J. Appl. Math. & Informatics Vol. Informatics Vol. **29***(2011), No. 5 - 6, pp. 1167 - 1178*
Website: http://www.kcam.biz

A GENERALIZATION OF LOCAL SYMMETRIC AND SKEW-SYMMETRIC SPLITTING ITERATION METHODS FOR GENERALIZED SADDLE POINT PROBLEMS

JIAN-LEI LI*, DANG LUO AND ZHI-JIANG ZHANG

Abstract. In this paper, we further investigate the local Hermitian and skew-Hermitian splitting (LHSS) iteration method and the modified LHSS (MLHSS) iteration method for solving generalized nonsymmetric saddle point problems with nonzero (2,2) blocks. When A is non-symmetric positive definite, the convergence conditions are obtained, which generalize some results of Jiang and Cao [M.-Q. Jiang and Y. Cao, On local Hermitian and Skew-Hermitian splitting iteration methods for generalized saddle point problems, J. Comput. Appl. Math., 2009(231): 973-982] for the generalized saddle point problems to generalized nonsymmetric saddle point problems with nonzero (2,2) blocks. Numerical experiments show the effectiveness of the iterative methods.

AMS Mathematics Subject Classification : 65F10, 65F50.
Key words and phrases : Matrix splitting, generalized saddle point problems, symmetric and skew-symmetric splitting, convergence, iterative method.

1. Introduction

We consider the following 2×2 block linear systems of the form:

$$\begin{pmatrix} A & B \\ B^T & -C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}, \tag{1}$$

where $A \in R^{m\times m}$ is a positive definite matrix and $A \neq A^T$, $C \in R^{n\times n}$ is symmetric positive semi-definite, $B \in R^{m\times n}$ is a matrix of full column rank and $m \geq n$, $f \in R^m$ and $g \in R^n$ are two given vectors, denotes B^T as the transpose of the matrix B. It is easy to see that the coefficient matrix of system (1) is nonsingular. The linear systems (1) are referred to as nonsymmetric generalized saddle point problems, which are important and arise in a large number of scientific and engineering applications, such as the field of computational fluid dynamics

Received June 1, 2010. Revised August 3, 2010. Accepted March 15, 2011. *Corresponding author.

[22], constrained and weighted least squares [12], interior point methods in constrained optimization [11], mixed finite element approximations of elliptic partial differential equations [16]. Especially, see [10] for a comprehensive survey and references therein.

In recent years, when A is symmetric positive definite, B is of full column rank, a large amount of work have been developed to solve the linear system (1). As is known, there exist two kinds of methods to solve the linear systems: direct methods and iterative methods. Direct methods are widely employed when the size of the coefficient matrix is not too large, and are usually regarded as robust methods. However, frequently, the matrices A and B are large and sparse, so iterative methods, such as Uzawa type methods [6, 7, 14, 18, 19, 20, 23, 26, 28, 30], HSS iteration methods [1, 2, 3, 4, 5], preconditioned Krylov subspace iteration methods [13, 27], become more attractive than direct methods for solving the systems (1).

When A is non-symmetric positive definite, B is of full column rank, various iterative methods also have been studied in [8, 9, 15, 17, 21, 24, 25]. For a broad overview of the numerical solution of linear systems (1), one can see [10] for more details. Recently, Jiang and Cao [24] presented local Hermitian and skew-Hermitian splitting (LHSS) iteration method and modified LHSS (MLHSS) iteration method for solving nonsingular systems (1) with $C = 0$. When A is non-symmetric positive definite, some convergence conditions of these methods were given under suitable preconditioners.

In this paper, we further investigate the LHSS and MLHSS iteration methods presented in [24] for solving generalized linear systems (1) with nonzero (2,2) blocks. When A is non-symmetric positive definite, the convergence conditions are obtained, which generalize some results of Jiang and Cao [24] for the generalized saddle point problems to generalized nonsymmetric saddle point problems with nonzero (2,2) blocks.

The paper is organized as follows. After describing the MLHSS method for systems (1), the convergence theorems are given in Section 2. In Section 3, several algorithms are presented. In Section 4 and Section 5, some numerical experiments and conclusions are given, respectively.

