

THE KARUSH-KUHN-TUCKER OPTIMALITY CONDITIONS IN INTERVAL-VALUED MULTIOBJECTIVE PROGRAMMING PROBLEMS

ELHAM HOSSEINZADE AND HASSAN HASSANPOUR*

ABSTRACT. The Karush-Kuhn-Tucker (KKT) necessary optimality conditions for nonlinear differentiable programming problems are also sufficient under suitable convexity assumptions. The KKT conditions in multiobjective programming problems with interval-valued objective and constraint functions are derived in this paper. The main contribution of this paper is to obtain the Pareto optimal solutions by resorting to the sufficient optimality condition.

AMS Mathematics Subject Classification : 90C29, 90C30.

Key words and phrases : Interval-valued function, KKT optimality conditions, Pareto optimal solution.

1. Introduction

The occurrence of randomness and imprecision in the real world is inevitable owing to some unexpected situations. Therefore, imposing the uncertainty upon the conventional optimization problems is an interesting research topic. The interval-valued optimization problems are closely related to the inexact linear programming problems. Charnes et al. [6] considered the linear programming problems in which the right-hand sides of linear inequality constraints are closed intervals. Ishibuchi and Tanaka [11] considered multiobjective programming problems with interval-valued objective functions and proposed the ordering relation between two closed intervals by considering the maximization and minimization problems separately. Inuiguchi and Kume [10] formulated and solved four kinds of goal programming problems with interval coefficients in which the target values were also assumed to be closed intervals. Urli and

Received September 27, 2010. Revised December 17, 2010. Accepted February 10, 2011.

*Corresponding author.

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Nadeau [19] used an interactive method to solve the multiobjective linear programming problems with interval coefficients. They also proposed a methodology to transform a nondeterministic problem into a deterministic problem. Chanas and Kuchta [5] presented an approach to unify the solution methods proposed by Ishibuchi and Tanaka [11] and Rommelfanger et al. [15]. Also the portfolio selection problem with interval objective functions were investigated by Ida [9]. Recently, Oliveria and Antunes [13] provided an overview of multiobjective linear programming problems with interval coefficients by illustrating many numerical examples. Su et al. [18] proposed two interval parameter fuzzy programming models for petroleum solid waste management. Qin and Huang et al. [14] proposed an interval parameter fuzzy nonlinear optimization model for stream water quality management under uncertainty. Benjamin [3] used interval programming for underwater projectile design optimization and multiobjective decision making.

The KKT optimality conditions for the optimization problems (single-objective and multiobjective programming problems) with interval-valued objective functions and real-valued constraint functions were investigated by Wu [21, 22]. Also, the necessary optimality conditions for single-objective nonlinear programming problems with interval-valued objective and constraint functions has been considered by Wu [20]. This paper focuses on multiobjective programming problems in which both objective and constraint functions are interval-valued.

The remainder of the paper is organised as follows:

In Section 2, some preliminaries of intervals arithmetics are introduced. In Section 3, multi-objective optimization problem with interval-valued objective and constraint functions is formulated, a solution concept for this problem is provided and the KKT optimality conditions for the problem are derived. Also a numerical example is solved for providing the basic techniques to compute the Pareto optimal solutions by resorting to KKT conditions. Finally, Section 4 is devoted to conclusion.

2. Preliminaries

Since the values of objective and constraint functions in our model are closed intervals, we need to compare the closed intervals.

Let us denote by φ the class of all closed and bounded intervals in \mathbb{R} . Throughout this paper, when we say that A is a closed interval, it implicitly means that A is also bounded. If A is a closed interval, we also adopt the notation $A = [a^L, a^U]$, where a^L and a^U are the lower and upper end points of A , respectively.

Definition 1 ([20]). Let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ be two closed intervals in \mathbb{R} . We say that A is less than or equal to B and write $A \preceq B$ if and only if $a^L \leq b^L$ and $a^U \leq b^U$.

