \( (n, q) = 1 \), we have the identity
\[ \sum_{\chi \text{ mod } q} |G(n, \chi; q)|^{2k} = 4^{(k-1)\omega(q)} \cdot q^{k-1} \cdot \phi^2(q). \]

The proposed method is supposed to be capable of proving this formula. However, the calculation will be so complex when \( k \geq 5 \) that such a general
conclusion cannot be obtained. For general positive integer \( q > 3 \), it is an open problem whether there is a formula to calculate the \( 2k \)-th power mean value of the generalized quadratic Gauss sums.

2. Several lemmas

To complete the proof of our theorems, we need the following several lemmas.

**Lemma 1.** For any integer \( q \geq 1 \), we have the identity

\[
G(1; q) = \frac{1}{2} \sqrt{q}(1 + i) \left( 1 + e^{-\frac{2\pi i}{q}} \right) = \begin{cases} \\
\sqrt{q}, & \text{if } q \equiv 1 \pmod{4}; \\
0, & \text{if } q \equiv 2 \pmod{4}; \\
\sqrt{q}, & \text{if } q \equiv 3 \pmod{4}; \\
(1 + i)\sqrt{q}, & \text{if } q \equiv 0 \pmod{4}.
\end{cases}
\]

**Proof.** This is a remarkable formula of Gauss. See Theorem 9.16 of [1]. □

**Lemma 2.** Let \( p \) be an odd prime and \( \alpha \geq 2 \) be an integer. Then for any integer \( n \) with \( (p, n) = 1 \), we have the identity

\[
\sum_{b=1}^{p^\alpha} e \left( \frac{nb^2}{p^\alpha} \right) = 0.
\]

**Proof.** First we know that for any positive integers \( q \geq 2 \) and integer \( n \) with \((n, q) = 1\), we have the identity

\[
\sum_{u=0}^{q-1} e \left( \frac{un}{q} \right) = 0.
\]

From this identity and the properties of reduce residue system we have

\[
\sum_{b=1}^{p^\alpha} e \left( \frac{nb^2}{p^\alpha} \right) = \sum_{u=0}^{p-1} \sum_{v=1}^{p^{\alpha-1}} e \left( \frac{n(u^{p^{\alpha-1}} + v^2)}{p^{\alpha}} \right) = \sum_{u=0}^{p-1} \sum_{v=1}^{p^{\alpha-1}} e \left( \frac{2nuvp^{\alpha-1} + v^2}{p^{\alpha}} \right)
\]

\[
= \sum_{v=1}^{p^{\alpha-1}} e \left( \frac{v^2}{p^{\alpha}} \right) \sum_{u=0}^{p-1} e \left( \frac{2nuv}{p} \right) = 0.
\]

This proves Lemma 2. □

**Lemma 3.** Let \( m, n \geq 2 \) and \( u \) be three integers with \((m, n) = 1\) and \((u, mn) = 1\). Then for any character \( \chi = \chi_1\chi_2 \) with \( \chi_1 \mod m \) and \( \chi_2 \mod n \), we have the identity

\[
G(u, \chi; mn) = \chi_1(n)\chi_2(m)G(u, \chi_1; m)G(um, \chi_2; n).
\]

**Proof.** See Lemma 6 of [6]. □

**Lemma 4.** Let \( p \) be an odd prime, \( \alpha \geq 2 \) and \( n \) be two integers with \((n, p) = 1\). Then we have

\[
\sum_{\chi \mod p^\alpha} |G(n, \chi; p^\alpha)|^6 = 16\phi^2(p^\alpha)p^{2\alpha}.
\]
Proof. From the definition of $G(n, \chi; p^\alpha)$ we have

$$|G(n, \chi; p^\alpha)|^2 = \sum_{a=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} \chi(a) \varphi(b) e \left( \frac{n(a^2 - b^2)}{p^\alpha} \right) = \sum_{a=1}^{p^\alpha} \chi(a) \sum_{b=1}^{p^\alpha} e \left( \frac{nb^2(a^2 - 1)}{p^\alpha} \right).$$