2. The convergence of the LHSS and MLHSS iteration methods

Denote $\rho(A)$ as the spectral radius of a square matrix A, $\lambda_{max}(W)$ and $\lambda_{min}(W)$ are the maximum and minimum eigenvalues of a symmetric positive definite matrix W, respectively. I is the identity matrix with appropriate dimension. $H = \frac{1}{2}(A + A^T)$ and $S = \frac{1}{2}(A - A^T)$ are the symmetric and the skew-symmetric parts of A, respectively. For the sake of simplicity, we rewrite the generalized saddle point problem (1) as

$$\begin{pmatrix} A & B \\ -B^T & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix}, \tag{2}$$

We make the following matrix splitting

$$\mathcal{A} = \begin{pmatrix} A & B \\ -B^T & C \end{pmatrix} = \mathcal{M} - \mathcal{N},$$

where

$$\mathcal{M} = \begin{pmatrix} Q_1 + H & 0 \\ -B^T & Q_2 \end{pmatrix}, \mathcal{N} = \begin{pmatrix} Q_1 - S & -B \\ 0 & Q_2 - C \end{pmatrix}.$$

Here, $Q_1 \in R^{m\times m}$ and $Q_2 \in R^{n\times n}$ are symmetric positive definite matrices. Then the MLHSS iterative scheme for solving the generalized saddle point problem (2), based on the matrix splitting, is

$$\begin{pmatrix} Q_1 + H & 0 \\ -B^T & Q_2 \end{pmatrix} \begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} Q_1 - S & -B \\ 0 & Q_2 - C \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix} + \begin{pmatrix} f \\ -g \end{pmatrix}, \tag{3}$$

or in block form,

$$\begin{cases} x_{k+1} = x_k + (Q_1 + H)^{-1}(f - (Ax_k + By_k)), \\ y_{k+1} = y_k + Q_2^{-1}(B^T x_{k+1} - Cy_k - g). \end{cases} \tag{4}$$

The corresponding iteration matrix of the iteration scheme (3) or (4) is given

$$\mathcal{T} = \begin{pmatrix} Q_1 + H & 0 \\ -B^T & Q_2 \end{pmatrix}^{-1} \begin{pmatrix} Q_1 - S & -B \\ 0 & Q_2 - C \end{pmatrix}, \tag{5}$$

or equivalently,

$$\mathcal{T} = I - \mathcal{M}^{-1}\mathcal{A}. \tag{6}$$

When $Q_1 = 0$, the MLHSS method becomes the LHSS method. We know that the iteration scheme (4) converges if and only if $\rho(\mathcal{T}) < 1$. To prove the convergence of the iteration scheme (4), we need the following lemma.

Lemma 1 ([29]). *Consider the quadratic equation $\lambda^2 + \phi\lambda + \psi = 0$. where ϕ and ψ are real numbers. Both roots of the equation are less than one in modulus if and only if $|\psi| < 1$ and $|\phi| < 1 + \psi$.*

The following theorem gives a sufficient and necessary condition for guaranteeing the convergence of the MLHSS method (4).

Theorem 1. *Assume that A is a non-symmetric matrix with the positive-definite symmetric part $H = \frac{1}{2}(A + A^T)$ and the skew-symmetric part $S = \frac{1}{2}(A - A^T)$. Let $Q_1 \in R^{m\times m}$ and $Q_2 \in R^{n\times n}$ be symmetric positive definite, and $B \in R^{m\times n}$ be of full column rank, with $m \geq n$. Then:*
(a) when $C = 0$, the MLHSS method is convergent if and only if

$$0 \leq c < 2a + 4b,$$

(b) when $C = \delta Q_2$ ($\delta \neq 0$), the MLHSS method is convergent if and only if

$$0 < \delta < 2 \text{ and } 0 \leq c < (2 - \delta)(a + 2b).$$

Here,

$$a = u^T H u,\ b = u^T Q_1 u,\ c = u^T B Q_2^{-1} B^T u,$$

and $[u^T, v^T]^T$ *is an eigenvector of iteration matrix* $\mathcal{T}$ *with* $u \in C^m$ *and* $v \in C^n$ *such that* $u^T u = 1$.