Also we say that A is less than B and write $A \prec B$ if and only if $A \preceq B$ and $A \neq B$. Equivalently, $A \prec B$ if and only if

$$\left\{ \begin{array}{l} a^L < b^L, \\ a^U \leq b^U \end{array} \right\} \text{ or } \left\{ \begin{array}{l} a^L \leq b^L, \\ a^U < b^U \end{array} \right\} \text{ or } \left\{ \begin{array}{l} a^L < b^L, \\ a^U < b^U \end{array} \right\}.$$

The function $f : \mathbb{R}^n \rightarrow \varphi$ is called an interval-valued function, i.e., $f(\mathbf{x}) = f(x_1, \dots, x_n)$ is a closed interval in \mathbb{R} for each $\mathbf{x} \in \mathbb{R}^n$. The interval-valued function f can also be written as $f(\mathbf{x}) = [f^L(\mathbf{x}), f^U(\mathbf{x})]$ where f^L and f^U are real-valued functions defined on \mathbb{R}^n and satisfy $f^L(\mathbf{x}) \leq f^U(\mathbf{x})$ for each $\mathbf{x} \in \mathbb{R}^n$.

Definition 2 ([22]). Let f be an interval-valued function defined on $X \subseteq \mathbb{R}^n$ and $\mathbf{x}_0 = (x_1^{(0)}, \dots, x_n^{(0)}) \in X$.

- (i) We say that f is weakly differentiable at \mathbf{x}_0 if the real-valued functions f^L and f^U are differentiable at \mathbf{x}_0 (which imply that all of the partial derivatives $\partial f^L / \partial x_i$ and $\partial f^U / \partial x_i$ exist at \mathbf{x}_0 for $i = 1, \dots, n$).
- (ii) We say that f is weakly continuously differentiable at \mathbf{x}_0 if the real-valued functions f^L and f^U are continuously differentiable at \mathbf{x}_0 (i.e., all of the partial derivatives of f^L and f^U exist on some neighborhoods of \mathbf{x}_0 and are continuous at \mathbf{x}_0).

If f be a differentiable real-valued function defined on a nonempty open convex subset X of \mathbb{R}^n , then f is convex at \mathbf{x}^* if and only if

$$f(\mathbf{x}) \geq f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^T (\mathbf{x} - \mathbf{x}^*)$$

for each $\mathbf{x} \in X$ [2].

Similar to the definition of convexity for real-valued function [2] the notion of convexity is defined for interval-valued function as follows:

Definition 3 ([20]). Let X be a nonempty convex subset of \mathbb{R}^n and f be an interval-valued function defined on X . We say that f is convex at \mathbf{x}^* if

$$f(\lambda \mathbf{x}^* + (1 - \lambda)\mathbf{x}) \preceq \lambda f(\mathbf{x}^*) + (1 - \lambda)f(\mathbf{x})$$

for each $\mathbf{x} \in X$ and $\lambda \in (0, 1)$.

Proposition 1 ([20]). Let X be a nonempty convex subset of \mathbb{R}^n and f be an interval-valued function defined on X . The interval-valued function f is convex at \mathbf{x}^* if and only if the real-valued functions f^L and f^U are convex at \mathbf{x}^* .

3. Karush-Kuhn-Tucker optimality conditions

Consider a multiobjective interval-valued optimization problem (MIVP) as follows:

$$\begin{aligned} \text{(MIVP1)} \quad & \min \quad f(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_r(\mathbf{x})) \\ & \text{subject to} \quad g_i(\mathbf{x}) \preceq b_i \quad i = 1, \dots, m, \end{aligned}$$

where $f_k(\mathbf{x}) = [f_k^L(\mathbf{x}), f_k^U(\mathbf{x})]$ and $g_i(\mathbf{x}) = [g_i^L(\mathbf{x}), g_i^U(\mathbf{x})]$ are interval-valued functions and $b_i = [b_i^L, b_i^U]$, for $k = 1, \dots, r$ and $i = 1, \dots, m$, and the relation " $<$ ", " \preceq " defined in Definition 1.

According to Definition 1, $\mathbf{x} = (x_1, \dots, x_n)$ is a feasible solution of problem

(MIVP1) if $g_i^L(\mathbf{x}) \leq b_i^L$ and $g_i^U(\mathbf{x}) \leq b_i^U$ for $i = 1, \dots, m$. Let us denote by X the set of feasible solutions of problem (MIVP1). Similar to usual multi-objective problems, often there does not exist a point $\mathbf{x} \in X$ to minimize all of the objective functions, simultaneously. Therefore we need to define another notion of optimal solution, named Pareto optimal (efficient) solution.