Then by this formula and the orthogonality relation for character sums modulo $p^\alpha$ we may get

$$\sum_{\chi \mod p^\alpha} |G(n, \chi; p^\alpha)|^6 = \phi(p^\alpha) \sum_{a=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} \sum_{c=1}^{p^\alpha} \left( \sum_{u=1}^{p^\alpha} e \left( \frac{nu^2(a^2 - 1)}{p^\alpha} \right) \right) \times \left( \sum_{v=1}^{p^\alpha} e \left( \frac{nu^2(b^2 - 1)}{p^\alpha} \right) \right) \times \left( \sum_{w=1}^{p^\alpha} e \left( \frac{nu^2(c^2 - 1)}{p^\alpha} \right) \right).$$

Let $(a^2 - 1, p^\alpha) = p^m$. If $m \leq \alpha - 2$, note that $(n(a^2 - 1)/p^m, p) = 1$, then from Lemma 2 we have

$$\sum_{u=1}^{p^\alpha} e \left( \frac{nu^2(a^2 - 1)}{p^\alpha} \right) = p^m \sum_{x=1}^{p^\alpha} e \left( \frac{nu^2(\chi_1^2 + \chi_2^2)}{p^\alpha} \right) = 0.$$

If $m = \alpha$, then

$$\sum_{u=1}^{p^\alpha} e \left( \frac{nu^2(a^2 - 1)}{p^\alpha} \right) = \phi(p^\alpha).$$

If $m = \alpha - 1$, then $a = r p^{\alpha-1} \pm 1$, $1 \leq r \leq p - 1$. Note that for any prime $p$ with $p \nmid n$, by Theorem 7.5.4 of [3] we have

$$G(n; p) = \left( \frac{n}{p} \right) G(1; p).$$

Then from (2) and Lemma 1 we may get

$$\sum_{u=1}^{p^\alpha} e \left( \frac{nu^2(a^2 - 1)}{p^\alpha} \right) = p^{\alpha - 1} \sum_{u=1}^{p^\alpha} e \left( \frac{nu^2(a^2 - 1)/p^{\alpha-1}}{p} \right) = p^{\alpha - 1} \left[ \left( \frac{\pm 2rm}{p} \right) G(1; p) - 1 \right].$$

Note that the number of the solutions of the congruent equation $1 \leq a, b, c \leq p^\alpha - 1$ with $p^\alpha \mid a^2 - 1, p^\alpha \mid b^2 - 1, p^\alpha \mid c^2 - 1$ and $abc \equiv 1 \mod p^\alpha$ are 4, the
number of the solutions of the congruent equation \(1 \leq a, b, c \leq p^a - 1\) with \(p^a \mid a^2 - 1, p^a \mid b^2 - 1, p^{a-1} \parallel c^2 - 1\) and \(abc \equiv 1 \mod p^a\) are 0, and

\[
\sum_{a=1}^{p^a} \sum_{b=1}^{p^a} \sum_{c=1}^{p^a} \left( \sum_{u=1}^{p^a} e \left( \frac{nu^2(a^2 - 1)}{p^a} \right) \right) \times \left( \sum_{u=1}^{p^a} e \left( \frac{nu^2(b^2 - 1)}{p^a} \right) \right) \times \left( \sum_{u=1}^{p^a} e \left( \frac{nu^2(c^2 - 1)}{p^a} \right) \right) = \phi(p^a) p^{2(a-1)} \sum_{r=1}^{p^a-1} \left[ \left( \frac{-1}{p} \right) G^2(1;p) + 1 \right] - 4 \phi(p^a) p^{2(a-1)} \sum_{r=1}^{p^a-1} \left[ \left( \frac{-2rn}{p} \right) + \left( \frac{2rn}{p} \right) \right] G(1;p) = 4 \phi(p^a) p^{2(a-1)}(p^2 - 1).
\]