Proof. Let λ be the eigenvalue of $\mathcal{T}$ and $[u^T, v^T]^T$ be the corresponding eigenvector. Then from equation (5), we have

$$\mathcal{N}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \mathcal{M}\begin{pmatrix} u \\ v \end{pmatrix},$$

or equivalently,

$$\begin{cases} (1-\lambda)(Q_1 + H)u - Au = Bv, \\ \lambda B^T u = (\lambda - 1)Q_2 v + Cv. \end{cases} \tag{7}$$

We can prove that $\lambda \neq 1$ and $u \neq 0$. If $\lambda = 1$, then the two equalities in (7) reduce to

$$\begin{cases} Au + Bv = 0, \\ B^T u - Cv = 0. \end{cases} \tag{8}$$

The nonsingular property of the matrix $\mathcal{A}$ implies $[u^T, v^T]^T = 0$, which contradicts the assumption that $[u^T, v^T]^T$ is an eigenvector. Besides, if $u = 0$, then, we have $Bv = 0$. Since B is a full column rank matrix, $Bv = 0$ implies $v = 0$, which also contradicts the assumption that $[u^T, v^T]^T$ is an eigenvector. Hence, $u \neq 0$. Without loos of generality, we assume that $u^T u = 1$.

For case (a), when $C = 0$, the MLHSS method (4) is the same as that in [24], we know that the result is true from Theorem 2.2 in [24].

Now, we prove the case (b). As $C = \delta Q_2$ and $\delta \neq 0$, equation (7) reduces to

$$\begin{cases} (1-\lambda)(Q_1 + H)u - Au = Bv, \\ \lambda B^T u = (\lambda + \delta - 1)Q_2 v. \end{cases} \tag{9}$$

If $\lambda = 1 - \delta$, then the above equation (9) leads to

$$\begin{cases} \delta(Q_1 + H)u - Au = Bv, \\ B^T u = 0, \end{cases}$$

or equivalently,

$$\begin{cases} u \in null(B^T), \\ v = (B^T(Q_1 + H)^{-1}B)^{-1}(\delta B^T - B^T(Q_1 + H)^{-1}A)u. \end{cases}$$

Here, *null* $(\cdot)$ is used to represent the null space of the corresponding matrix.

If $\lambda \neq 1 - \delta$, from the second equality in (9), we have

$$v = \frac{\lambda}{\lambda + \delta - 1} Q_2^{-1} B^T u.$$

By substituting v into the first equality of equation (9), we get

$$(Q_1 - S)u - \frac{\lambda}{\lambda + \delta - 1} BQ_2^{-1}B^T u = \lambda(Q_1 + H)u. \tag{10}$$

Note that S is a skew-symmetric matrix, then $u^T Su = 0$ for all $u \in C^m$. Multiplying both sides of this equality from left with u^T, after rearranging we immediately obtain

$$\lambda^2 + \lambda\left(\delta - 1 + \frac{c-b}{a+b}\right) + \frac{(1-\delta)b}{a+b} = 0.$$

From Lemma 1, it then follows that $|\lambda| < 1$ if and only if

$$\begin{cases} |1-\delta| < 1, \\ |\frac{(1-\delta)b}{a+b}| < 1, \\ |\delta - 1 + \frac{c-b}{a+b}| < 1 + \frac{(1-\delta)b}{a+b}. \end{cases} \tag{11}$$

By straightforwardly solving (11), we immediately get that the MLHSS method is convergent if and only if

$$0 < \delta < 2 \ \textit{and} \ 0 \le c < (2-\delta)(a+2b).$$

Up to now, the proof has been completed. □

When $Q_1 = 0$, the MLHSS iteration method becomes the LHSS iteration method. Hence, by Theorem 1, the following theorem gives a description on the convergence of the LHSS method.