Definition 4 ([16]). Let \mathbf{x}^* be a feasible solution of problem (MIVP1). We say that \mathbf{x}^* is a Pareto optimal solution of problem (MIVP1) if there exists no $\bar{\mathbf{x}} \in X$ such that $f_k(\bar{\mathbf{x}}) \preceq f_k(\mathbf{x}^*)$ for each $k \in \{1, \dots, r\}$ and $f_h(\bar{\mathbf{x}}) \prec f_h(\mathbf{x}^*)$ for at least one index $h \in \{1, \dots, r\}$.

Let f and g_i , $i = 1, \dots, m$, be real-valued functions defined on \mathbb{R}^n , and consider the following optimization problem

$$(P) \quad \min \quad f(\mathbf{x}) = f(x_1, \dots, x_n) \\ \text{subject to} \quad g_i(\mathbf{x}) \leq 0 \quad i = 1, \dots, m.$$

Suppose that the constraint functions g_i are convex on \mathbb{R}^n for $i = 1, \dots, m$. Then the feasible set $X = \{\mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \leq 0, i = 1, \dots, m\}$ is a convex subset of \mathbb{R}^n . The well-known Karush-Kuhn-Tucker conditions for problem (P) is stated as follows.

Theorem 1 ([8]). Assume that the constraint functions $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$ are convex on \mathbb{R}^n for $i = 1, \dots, m$. Let $X = \{\mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \leq 0, i = 1, \dots, m\}$ be the feasible set and $\mathbf{x}^* \in X$. Suppose that the objective function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex at \mathbf{x}^* , and f and g_i , $i = 1, \dots, m$, are continuously differentiable at \mathbf{x}^* . If there exist (Lagrange) multipliers $0 \leq \mu_i \in \mathbb{R}$, $i = 1, \dots, m$, such that

$$i) \quad \nabla f(\mathbf{x}^*) + \sum_{i=1}^m \mu_i \nabla g_i(\mathbf{x}^*) = \mathbf{0},$$

$$ii) \quad \mu_i g_i(\mathbf{x}^*) = 0 \quad i = 1, \dots, m,$$

then \mathbf{x}^* is an optimal solution of problem (P).

By using the ordering relation " \preceq ", the problem (MIVP1) can be written as follows:

$$(MIVP2) \quad \min \quad f(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_r(\mathbf{x})) \\ \text{subject to} \quad g_i^L(\mathbf{x}) \leq b_i^L \quad i = 1, \dots, m, \\ g_i^U(\mathbf{x}) \leq b_i^U \quad i = 1, \dots, m.$$

It is obvious that the feasible sets of problems (MIVP1) and (MIVP2) are the same. By re-nomination of the constraint functions, the problem (MIVP2) can be written as follows:

$$(MIVP3) \quad \min \quad f(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_r(\mathbf{x})) \\ \text{subject to} \quad g_i(\mathbf{x}) \leq 0 \quad i = 1, \dots, 2m,$$

where $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1, \dots, 2m$, are real-valued functions.

Let $X = \{\mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \leq 0, i = 1, \dots, 2m\}$ be the feasible set and $J(\mathbf{x}^*) = \{i : g_i(\mathbf{x}^*) = 0, i = 1, \dots, 2m\}$. We say that the constraint functions $g_i, i = 1, \dots, 2m$, satisfy the *Kuhn-Tucker constraint qualification* at \mathbf{x}^* when if $\nabla g_i(\mathbf{x}^*)^T \mathbf{d} \leq 0$ for all $i \in J(\mathbf{x}^*)$, where $\mathbf{d} \in \mathbb{R}^n$, then there exists an n -dimensional vector function $\mathbf{a} : [0, 1] \rightarrow \mathbb{R}^n$ such that \mathbf{a} is right-differentiable at 0, $\mathbf{a}(0) = \mathbf{x}^*$, $\mathbf{a}(t) \in X$ for all $t \in [0, 1]$, and there exists a real number $\alpha > 0$ with $\mathbf{a}'_+(0) = \alpha \mathbf{d}$ [20].

