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A NEW METHOD FOR A FINITE FAMILY OF PSEUDOCONTRACTIONS AND EQUILIBRIUM PROBLEMS†

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Abstract. In this paper, we introduce a new iterative scheme for finding a common element of the set of fixed points of a finite family of strict pseudocontractions and the solution set of pseudomonotone and Lipschitz-type continuous equilibrium problems. The scheme is based on the idea of extragradient methods and fixed point iteration methods. We show that the iterative sequences generated by this algorithm converge strongly to the common element in a real Hilbert space.

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1. Introduction

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of $\mathcal{H}$ and f be a bifunction from $C \times C$ to $\mathbf{R}$ such that $f(x,x) = 0$ for all $x \in C$. We consider the following equilibrium problems $\big($shortly $EP(f,C)\big)$:

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0 \quad \forall y \in C.$$

The set of solutions of problem $EP(f,C)$ is denoted by $Sol(f,C)$.

If $f(x,y) := \langle F(x), y - x \rangle$ for all $x, y \in C$, where F is a mapping from C to $\mathcal{H}$, then problem $EP(f,C)$ becomes the following variational inequalities:

$$\text{Find } x^* \in C \text{ such that } \langle F(x^*), y - x^* \rangle \geq 0 \quad \forall y \in C.$$

It is well-known that problem $EP(f,C)$ covers many important problems in optimization and nonlinear analysis as well as has found many applications in

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economic, transportation and engineering (see [7, 10] and the references quoted therein). Theory and methods for solving this problem have been well developed by many researchers [3, 4, 12, 15].

Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. A mapping $S : C \to C$ is said to be a *strict pseudocontraction* if there exists a constant $0 \leq L < 1$ such that

$$\|S(x) - S(y)\|^2 \leq \|x - y\|^2 + L\|(I - S)(x) - (I - S)(y)\|^2 \ \forall x, y \in C,$$

where I is the identity mapping on $\mathcal{H}$. If $L = 0$ then S is called *nonexpansive* on C.

The problem of finding a common fixed point element of a finite family of strict contractions $\{S_i\}_{i=1}^p$ $(p \geq 1)$ is described as follows:

$$\text{Find } x^* \in C \text{ such that } x^* \in \cap_{i=1}^p Fix(S_i), \qquad (Fix)$$

where $Fix(S_i)$ is the set of the fixed points of the mapping S_i $(i = 1, \cdots, p)$. This problem now becomes a mature subject in nonlinear analysis. The theory and solution methods of this problem can be found in many research papers and monographs (see [11]).

We are interested in the problem of find a common element of the solution set of the equilibrium problem $EP(f, C)$ and the solution set of the fixed problem (Fix), namely:

$$\text{Find } x^* \in \cap_{i=1}^p Fix(S_i) \cap Sol(f, C). \qquad (1)$$

An important special case of problem (1) is that $f(x, y) = \langle F(x), y - x\rangle$ and this problem is reduced to finding a common element of the solution set of variational inequalities and the solution set of a fixed point problem (see [5, 6, 14, 18, 19, 23]).

In this paper, we propose a new iterative scheme for solving problem (1). This method can be considered as an improvement of the viscosity approximation method in [18], the iterative method in [9] via an improvement set of extragradient methods [3, 4] and extended the algorithm in [2]. The algorithm is then modified by projecting on a suitable set to obtain the strongly convergence. The main iterations of the algorithm, we only solve strongly convex problems.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. A bifunction $f : C \times C \to \mathbf{R}$ is said to be

a) *monotone* on C if

$$f(x, y) + f(y, x) \leq 0 \ \forall x, y \in C;$$

b) *pseudomonotone* on C if

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0 \ \forall x, y \in C;$$

c) *Lipschitz-type continuous* on C with two constants $c_1 > 0$ and $c_2 > 0$ if

$$f(x, y) + f(y, z) \geq f(x, z) - c_1\|x - y\|^2 - c_2\|y - z\|^2 \ \forall x, y, z \in C. \qquad (2)$$

It is clear that every monotone bifunction f is pseudomonotone. However, if f is pseudomonotone, f might not be monotone. It is not difficult to check such examples (see [16]).

The following proposition lists some useful properties for strict pseudocontractions.