Theorem 2. *Assume that A is a non-symmetric matrix with the positive-definite symmetric part $H = \frac{1}{2}(A + A^T)$ and the skew-symmetric part $S = \frac{1}{2}(A - A^T)$. Let $Q_2 \in R^{n\times n}$ be symmetric positive definite, and $B \in R^{m\times n}$ be of full column rank, with $m \ge n$. Then:*
(a) when $C = 0$, the LHSS method is convergent if and only if

$$0 \le c < 2a,$$

(b) when $C = \delta Q_2$ ($\delta \neq 0$), the LHSS method is convergent if and only if

$$0 < \delta < 2 \ \textit{and} \ 0 \le c < (2-\delta)a.$$

Here,

$$a = u^T Hu, \ c = u^T BQ_2^{-1}B^T u,$$

and $[u^T, v^T]^T$ is an eigenvector of iteration matrix $\mathcal{T}$ with $u \in C^m$ and $v \in C^n$ such that $u^T u = 1$.

Based upon the proof of Theorem 1, we can easily derive the following convergence condition of the MLHSS method, which can be used in practical applications.

Theorem 3. *Assume that A is a non-symmetric matrix with the positive-definite symmetric part $H = \frac{1}{2}(A + A^T)$ and the skew-symmetric part $S = \frac{1}{2}(A - A^T)$.*

Let $Q_1 \in R^{m\times m}$ and $Q_2 \in R^{n\times n}$ be symmetric positive definite, and $B \in R^{m\times n}$ be of full column rank, with $m \geq n$. Then:
(a) when $C = 0$, the MLHSS method is convergent if $2H + 4Q_1 - BQ_2^{-1}B^T$ is a positive definite matrix.
(b) when $C = \delta Q_2$ $(0 < \delta < 2)$, the MLHSS method is convergent if $(2-\delta)(2H + 4Q_1) - BQ_2^{-1}B^T$ is a positive definite matrix.

Proof. When $C = 0$, the MLHSS method is convergent if

$$u^T(2H + 4Q_1 - BQ_2^{-1}B^T)u > 0$$

or in other words $2H + 4Q_1 - BQ_2^{-1}B^T$ is positive definite.

when $C = \delta Q_2$ $(0 < \delta < 2)$, the MLHSS method is convergent if

$$u^T((2-\delta)(2H + 4Q_1) - BQ_2^{-1}B^T)u > 0$$

or in other words $(2-\delta)(2H + 4Q_1) - BQ_2^{-1}B^T$ is positive definite. □

Corollary 1. *Under the assumption conditions of Theorem 2, Then:*
(a) when $C = 0$, the MLHSS method is convergent if $2H - BQ_2^{-1}B^T$ is a positive definite matrix.
(b) when $C = \delta Q_2$ $(0 < \delta < 2)$, the MLHSS method is convergent if $2(2-\delta)H - BQ_2^{-1}B^T$ is a positive definite matrix.

Corollary 2. *Under the assumption conditions of Theorem 3, Then:*
(a) when $C = 0$, the MLHSS method is convergent if

$$2\lambda_{max}(H) + 4\lambda_{max}(Q_1) > \lambda_{min}(BQ_2^{-1}B^T).$$

(b) when $C = \delta Q_2$ $(0 < \delta < 2)$, the MLHSS method is convergent if

$$(2-\delta)(2\lambda_{max}(H) + 4\lambda_{max}(Q_1)) > \lambda_{min}(BQ_2^{-1}B^T).$$

3. Several algorithms

In Section 2, the convergence of the LHSS method and MLHSS method are given for nonsymmetric generalized saddle point problems with nonzero (2,2) blocks. Now, we give other formal MLHSS methods. Since the LHSS method is the special case of the MLHSS method, we only give MLHSS method.