In the proof of Theorem 2, the Motzkin’s theorem of the alternative is needed. It states that, given matrices $A \neq 0$ and C , exactly one of the following systems has a solution:

System I: $A\mathbf{x} < 0, C\mathbf{x} \leq \mathbf{0}$ for some $\mathbf{x} \in \mathbb{R}^n$;

System II: $A^T \boldsymbol{\lambda} + C^T \boldsymbol{\mu} = \mathbf{0}$ for some $\boldsymbol{\mu} \geq \mathbf{0}$ and $\boldsymbol{\lambda} \geq \mathbf{0}$ with $\boldsymbol{\lambda} \neq \mathbf{0}$.

Theorem 2 (KKT Optimality Conditions). *Suppose that \mathbf{x}^* is a pareto optimal solution of problem (MIVP3) and $f_k : \mathbb{R}^n \rightarrow \varphi, k = 1, \dots, r$ and $g_i, i = 1, \dots, 2m$, are weakly differentiable at \mathbf{x}^* . Also assume that the constraint functions $g_i, i = 1, \dots, 2m$, satisfy the Kuhn-Tucker constraint qualification at \mathbf{x}^* . Then there exist multipliers $0 \leq \mu_i \in \mathbb{R}, i = 1, \dots, 2m$, and $\mathbf{0} \leq \boldsymbol{\zeta}_k = (\zeta_k^L, \zeta_k^U)$, with $\boldsymbol{\zeta}_k \neq \mathbf{0}$ for some $k \in \{1, \dots, r\}$, such that*

$$\sum_{k=1}^r \zeta_k^L \nabla f_k^L(\mathbf{x}^*) + \sum_{k=1}^r \zeta_k^U \nabla f_k^U(\mathbf{x}^*) + \sum_{i=1}^{2m} \mu_i \nabla g_i(\mathbf{x}^*) = \mathbf{0}, \tag{1}$$

$$\mu_i g_i(\mathbf{x}^*) = 0 \quad i = 1, \dots, 2m. \tag{2}$$

Proof. Since each f_k is weakly differentiable at \mathbf{x}^* , by Definition 2, f_k^L and f_k^U are differentiable at \mathbf{x}^* . Suppose that there exists $\mathbf{d} \in \mathbb{R}^n$ such that

$$\begin{cases} \nabla f_k^L(\mathbf{x}^*)^T \mathbf{d} < 0, \\ \nabla f_k^U(\mathbf{x}^*)^T \mathbf{d} < 0, \\ \nabla g_i(\mathbf{x}^*)^T \mathbf{d} \leq 0 \end{cases} \quad \text{for } i \in J(\mathbf{x}^*). \tag{3}$$

Since $g_i, i = 1, \dots, 2m$, satisfy the Kuhn-Tucker constraint qualification at \mathbf{x}^* and f_k^L is differentiable at \mathbf{x}^* , we have

$$\begin{aligned} f_k^L(\mathbf{a}(t)) &= f_k^L(\mathbf{x}^*) + \nabla f_k^L(\mathbf{x}^*)^T (\mathbf{a}(t) - \mathbf{x}^*) + \|\mathbf{a}(t) - \mathbf{x}^*\| \varepsilon(\mathbf{a}(t), \mathbf{x}^*) \\ &= f_k^L(\mathbf{x}^*) + \nabla f_k^L(\mathbf{x}^*)^T (\mathbf{a}(t) - \mathbf{a}(0)) + \|\mathbf{a}(t) - \mathbf{a}(0)\| \varepsilon(\mathbf{a}(t), \mathbf{a}(0)) \\ &= f_k^L(\mathbf{x}^*) + t \nabla f_k^L(\mathbf{x}^*)^T \left(\frac{\mathbf{a}(0+t) - \mathbf{a}(0)}{t} \right) + \|\mathbf{a}(t) - \mathbf{a}(0)\| \varepsilon(\mathbf{a}(t), \mathbf{a}(0)) \end{aligned}$$

where $\varepsilon(\mathbf{a}(t), \mathbf{a}(0)) \rightarrow 0$ as $\|\mathbf{a}(t) - \mathbf{a}(0)\| \rightarrow 0$. Therefore, when $t \rightarrow 0^+$, we have

$$\frac{a(0+t) - a(0)}{t} \rightarrow \mathbf{a}'_+(0) = \alpha \mathbf{d}, \quad \text{where } \alpha > 0.$$