Proposition 2.1 ([1])**.** *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, $S : C \to C$ be an L-strict pseudocontraction and for each $i = 1, \cdots, p$, $S_i : C \to C$ is a L_i-strict pseudocontraction for some $0 \le L_i < 1$. Then:*

(a) *S satisfies the following Lipschitz condition:*

$$\|S(x) - S(y)\| \le \frac{1+L}{1-L}\|x - y\| \ \forall x, y \in C;$$

(b) *$I - S$ is demiclosed at 0. That is, if the sequence $\{x^k\}$ contains in C such that $x^k \rightharpoonup \bar{x}$ and $(I - S)(x^k) \to 0$ then $(I - S)(\bar{x}) = 0$;*

(c) *the set of fixed points $Fix(S)$ is closed and convex;*

(d) *if $\lambda_i > 0$ $(i = 1, \cdots, p)$ and $\sum\limits_{i=1}^{p} \lambda_i = 1$ then $\sum\limits_{i=1}^{p} \lambda_i S_i$ is a $\bar{L}$-strict pseudocontraction with $\bar{L} := \max\{L_i \mid 1 \le i \le L\}$;*

(e) *if λ_i is chosen as in (d) and $\{S_i \mid i = 1, \cdots, p\}$ has a common fixed point then*

$$Fix\Big(\sum_{i=1}^{p} \lambda_i S_i\Big) = \cap_{i=1}^{p} Fix(S_i).$$

Before presenting our main contribution, let us briefly look at the recently literature related to the methods for solving problem (1). In [18] Takahashi and Takahashi proposed an viscosity approximation method for finding a common element of set of solutions of problem $EP(f, C)$ and the set of fixed points of a nonexpansive mapping S in a real Hilbert space $\mathcal{H}$. This method generated an iteration sequence $\{x^k\}$ starting from a given intial point $x^0 \in \mathcal{H}$ and computed x^{k+1} as

$$\begin{cases} \text{Find } u^k \in C \text{ such that } f(u^k, y) + \frac{1}{r_k}\langle y - u^k, u^k - x^k\rangle \ge 0, \text{ for all } y \in C, \\ \text{Compute } x^{k+1} = \alpha_k g(x^k) + (1 - \alpha_k) S(u^k), \end{cases} \tag{3}$$

where g is a contraction of $\mathcal{H}$ into itself, the sequences of parameters $\{r_k\}$ and $\{\alpha_k\}$ were chosen appropriately. The authors showed that two iterative sequences $\{x^k\}$ and $\{u^k\}$ converged strongly to $z = Pr_{Fix(S)\cap Sol(f,C)}\big(g(z)\big)$, where Pr_C denotes the projection onto C.

The problem of finding a common fixed point of a finite sequence of mappings has been studied by many researchers. For instance, Marino and Xu in [13] proposed an iterative algorithm for finding a common fixed point of p strict pseudocontractions S_i $(i = 1, \cdots, p)$. The method computed a sequence $\{x^k\}$

starting from $x^0 \in \mathcal{H}$ and taking

$$x^{k+1} = \alpha_k x^k + (1-\alpha_k)\sum_{i=1}^{p}\lambda_{k,i}S_i(x^k), \tag{4}$$

where the sequence of parameters $\{\alpha_k\}$ and $\{\lambda_{k,i}\}$ was chosen in a specific way to ensure the convergence of the iterative sequence $\{x^k\}$. The authors showed that the sequence $\{x^k\}$ converged weakly to the same point $\bar{x} \in \cap_{i=1}^p Fix(S_i)$.

Recently, Chen et al. in [9] proposed a new iterative scheme for finding a common element of the set of common fixed points of a strict pseudocontraction sequence $\{\bar{S}_i\}$ and the set of solutions of problem $EP(f,C)$ in a real Hilbert space $\mathcal{H}$. This method is briefly described as follows. Given a starting point $x^0 \in \mathcal{H}$ and generates three iterative sequences $\{x^k\}, \{y^k\}$ and $\{z^k\}$ using the following scheme:

$$\begin{cases} \text{Compute } y^k = \alpha_k x^k + (1-\alpha_k)\bar{S}_k(x^k), \\ \text{Find } z^k \in C \text{ such that } f(z^k, y) + \frac{1}{r_k}\langle y - z^k, z^k - y^k\rangle \geq 0 \ \forall y \in C, \\ \text{Compute } x^{k+1} = Pr_{C_k}(x^0), \text{ where } C_k := \{v \in C \mid \|z^k - v\| \leq \|x^k - v\|\}. \end{cases} \tag{5}$$

Here, two sequences $\{\alpha_k\}$ and $\{r_k\}$ are given as control parameters. Under certain conditions imposed on $\{\alpha_k\}$ and $\{r_k\}$, the authors showed that the sequences $\{x^k\}, \{y^k\}$ and $\{z^k\}$ converged strongly to the same point x^* such that $x^* \in Pr_{Sol(f,C)\cap Fix(S)}(x^0)$, where S is a nonexpansive mapping of C into itself defined by $S(x) = \lim_{j\to\infty} \bar{S}_j(x)$ for all $x \in C$.