Case 1. Motivated by the generalized inexact parameterized Uzawa method presented in [19], which is mainly about the Hermitian saddle point problems, for the nonsymmetric generalized saddle point problems with nonzero (2,2) blocks, the generalized MLHSS method can be taken as follows, denoted as **Algorithm 1**:

$$\begin{cases} x_{k+1} = x_k + (Q_1 + H)^{-1}(f - (Ax_k + By_k)), \\ y_{k+1} = y_k + Q_2^{-1}((1-t)B^T x_{k+1} + tB^T x_k - Cy_k - g). \end{cases}$$

Case 2. By adding a correction iteration step (see [25]) for the **Algorithm 1**, we can have the following algorithm, denoted as **Algorithm 2**:

$$\begin{cases} \overline{x}_{k+1} = x_k + (Q_1 + H)^{-1}(f - (Ax_k + By_k)), \\ y_{k+1} = y_k + Q_2^{-1}((1-t)B^T\overline{x}_{k+1} + tB^T x_k - Cy_k - g), \\ x_{k+1} = \overline{x}_{k+1} - (Q_1 + H)^{-1}B(y_{k+1} - y_k). \end{cases}$$

Case 3. For the **Algorithm 2**, if we use different relaxed factors for x and y, we can have the following algorithm, denoted as **Algorithm 3**:

$$\begin{cases} \overline{x}_{k+1} = x_k + \omega(Q_1 + H)^{-1}(f - (Ax_k + By_k)), \\ y_{k+1} = y_k + \tau Q_2^{-1}((1-t)B^T\overline{x}_{k+1} + tB^T x_k - Cy_k - g), \\ x_{k+1} = \overline{x}_{k+1} - (Q_1 + H)^{-1}B(y_{k+1} - y_k). \end{cases}$$

4. Numerical experiments

In this section, we illustrate the feasibility and effectiveness of those iteration algorithms by using numerical examples. We only list the number of iterations (denoted by IT), CPU time is canceled because it's small. "RES" are defined as

$$RES := \frac{\sqrt{\|f - Ax_k - By_k\|^2 + \|B^T x_k - Cy_k - g\|^2}}{\sqrt{f^2 + g^2}}.$$

In our computations, all runs of Algorithms are started from the initial vector $(x_0^T, y_0^T)^T = 0$ and terminated if the current iteration satisfies either $RES < 10^{-5}$ or the number of the prescribed iteration $kmax = 1000$ are exceeded.

Consider the linear system (2), in which

$$A = \begin{pmatrix} I \otimes T & 0 \\ 0 & I \otimes T + T \otimes I \end{pmatrix} \in R^{2p^2 \times 2p^2},$$

$$B = \begin{pmatrix} I \otimes F \\ F \otimes I \end{pmatrix} \in R^{2p^2 \times p^2},\ C = I \in R^{p^2 \times p^2}$$

and

$$T = \frac{1}{h^2} \cdot tridiag(-1, 2, -1) \in R^{p \times p},\ F = \frac{1}{h} \cdot tridiag(-1, 1, 0) \in R^{p \times p},$$

with $\otimes$ being the Kronecker product symbol, $h = \frac{1}{p+1}$ the discretization mesh-size. We set $m = 2p^2$ and $n = p^2$, hence, the total number of variables is $m + n = 3p^2$. In our computations, we choose the right-hand-side vector $(f^T, g^T)^T \in R^{m+n}$ such that the exact solution of the linear system (2) is $(x^T, y^T)^T = (1, 1, ..., 1)^T \in R^{m+n}$. All the experiment are performed in MATLAB and $Q_2 = C$. We list the computed results in Tables with different choices of Q_1, ω, τ and t. From these tables, we can see that the Algorithm 3 is best if the parameters are chosen appropriately.

Table 1. Iterations numbers for algorithm 1

t	$Q_1 = H$			$Q_1 = I$			$Q_1 = 0$		
	$p = 8$	$p = 16$	$p = 24$	$p = 8$	$p = 16$	$p = 24$	$p = 8$	$p = 16$	$p = 24$
-2.5	137	110	99	127	104	94	121	100	90
-1.5	147	116	103	135	109	98	128	105	94
-1	154	121	107	141	113	101	134	108	97
-0.8	157	123	109	144	115	102	136	110	98
-0.6	161	125	111	148	117	104	139	112	100
-0.5	164	127	112	150	118	105	141	113	101
-0.4	166	128	113	152	120	106	143	114	102
-0.2	172	132	116	157	123	109	148	117	104
0	180	137	119	164	127	112	153	121	107
0.1	186	139	121	168	129	114	157	123	109
0.3	199	147	127	179	135	118	166	128	113
0.5	222	158	135	196	145	125	181	137	119
0.7	282	179	149	235	162	138	211	152	130
0.8	379	204	164	342	181	150	257	167	141
0.9	368	277	216	334	257	182	312	225	167
1	358	272	237	326	253	222	306	240	213
1.2	343	207	165	313	182	150	294	169	141
1.5	226	158	135	199	145	125	183	137	119
1.8	193	143	124	174	132	116	162	126	111
2.2	174	132	116	158	123	109	148	117	104
2.5	165	127	112	150	118	105	142	113	101