Since $\nabla f_k^L(\mathbf{x}^*) \mathbf{d} < 0$, we have $f_k^L(\mathbf{a}(t_1)) < f_k^L(\mathbf{x}^*)$ for a sufficiently small $t_1 > 0$. Similar statements are hold for f_k^U , so $f_k^U(\mathbf{a}(t_2)) < f_k^U(\mathbf{x}^*)$ for a sufficiently small $t_2 > 0$. Therefore, we have $f_k^L(\mathbf{a}(t)) < f_k^L(\mathbf{x}^*)$ and $f_k^U(\mathbf{a}(t)) < f_k^U(\mathbf{x}^*)$ for a sufficiently small t ($t < \min\{t_1, t_2\}$); consequently, $f_k(\mathbf{a}(t)) \prec f_k(\mathbf{x}^*)$ for

a sufficiently small t , which contradicts the fact that \mathbf{x}^* is a Pareto optimal solution of problem (MIVP3). Therefore, the system (3) has no solution.

Now let A be the matrix whose rows are $\nabla f_k^L(\mathbf{x}^*)^T$ and $\nabla f_k^U(\mathbf{x}^*)^T$ and C be the matrix whose rows are $\nabla g_i(\mathbf{x}^*)^T$ for $i \in J(\mathbf{x}^*)$. According to the Motzkin's theorem of the alternative, since the system (3), has no solution, there exist multipliers $\mathbf{0} \leq \zeta_k = (\zeta_k^L, \zeta_k^U)$ with $\zeta_k \neq \mathbf{0}$ for some $k \in \{1, \dots, r\}$, and $0 \leq \mu_i \in \mathbb{R}$ for $i \in J(\mathbf{x}^*)$ such that

$$\sum_{k=1}^r \zeta_k^L \nabla f_k^L(\mathbf{x}^*) + \sum_{k=1}^r \zeta_k^U \nabla f_k^U(\mathbf{x}^*) + \sum_{i \in J(\mathbf{x}^*)} \mu_i \nabla g_i(\mathbf{x}^*) = \mathbf{0}.$$

Set $\mu_i = 0$ for $i \in \{1, \dots, 2m\} \setminus J(\mathbf{x}^*)$. Then we have

$$\begin{aligned} \sum_{k=1}^r \zeta_k^L \nabla f_k^L(\mathbf{x}^*) + \sum_{k=1}^r \zeta_k^U \nabla f_k^U(\mathbf{x}^*) + \sum_{i=1}^{2m} \mu_i \nabla g_i(\mathbf{x}^*) &= \mathbf{0}, \\ \mu_i g_i(\mathbf{x}^*) &= 0 \quad \text{for all } i = 1, \dots, 2m \end{aligned}$$

and the proof is completed. □

The above theorem states the necessary conditions for Pareto optimality of the feasible point \mathbf{x}^* . The following theorem states some sufficient conditions for Pareto optimality.

Theorem 3 (sufficient conditions). *Assume that the real-valued constraint functions $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$ are convex on \mathbb{R}^n and continuously differentiable at $\mathbf{x}^* \in X$ for $i = 1, \dots, 2m$. Also suppose that the interval-valued objective functions f_k are convex and continuously differentiable at \mathbf{x}^* for $k = 1, \dots, r$. If there exist (Lagrange) multipliers $\zeta_k^L, \zeta_k^U > 0$, $k = 1, \dots, r$, and $\mu_i \geq 0$, $i = 1, \dots, 2m$, such that*

$$\begin{aligned} i) \quad & \sum_{k=1}^r \zeta_k^L \nabla f_k^L(\mathbf{x}^*) + \sum_{k=1}^r \zeta_k^U \nabla f_k^U(\mathbf{x}^*) + \sum_{i=1}^{2m} \mu_i \nabla g_i(\mathbf{x}^*) = \mathbf{0}, \\ ii) \quad & \mu_i g_i(\mathbf{x}^*) = \mathbf{0} \quad i = 1, \dots, 2m, \end{aligned}$$

then \mathbf{x}^* is a Pareto optimal solution of problem (MIVP3).