So combining the above several cases and (1) we have

\[
\sum_{\chi \mod p^a} |G(n, \chi; p^a)|^6 = 4\phi^4(p^a) + 12\phi^2(p^a) p^{2(a-1)}(p^2 - 1) + \phi(p^a) p^{3(a-1)} \sum_{r=1}^{p^a-1} \sum_{s=1}^{p^a-1} \sum_{t=1}^{p^a-1} \left[ \left( \frac{\pm 2rn}{p} \right) G(1;p) - 1 \right] \times \left[ \left( \frac{\pm 2sn}{p} \right) G(1;p) - 1 \right] = 4\phi^4(p^a) + 12\phi^2(p^a) p^{2(a-1)}(p^2 - 1) + \phi(p^a) p^{3(a-1)} \sum_{r=1}^{p^a-1} \sum_{s=1}^{p^a-1} \sum_{t=1}^{p^a-1} \left[ \left( \frac{2rn}{p} \right) G(1;p) - 1 \right] \times \left[ \left( \frac{2sn}{p} \right) G(1;p) - 1 \right] + 3\phi(p^a) p^{3(a-1)} \sum_{r=1}^{p^a-1} \sum_{s=1}^{p^a-1} \sum_{t=1}^{p^a-1} \left[ \left( \frac{2rn}{p} \right) G(1;p) - 1 \right] = 4 \phi(p^a) p^{2(a-1)}(p^2 - 1).
\]
This proves Lemma 4. □

From the properties of the Legendre symbol (see reference [1]) we know that

\[
\sum_{r+s \pm t \equiv 0 \mod p} \left( \frac{r}{p} \right) \left( \frac{s}{p} \right) \left( \frac{t}{p} \right) = \sum_{r=1}^{p-1} \sum_{s=1}^{p-1} \sum_{t=1}^{p-1} \left( \frac{r\tau s}{p} \right) = \sum_{r=1}^{p-1} \sum_{s=1}^{p-1} \left( \frac{r\tau}{p} \right) \sum_{s=1}^{p-1} \left( \frac{s}{p} \right) = 0,
\]

(4)

\[
\sum_{r+s \pm t \equiv 0 \mod p} \left( \frac{r}{p} \right) \left( \frac{s}{p} \right) = \sum_{r=1}^{p-1} \sum_{s=1}^{p-1} \sum_{t=1}^{p-1} \left( \frac{r}{p} \right) = \sum_{r=1}^{p-1} \sum_{s=1}^{p-1} \left( \frac{r}{p} \right) - \sum_{r+s \equiv 0 \mod p} \left( \frac{r}{p} \right) = 0
\]

(5)

\[
\sum_{r+s \pm t \equiv 0 \mod p} 1 = \sum_{r=1}^{p-1} \sum_{s=1}^{p-1} 1 - \sum_{r+s \equiv 0 \mod p} 1 = (p-1)(p-2). \]

(7)

Note that \( \left( \frac{-1}{p} \right) G^2(1;p) = p \), from (3), (4), (5), (6) and (7) we may get

\[
\sum_{\chi \mod p^\alpha} |G(n, \chi; p^\alpha)|^6 = 4\phi^4 (p^\alpha) + 12\phi^2 (p^\alpha) p^2(\alpha-1)(p^2 - 1)
\]

\[+12\phi^2 (p^\alpha) p^{2\alpha-1} - 4\phi^2 (p^\alpha) p^{2(\alpha-1)} (p - 2) = 16\phi^2 (p^\alpha) p^{2\alpha}. \]

This proves Lemma 4.
3. Proof of theorems

In this section, we shall complete the proof of our theorems. We only prove Theorem 1. Similarly, we can deduce Theorem 2. In fact if \( q \) is an odd square-full number, let \( q = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k} \) be the factorization of \( q \) into prime powers, then \( \alpha_i \geq 2, \ i = 1, 2, \ldots, k \). Then for any integer \( n \) with \( (n, q) = 1 \), from Lemma 3, Lemma 4 and the properties of Dirichlet character we have

\[
\sum_{\chi \mod q} |G(n, \chi; q)|^6 = \prod_{p^a \| q} \left[ \sum_{\chi \mod p^a} |G(nq/p^a, \chi; p^a)|^6 \right] = \prod_{p^a \| q} \left[ \frac{8\phi^2 (p^a)}{p^{2a}} \right] = \frac{8^{\omega(q)} q^{2} \phi^2 (q)}{q},
\]

where \( \prod_{p^a \| q} \) denotes that \( p^a \mid q \) and \( p^{a+1} \nmid q \).

This completes the proof of Theorem 1.

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