5. Conclusion

In this paper, we further investigate the LHSS and MLHSS iteration methods presented in [24] for solving generalized nonsymmetric saddle point problems with nonzero (2,2) blocks. When A is non-symmetric positive definite, the convergence conditions are obtained, which generalize some results of Jiang and Cao [24] for the generalized saddle point systems to the generalized nonsymmetric saddle point problems with nonzero (2,2) blocks.

References

1. Z. Z. Bai and G. H. Golub, *Accelerated Hermitian and skew-Hermitian splitting iteration methods for saddle-point problems*, IMA J. Numer. Anal., **27** (2007), 1-23.
2. Z. Z. Bai, G. H. Golub, and C. K. Li, *Optimal parameter in Hermitian and skew-Hermitian splitting method for certain two-by-two block matrices*, SIAM J. Sci. Comput., **28** (2006), 583-603.
3. Z. Z. Bai, G. H. Golub, and C. K. Li, *Convergence properties of preconditioned Hermitian and skew-Hermitian splitting methods for non-Hermitian positive semidefinite matrices*, Math. Comput., **76** (2007), 287-298.

TABLE 2. Iterations numbers for algorithm 2

t	$Q_1 = H$			$Q_1 = I$			$Q_1 = 0$		
	$p = 8$	$p = 16$	$p = 24$	$p = 8$	$p = 16$	$p = 24$	$p = 8$	$p = 16$	$p = 24$
-2.5	146	114	101	144	114	101	142	112	100
-1.5	152	118	104	149	117	103	149	117	103
-1	155	120	106	152	119	105	152	119	105
-0.8	156	121	107	153	120	105	154	120	106
-0.6	158	122	108	154	120	105	155	121	106
-0.4	159	123	109	156	121	106	156	121	107
-0.1	162	125	110	157	121	107	157	122	107
0	163	126	111	157	122	107	158	122	107
0.2	166	128	112	158	122	108	158	122	106
0.3	167	128	113	158	123	108	157	121	106
0.5	170	130	114	159	124	109	155	120	106
0.7	174	133	116	161	125	110	153	120	106
0.8	176	134	117	161	125	110	153	120	106
0.9	178	135	118	162	126	111	153	120	107
1	180	136	119	163	127	111	153	120	107
1.2	185	139	121	165	128	112	153	120	106
1.5	196	144	124	169	129	113	155	120	106
1.8	217	153	130	177	132	115	158	122	106
2.2	241	167	141	189	140	120	157	122	107
2.5	245	171	144	194	143	123	156	121	107