Proof. Since f_k are weakly continuously differentiable at \mathbf{x}^* , f_k^L and f_k^U are continuously differentiable at \mathbf{x}^* for $k = 1, \dots, r$. Define the real-valued function

$$\bar{f}(\mathbf{x}) = \zeta_1^L f_1^L(\mathbf{x}) + \dots + \zeta_r^L f_r^L(\mathbf{x}) + \zeta_1^U f_1^U(\mathbf{x}) + \dots + \zeta_r^U f_r^U(\mathbf{x}). \tag{4}$$

Since f_k 's are convex, according to Proposition 1 f_k^L 's and f_k^U 's are also convex. Therefore, \bar{f} is convex and continuously differentiable at \mathbf{x}^* . Now we have

$$\nabla \bar{f}(\mathbf{x}) = \zeta_1^L \nabla f_1^L(\mathbf{x}) + \dots + \zeta_r^L \nabla f_r^L(\mathbf{x}) + \zeta_1^U \nabla f_1^U(\mathbf{x}) + \dots + \zeta_r^U \nabla f_r^U(\mathbf{x}).$$

By the assumptions (i) and (ii) and Theorem 1, \mathbf{x}^* is an optimal solution of the real-valued objective function $\bar{f}(\mathbf{x})$ subject to the same constraints of problem (MIVP3), i.e., $\bar{f}(\mathbf{x}^*) \leq \bar{f}(\bar{\mathbf{x}})$ for each $\bar{\mathbf{x}} (\neq \mathbf{x}^*) \in X$. Now suppose that \mathbf{x}^* is not a Pareto optimal solution of problem (MIVP3). Then, based on Definition 3, there

exists $\bar{\mathbf{x}} \in X$ such that $f_k(\bar{\mathbf{x}}) \preceq f_k(\mathbf{x}^*)$ for each $k = 1, \dots, r$ and $f_h(\bar{\mathbf{x}}) \prec f_h(\mathbf{x}^*)$ for at least one index h . Therefore, from (4), we have $\bar{f}(\bar{\mathbf{x}}) < \bar{f}(\mathbf{x}^*)$, since $\zeta_k^L > 0$ and $\zeta_k^U > 0$ for all $k = 1, \dots, r$, which contradicts the fact that $\bar{f}(\mathbf{x}^*) \leq \bar{f}(\bar{\mathbf{x}})$. This shows that \mathbf{x}^* is a Pareto optimal solution of problem (MIVP3) which proves the theorem. \square

4. Numerical example

We are going to solve a numerical example by applying Theorem 3. Consider the following biobjective programming problems with interval-valued objective and constraint functions:

$$\begin{aligned} \min \quad & (f_1(x_1, x_2), f_2(x_1, x_2)) = ([x_1^2 + x_2^2 + 1, x_1^2 + x_2^2 + 2], \\ & [2x_1^2 + 2x_2^2 + 3, 2x_1^2 + 2x_2^2 + 4]) \\ \text{subject to} \quad & [1, 6]x_1 + [1, 2]x_2 \succeq [1, 12] \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

Then we have

$$\begin{aligned} f_1^L(x_1, x_2) &= x_1^2 + x_2^2 + 1 & , & & f_1^U(x_1, x_2) &= x_1^2 + x_2^2 + 2 \\ f_2^L(x_1, x_2) &= 2x_1^2 + 2x_2^2 + 3 & , & & f_2^U(x_1, x_2) &= 2x_1^2 + 2x_2^2 + 4 \end{aligned}$$

and

$$\begin{aligned} g_1(x_1, x_2) &= -x_1 - x_2 + 1 & , & & g_2(x_1, x_2) &= -6x_1 - 2x_2 + 12 \\ g_3(x_1, x_2) &= -x_1 & , & & g_4(x_1, x_2) &= -x_2 \end{aligned}$$

