DUALITY OF CO-POISSON HOPF ALGEBRAS

Sei-Qwon Oh and Hyung-Min Park

Abstract. Let $A$ be a co-Poisson Hopf algebra with Poisson co-bracket $\delta$. Here it is shown that the Hopf dual $A^\circ$ is a Poisson Hopf algebra with Poisson bracket $\{f, g\}(x) = \langle \delta(x), f \otimes g \rangle$ for any $f, g \in A^\circ$ and $x \in A$ if $A$ is an almost normalizing extension over the ground field. Moreover we get, as a corollary, the fact that the Hopf dual of the universal enveloping algebra $U(g)$ for a finite dimensional Lie bialgebra $g$ is a Poisson Hopf algebra.

Let $G$ be a Lie group with Lie algebra $g$. Then its coordinate ring $O(G)$ is a Hopf algebra and can be replaced by the Hopf dual $U(g)^\circ$ of the universal enveloping algebra $U(g)$. In fact, it is well-known that $U(g)^\circ$ is equal to $O(G)$ if $G$ is connected and simply connected. Moreover it is convenient to work on $U(g)^\circ$ instead of $O(G)$ since $U(g)^\circ$ has a natural grading. For instance, see [3, Chapter 2] and [2].

Recall that a Lie group $G$ is said to be a Poisson Lie group if its coordinate ring $O(G)$ is a Poisson Hopf algebra. If $G$ is a Poisson Lie group, then its Lie algebra $g$ becomes a finite dimensional Lie bialgebra with a co-bracket $\delta$ and the universal enveloping algebra $U(g)$ is a co-Poisson Hopf algebra with Poisson co-bracket extended naturally from $\delta$ (See [1, §6.2]). Thus its Hopf dual $U(g)^\circ$ would be a Poisson Hopf algebra with Poisson bracket induced by $\delta$. At this moment, we would show the fact $\{f, g\} \in U(g)^\circ$ for all $f, g \in U(g)^\circ$.

Let $A$ be a co-Poisson Hopf algebra. Since the concept of a co-Poisson Hopf algebra is a dual concept of Poisson Hopf algebra, the Hopf dual $A^\circ$ of $A$ is anticipated a Poisson Hopf algebra. Here we give a complete proof that the Hopf dual $A^\circ$ is a Poisson Hopf algebra in the case that $A$ is an almost normalizing extension over the ground field and we get, as a corollary, the fact that $U(g)^\circ$ is a Poisson Hopf algebra if $g$ is a finite dimensional Lie bialgebra.

Assume throughout that $k$ denotes a field of characteristic zero and all vector spaces are over $k$.

Received April 13, 2009.
2010 Mathematics Subject Classification. 17B62, 17B63, 16W30.
Key words and phrases. co-Poisson Hopf algebra, Poisson Hopf algebra.

This work was supported by the Korea Research Foundation Grant funded by the Korean Government, KRF-2008-313-C00021.

©2011 The Korean Mathematical Society
Recall the definition of co-Poisson Hopf algebra. Let $A = (A, \iota, \mu, \epsilon, \Delta, S)$ be a Hopf algebra over $k$. Let $\tau$ be the flip on $A \otimes A$, that is, $\tau$ is a $k$-linear map defined by
\[
\tau : A \otimes A \to A \otimes A, \quad x \otimes y \mapsto y \otimes x,
\]
and set
\[
\tau_{12} = \tau \otimes 1, \quad \tau_{23} = 1 \otimes \tau.
\]
A Hopf algebra $A$ is said to be a co-Poisson Hopf algebra if there exists a skew-symmetric $k$-linear map $\delta : A \to A \otimes A$, called a Poisson co-bracket, satisfying the following conditions:

(i) (co-Jacobi identity)
\[
(\delta \otimes 1) \circ \delta + \tau_{12} \circ (\delta \otimes 1) \circ \delta + \tau_{23} \circ (\delta \otimes 1) \circ \delta = 0.
\]

(ii) (co-Leibniz rule)
\[
(\Delta \otimes \text{id}) \circ \delta = (\text{id} \otimes \delta) \circ \Delta + \tau_{23} \circ (\delta \otimes \text{id}) \circ \Delta.
\]

(iii) ($\Delta$-derivation)
\[
\delta(ab) = \delta(a)\Delta(b) + \Delta(a)\delta(b)
\]
for all $a, b \in A$.

**Definition 1** ([4, 1.6.10]). An algebra $R$ over $k$ is said to be an almost normalizing extension over $k$ if $R$ is a finitely generated $k$-algebra with generators $x_1, \ldots, x_n$ satisfying the condition
\[
x_{i} x_{j} - x_{j} x_{i} \in \sum_{\ell=1}^{n} k x_{\ell} + k
\]
for all $i, j$.

**Lemma 2.** Let $R$ be an almost normalizing extension of $k$ with generators $x_1, \ldots, x_n$. Then $R$ is spanned by all standard monomials
\[
x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}, \quad r_i = 0, 1, \ldots
\]
together with the unity $1$.