4. Z. Z. Bai, G. H. Golub, and M. K. Ng, *Hermitian and skew-Hermitian splitting methods for non- Hermitian positive definite linear systems*, SIAM J. Matrix Anal. Appl., **24** (2003), 603-626.
5. Z. Z. Bai, G. H. Golub, and J. Y. Pan, *Preconditioned Hermitian and skew-Hermitian splitting methods for non-Hermitian positive semidefinite linear systems*, Numer. Math., **98** (2004), 1-32.
6. Z. Z. Bai, B. N. Parlett, Z. Q. Wang, *On generalized successive overrelaxation methods for augmented linear systems*, Numer. Math., **102** (2005), 1-38.
7. Z. Z. Bai and Z. Q. Wang. *On parameterized inexact Uzawa methods for generalized saddle point problems*, Linear Algebra Appl., **428** (2008), 2900-2932.
8. M. Benzi, M. J. Gander, and G. H. Golub, *Optimization of the Hermitian and skew-Hermitian splitting iteration for saddle-point problems*, BIT., **43** (2003), 881-900.
9. M. Benzi and G. H. Golub, *A preconditioner for generalized saddle point problems*, SIAM J. Matrix Anal. Appl., **26** (2004), 20-41.
10. M. Benzi, G. H. Golub, and J. Liesen. *Numerical solution of saddle point problems*, Acta Numerical., **14** (2005), 1-137.
11. L. Bergamaschi, J. Gondzio, and G. Zilli. *Preconditioning indefinite systems in interior point methods for optimization*, Comput. Optim. Appl., **28** (2004), 149-171.
12. Å. Björck. *Numerical Methods for Least Squares Problems*. SIAM, Phil-adelphia, PA, 1996.
13. J. Bramble and J. Pasciak, *A preconditioned technique for indefinite systems resulting from mixed approximations of elliptic problems*, Math. Comput., **50** (1988), 1-17.
14. J. Bramble, J. Pasciak, and A. Vassilev, *Analysis of the inexact Uzawa algorithm for saddle point problems*, SIAM J. Numer. Anal., **34** (1997), 1072-1092.

TABLE 3. Iterations numbers for algorithm 3

(ω, τ, t)	$Q_1 = H$			$Q_1 = I$			$Q_1 = 0$		
	$p = 8$	$p = 16$	$p = 24$	$p = 8$	$p = 16$	$p = 24$	$p = 8$	$p = 16$	$p = 24$
(0.8,0.2,-1)	112	93	84	112	93	84	111	92	84
(0.8,0.2,-0.6)	114	94	86	113	94	84	113	94	85
(0.8,0.2,-0.3)	116	96	87	114	94	85	115	95	86
(0.8,0.2,0)	118	97	88	115	95	86	116	95	86
(0.8,0.2,0.2)	119	98	89	116	95	86	116	95	86
(0.8,0.2,0.3)	120	99	89	116	96	87	116	95	86
(0.8,0.2,0.4)	121	99	90	116	96	87	116	95	86
(0.8,0.2,0.5)	122	100	90	117	96	87	115	95	86
(0.8,0.2,0.7)	124	102	92	117	97	88	114	95	86
(0.8,0.2,1)	128	104	94	119	99	89	114	95	86
(0.6,0.8,-1)	140	111	99	138	110	98	137	109	97
(0.6,0.8,-0.6)	144	113	100	142	112	100	141	112	99
(0.6,0.8,-0.4)	145	114	101	143	113	100	143	113	100
(0.6,0.8,0)	150	117	104	147	116	102	147	116	103
(0.6,0.8,0.2)	152	119	105	149	116	103	149	117	103
(0.6,0.8,0.4)	155	121	107	150	117	104	150	118	103
(0.6,0.8,0.6)	159	124	109	151	118	105	149	116	103
(0.6,0.8,0.8)	164	127	112	153	120	106	146	116	103
(0.6,0.8,1)	170	131	115	155	122	108	146	116	103
(0.4,1.2,-1)	144	114	100	143	113	100	139	111	99
(0.4,1.2,-0.6)	149	116	103	147	116	103	144	114	101
(0.4,1.2,-0.4)	151	118	104	149	117	104	147	116	103
(0.4,1.2,0)	158	121	107	155	121	106	154	120	106
(0.4,1.2,0.2)	161	124	109	158	123	108	158	123	108
(0.4,1.2,0.4)	165	127	111	161	124	109	162	125	109
(0.4,1.2,0.6)	171	130	114	164	126	110	164	126	110
(0.4,1.2,0.8)	179	135	118	166	128	112	161	124	109
(0.4,1.2,1)	190	141	122	171	131	115	159	124	110
(1.2,0.6,-1)	141	112	100	139	110	98	139	111	98
(1.2,0.6,-0.6)	143	114	101	140	111	99	141	112	99
(1.2,0.6,-0.4)	145	115	102	141	111	99	141	112	99
(1.2,0.6,0)	148	117	104	142	112	100	141	112	99
(1.2,0.6,0.2)	150	118	105	142	113	101	141	111	99
(1.2,0.6,0.4)	152	119	106	143	114	101	140	111	99
(1.2,0.6,0.6)	154	121	107	144	114	102	138	111	98
(1.2,0.6,0.8)	156	122	108	145	115	102	138	111	99
(1.2,0.6,1)	159	124	109	146	116	103	138	111	99