It is easy to see that the above functions satisfy the assumptions of Theorem 3. We have to find x_1, x_2 and $\mu_i, \zeta_i^L, \zeta_i^U$ for $i = 1, 2$ such that:

$$\begin{aligned} \zeta_1^L \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} + \zeta_2^L \begin{bmatrix} 4x_1 \\ 4x_2 \end{bmatrix} + \zeta_1^U \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} + \zeta_2^U \begin{bmatrix} 4x_1 \\ 4x_2 \end{bmatrix} + \mu_1 \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \mu_2 \begin{bmatrix} -6 \\ -2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left\{ \begin{aligned} 1 - x_1 - x_2 &\leq 0 \\ 12 - 6x_1 - 2x_2 &\leq 0 \\ \mu_1(1 - x_1 - x_2) &= 0 \\ \mu_2(12 - 6x_1 - 2x_2) &= 0 \\ x_i, \mu_i \geq 0, \zeta_i^L, \zeta_i^U &> 0 \quad i = 1, 2 \end{aligned} \right. & \quad (5) \end{aligned}$$

that is, we have to find a solution for the following simultaneous equations which satisfy the conditions (5).

$$\begin{cases} 2\zeta_1^L x_1 + 4\zeta_2^L x_1 + 2\zeta_1^U x_1 + 4\zeta_2^U x_1 - \mu_1 - 6\mu_2 = 0 \\ 2\zeta_1^L x_2 + 4\zeta_2^L x_2 + 2\zeta_1^U x_2 + 4\zeta_2^U x_2 - \mu_1 - 2\mu_2 = 0 \end{cases}$$

After some algebraic calculations, we obtain

$$\begin{aligned} (x_1^*, x_2^*) &= (9/5, 3/5) \\ \zeta_1^L = \zeta_1^U &= 1/2 \quad \text{and} \quad \zeta_2^L = \zeta_2^U = 1/4, \\ \mu_1 &= 0 \quad \text{and} \quad \mu_2 = 6/5. \end{aligned}$$

Since $g_2(9/5, 3/5) = 0$, condition (ii) in Theorem 3 is satisfied, i.e., $\mu_2 g_2(\mathbf{x}^*) = 0$. Therefore, $(x_1^*, x_2^*) = (9/5, 3/5)$ is a Pareto optimal solution.

5. Conclusion

The Karush-Kuhn-Tucker (KKT) optimality conditions in multiobjective programming problems with interval-valued objective and constraint functions have been successfully obtained in this paper.

The main contribution of this paper is to obtain the Pareto optimal solutions by applying the KKT conditions. Of course, many other methodologies can be developed to solve the interval-valued optimization problems based on the classical techniques in conventional optimization problems. We have to mention that, although the equality constraints are not concerned in this paper, the similar methodology can be used to handle the equality constraints. Furthermore, the KKT optimality conditions can be derived by defining different ordering relations between closed intervals.

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Elham Hosseinzade received her BSc in Applied Mathematics from Ferdowsi University of Mashhad, Mashhad, Iran, and MSc degree with rank first in Applied Mathematics/Operational Research from the University of Birjand, Birjand, Iran under direction of Dr. Hassan Hassanpour. Her research interests are in the field of multiobjective optimization, fuzzy goal programming, linear and nonlinear programming, and network optimization.

Department of Mathematics, University of Birjand, Birjand, I.R. of Iran.
e-mail: e.hosseinzade65@gmail.com

Hassan Hassanpour received his BSc degree in pure Mathematics from University of Birjand, Birjand, Iran, MSc degree in Applied Mathematics from Ferdowsi University of Mashhad, Mashhad, Iran and PhD degree in Applied Mathematics/Operational Research from Shahid Bahonar University of Kerman. He is an Assistant Professor in Department of Mathematics at the University of Birjand. His main research interests are in the field of fuzzy linear regression, multiple criteria optimization, goal programming and fuzzy goal programming, linear and nonlinear programming, and fuzzy regression.

Department of Mathematics, University of Birjand, Birjand, I.R. of Iran.
e-mail: hassanhassanpur@yahoo.com