**Proof.** This follows immediately from induction on the degree of monomials. \qed

Note that the Hopf dual $A^\circ$ of a Hopf algebra $A$ consists of
\[
A^\circ = \{ f \in A^* \mid f(I) = 0 \text{ for some cofinite ideal } I \text{ of } A \},
\]
where $A^*$ is the dual vector space of $A$. 
Theorem 3. Let $A$ be a co-Poisson Hopf algebra with Poisson co-bracket $\delta$. If $A$ is an almost normalizing extension over $\mathbf{k}$, then the Hopf dual $A^\circ$ is a Poisson Hopf algebra with Poisson bracket
\[
\{f, g\}(x) = \langle \delta(x), f \otimes g \rangle, \quad x \in A
\]
for any $f, g \in A^\circ$, where $\langle \cdot, \cdot \rangle$ is the natural pairing between the vector space $A \otimes A$ and its dual vector space.

Proof. Step 1. The Poisson bracket (1) is well-defined. That is, $\{f, g\} \in A^\circ$ for every $f, g \in A^\circ$. There exist cofinite ideals $I, J$ of $A$ such that $f(I) = 0$ and $g(J) = 0$. Since the canonical map $A/I \otimes A/J \to A/I \times A/J$ is a monomorphism, the ideal $I \cap J$ is also cofinite. Set
\[K = (I \cap J) \otimes A + A \otimes (I \cap J).
\]
Note that $\langle K, f \otimes g \rangle = 0$. The canonical map from $[A/(I \cap J)] \otimes [A/(I \cap J)]$ into $(A \otimes A)/K$ is surjective and thus $(A \otimes A)/K$ is finite dimensional.

Note that $A$ is spanned by the standard monomials
\[x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}, \quad r_i = 0, 1, \ldots \]
together with the unity 1 by Lemma 2. For each $i = 1, 2, \ldots, n$, the set of cosets
\[\{\delta(x_i^k) + K \mid k = 1, 2, \ldots\}
\]
is linearly dependent since $(A \otimes A)/K$ is finite dimensional and thus there exists a nonzero polynomial $h \in \mathbf{k}[x]$ such that $\delta(h(x_i)) \in K$, where $x$ is an indeterminate. Consider the set
\[S = \{0 \neq h \in \mathbf{k}[x] \mid \delta(h(x_i)) \in K\}.
\]
Note that $S$ is an infinite set since $K$ is an ideal and $S$ is not empty. For instance, if $h \in S$, then $h^k \in S$ for all positive integer $k$ by the $\Delta$-derivation of $\delta$. Since $S$ is an infinite set and $(A \otimes A)/K$ is finite dimensional, there exists a nonzero polynomial $h_i \in S$ such that $\Delta(h_i(x_i)) \in K$. That is,
\[\delta(h_i(x_i)) \in K, \quad \Delta(h_i(x_i)) \in K.
\]
Let $s_i = \deg(h_i)$ and let $L$ be the ideal of $A$ generated by
\[h_1(x_1), h_2(x_2), \ldots, h_n(x_n).
\]
For any standard monomial $X = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$, there exist polynomials $q_1, \ldots, q_n, t_1, \ldots, t_n$ of $\mathbf{k}[x]$ such that
\[x_i^{r_i} = q_i(x_i) h_i(x_i) + t_i(x_i), \quad \deg(t_i) < s_i
\]
for $i = 1, 2, \ldots, n$. Replacing each factor $x_i^{r_i}$ in $X$ by the right hand of the equation (2), we have the fact that $X$ is congruent to a $\mathbf{k}$-linear combination of finite standard monomials
\[x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}, \quad p_i < s_i \text{ for } i = 1, 2, \ldots, n
\]
modulo $L$. Thus $A/L$ is finite dimensional and hence $L$ is a cofinite ideal.
Note that every element of $L$ is a sum of elements of the form $a h_i(x_i) b$, where $a, b \in A$ and $i = 1, \ldots, n$. For every element $a h_i(x_i) b$, we have
\[
\delta(a h_i(x_i) b) = \delta(a) \Delta(h_i(x_i)) \Delta(b) + \Delta(a) \delta(h_i(x_i)) \Delta(b) + \Delta(a) \Delta(h_i(x_i)) \delta(b) \in K.
\]
Hence $\{f, g\}(L) = \langle \delta(L), f \otimes g \rangle = 0$ and thus $\{f, g\} \in A^\circ$.

Step 2. For every $f, g \in A^\circ$, $\{f, g\} = -\{g, f\}$: Since $\tau \circ \delta = -\delta$, we have immediately that
\[
\{f, g\}(x) = \langle \delta(x), f \otimes g \rangle = \langle \tau \circ \delta(x), g \otimes f \rangle
\]
for all $x \in A$. Thus we have $\{f, g\} = -\{g, f\}$.