15. J. Bramble, J. Pasciak, and A. Vassilev, *Uzawa type algorithm for nonsymmetric saddle point problems*, Math. Comput., **69** (1999), 667-689.
16. F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Vol. **15** of Springer Series in Computational Mathematics, Springer, New York, 1991.
17. Z. H. Cao, *Fast Uzawa algorithms for solving non-symmetric stabilized saddle point problems*, Numer. Linear Algebra Appl., **11** (2004), 1-24.
18. X. J. Chen, *On preconditioned Uzawa methods and SOR methods for saddle point problems*, J. Comput. Appl. Math., **100** (1998), 207-224.
19. F. Chen and Y. L. Jiang, *A generalization of the inexact parameterized Uzawa methods for saddle point problems*, Appl. Math. Comput., **206** (2008), 765-771.
20. M. R. Cui, *A sufficient condition for the inexact Uzawa algorithm for saddle point problems*, J. Comput. Appl. Math., **139** (2002), 189-196.
21. M. R. Cui, *Analysis of iterative algorithms of Uzawa type for saddle point problems*, Appl. Numer. Math., **50** (2004), 133-146.
22. H. C. Elman, D. J. Silvester, and A. J. Wathen, *Finite Elements and Fast Iterative Solvers*, Numerical Mathematics and Scientific Computation, Oxford University Press, Oxford, 2005.
23. G. H. Golub, X. Wu, and J.Y. Yuan, *SOR-like methods for augmented systems*, BIT., **41** (2001), 71-85.
24. M. Q. Jiang and Y. Cao, *On local Hermitian and Skew-Hermitian splitting iteration methods for grneralized saddle point problems*, J. Comput. Appl. Math., **231** (2009), 973-982.
25. Y. Q. Lin and Y. M. Wei, *Corrected Uzawa methods for solving large nonsymmetric saddle point problems*, Appl. Math. Comput., **183** (2006), 1108-1120.
26. X. F. Ling and X. Z. Hu, *On the iterative algorithm for large sparse saddle point problems*, Appl. Math. Comput., **178** (2006), 372-379.
27. T. Rusten and R. Winther, *A preconditioned iterative method for saddle point problems*, SIAM J. Matrix. Anal. Appl., **13** (1992), 887-904.
28. Z. Q. Wang, *Optimization of the parameterized Uzawa preconditioners for saddle point matrices*, J. Comput. Appl. Math., **226** (2009), 136-154.
29. D. M. Young, *Iterative Solution for Large Linear Syatems*, Academic press, New York, 1971.
30. Y. Y. Zhou and G. F. Zhang, *A generalization of parameterized inexact Uzawa method for generalized saddle point problems*, Appl. Math. Comput., **215** (2009), 599-607.

Jian-Lei Li received his master degree and Ph.D. from University of Electronic Science and Technology of China. He works at North China University of Water Resources and Electric Power. His research interests focus on the numerical linear algebra and matrix analysis, especially, focus on the numerical methods for saddel point problems.

College of Mathematics and Information Science, North China University of Water Resources and Electric Power, Zhengzhou, Henan, 450011, P. R. China.
e-mail: hnmaths@163.com

Dang Luo received his Ph.D. from Nanjing University of Aeronautics and Astronautics. He works at North China University of Water Resources and Electric Power. His research interests are investment decision and economic evaluation, quantity economy, project management and decision-making technique.

College of Mathematics and Information Science, North China University of Water Resources and Electric Power, Zhengzhou, Henan, 450011, P. R. China.
e-mail: iamld99@163.com

Zhi-Jiang Zhang received his master degree from Guangxi Normal University. He works at Minsheng College of Henan University. His research interests are computational mathematics, iterative method and parallel computation.

Minsheng College of Henan University, Kaifeng, Henan, 475004, P. R. China.
e-mail: zhangzhijiang@sohu.com