Step 3. The equation (1) satisfies the Leibniz rule: Since
\[
\{f, g\}(x) = \langle \Delta \circ 1 \circ \delta(x), f \otimes g \otimes h \rangle
\]
and
\[
(f \{g, h\} + \{f, h\} g)(x) = \langle (1 \otimes \delta) \circ \Delta(x), f \otimes g \otimes h \rangle + \langle \tau_{23} \circ (\delta \otimes 1) \circ \Delta(x), f \otimes g \otimes h \rangle
\]
for $x \in A$ and $f, g, h \in A^\circ$, it is enough to show that
\[
(\Delta \circ 1) \circ \delta = (1 \otimes \delta) \circ \Delta + \tau_{23} \circ (\delta \otimes 1) \circ \Delta.
\]
But the equation (3) is just the co-Leibniz rule of $\delta$.

Step 4. The equation (1) satisfies the Jacobi identity: Observe that
\[
\{\{f, g\}, h\}(x) = \langle \delta(\delta(x), f \otimes g \otimes h),
\{g, h\}, f\rangle(x) = \langle \delta_{12} \circ \tau_{23} \circ (\delta \otimes 1) \circ \delta(x), f \otimes g \otimes h \rangle,
\{h, f \}, g\rangle(x) = \langle \tau_{23} \circ \delta_{12} \circ (\delta \otimes 1) \circ \delta(x), f \otimes g \otimes h \rangle
\]
for $x \in A$ and $f, g, h \in A^\circ$. Hence (1) satisfies the Jacobi identity if and only if $\delta$ satisfies
\[
(\delta \otimes 1) \circ \delta + \tau_{12} \circ \tau_{23} \circ (\delta \otimes 1) \circ \delta + \tau_{23} \circ \tau_{12} \circ (\delta \otimes 1) \circ \delta = 0.
\]
But the equation (4) is the co-Jacobi identity of $\delta$. Hence (1) satisfies the Jacobi identity.

Step 5. $\Delta(\{f, g\}) = \{\Delta(f), \Delta(g)\}$ for all $f, g \in A^\circ$: For any $x, y \in A$,
\[
\Delta(\{f, g\})(x \otimes y) = \{f, g\}(xy) = \langle \delta(xy), f \otimes g \rangle
\]
\[
= \langle \delta(x) \Delta(y), f \otimes g \rangle + \langle \Delta(x) \delta(y), f \otimes g \rangle
\]
\[
= \sum \langle \delta(x), f' \otimes g' \rangle (\Delta(y), f'' \otimes g'')
\]
\[
+ \sum \langle \Delta(x), f' \otimes g' \rangle (\delta(y), f'' \otimes g'')
\]
\[
= \{f', g'\}(x)(f''g'')(y) + (f'g')(x)(\{f'', g''\})(y)
\]
\[
= \{\Delta(f), \Delta(g)\}(x \otimes y),
\]
where $\Delta(f) = \sum f' \otimes f''$, $\Delta(g) = g' \otimes g''$. Thus we have
$$\Delta(\{f, g\}) = \{\Delta(f), \Delta(g)\}$$
for $f, g \in A^\circ$. This completes the proof of Theorem 3. \hfill \Box

Refer to [1, 1.3] for the definition of Lie bialgebra. Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra, $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ and $\Delta$ the comultiplication of $U(\mathfrak{g})$. The cobracket $\delta$ is extended uniquely to a $\Delta$-derivation $\overline{\delta}$. That is,
$$\overline{\delta} : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$$
is a $k$-linear map such that $\overline{\delta}|_{\mathfrak{g}} = \delta$ and $\overline{\delta}(xy) = \overline{\delta}(x)\Delta(y) + \Delta(x)\overline{\delta}(y)$ for all $x, y \in U(\mathfrak{g})$. Then, by [1, Proposition 6.2.3], $U(\mathfrak{g})$ is a co-Poisson Hopf algebra with Poisson co-bracket $\overline{\delta}$.

**Corollary 4.** Let $(\mathfrak{g}, \delta)$ be a finite dimensional Lie bialgebra. Then the Hopf dual $U(\mathfrak{g})^\circ$ of the universal enveloping algebra $U(\mathfrak{g})$ is a Poisson Hopf algebra with Poisson bracket
$$\{f, g\}(x) = \langle \overline{\delta}(x), f \otimes g \rangle, \quad x \in U(\mathfrak{g})$$
for $f, g \in U(\mathfrak{g})^\circ$.

**Proof.** Let $\{x_1, \ldots, x_n\}$ be a basis of $\mathfrak{g}$. Then $U(\mathfrak{g})$ is an almost normalizing extension over $k$ with generators $x_1, \ldots, x_n$. Thus the result follows immediately from Theorem 3. \hfill \Box

**References**


Sei-Qwon Oh
Department of Mathematics
Chungnam National University
Daejeon 305-764, Korea
E-mail address: sqoh@cnu.ac.kr

Hyung-Min Park
Department of Mathematics
Chungnam National University
Daejeon 305-764, Korea
E-mail address: my-love@hanmail.net