The solution methods for finding a common element of the set of solutions of problem $EP(f,C)$ and $\cap_{i=1}^p Fix(S_i)$ in a real Hilbert space have been recently studied in many research papers (see [8, 15, 20, 21, 22, 23] and many other references cited therein). At each iteration n of all current algorithms, it requires to solve approximation equilibrium problems for a *monotone* and *Lipschitz-type continuous* bifunction on C.

Recall the following assumptions that will be used to prove the convergence of the algorithms.

Assumption 2.2. *The bifunction f satisfies the following conditions:*

(i) *f is pseudomonotone and weakly continuous on C;*
(ii) *f is Lipschitz-type continuous on C;*
(iii) *for each $x \in C$, $f(x,\cdot)$ is convex and subdifferentiable on C.*

Assumption 2.3. *For each $i = 1, \cdots, p$, S_i is L_i-strict pseudocontraction for some $0 \leq L_i < 1$.*

Assumption 2.4. *The solution set of* (1) *is nonempty, i.e.,*

$$\cap_{i=1}^p Fix(S_i) \cap Sol(f,C) \neq \emptyset.$$

The algorithm is now described as follows.

Algorithm 2.5.
Initialization: *Choose positive sequences $\{\lambda_k\}, \{\lambda_{k,i}\}$ and $\{\alpha_k\}$ satisfy the conditions:*

$$\begin{cases} \{\lambda_k\} \subset [a,b] \text{ for some } a,b \in (0, \frac{1}{L}), \text{ where } L := \max\{2c_1, 2c_2\}, \\ \{\alpha_k\} \subset [\alpha, \beta] \text{ for some } \alpha, \beta \in (\bar{L}, 1), \text{ where } \bar{L} := \max\{L_i \mid 1 \le i \le p\}, \\ \sum_{i=1}^{p} \lambda_{k,i} = 1 \text{ for all } k \ge 1. \end{cases} \tag{6}$$

Find an initial point $x^0 \in C$ and Set $k := 0$.
Iteration k: *Perform the steps below:*

- *Step 1. Solve two strongly convex programs:*

$$\begin{cases} y^k := \mathrm{argmin}\left\{\lambda_k f(x^k, y) + \frac{1}{2}\|y - x^k\|^2 \mid y \in C\right\}, \\ t^k := \mathrm{argmin}\left\{\lambda_k f(y^k, y) + \frac{1}{2}\|y - x^k\|^2 \mid y \in C\right\}. \end{cases}$$

- *Step 2. Set $z^k := \alpha_k t^k + (1 - \alpha_k) \sum_{i=1}^{p} \lambda_{k,i} S_i(t^k)$.*
- *Step 3. Set*

$$\begin{cases} P_k := \{z \in C \mid \|z^k - z\| \le \|x^k - z\|\}, \\ Q_k := \{z \in C \mid \langle x^k - z, x^0 - x^k\rangle \ge 0\}. \end{cases}$$

Compute $x^{k+1} := Pr_{P_k \cap Q_k}(x^0)$. Increase k by 1 and go to Step 1.

3. Convergence of the algorithms

This section investigates the convergence of Algorithm 2.5. For this purpose, let us recall the following technical lemma which will be used in the sequel.

Lemma 3.1 ([10])**.** *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and $g : C \to \mathbf{R}$ be subdifferentiable on C. Then x^* is a solution to the following convex problem*

$$\min\{g(x) \mid x \in C\}$$

if and only if $0 \in \partial g(x^*) + N_C(x^*)$, where $\partial g(\cdot)$ denotes the subdifferential of g and $N_C(x^*)$ is the (outward) normal cone of C at $x^* \in C$.

Lemma 3.2 ([11])**.** *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and $x^0 \in \mathcal{H}$. Let the sequence $\{x^k\}$ be bounded such that every weakly cluster point $\bar{x}$ of $\{x^k\}$ belongs to C and*

$$\|x^k - x^0\| \le \|x^0 - Pr_C(x^0)\| \ \forall k \ge 0.$$

Then $\{x^k\}$ converges strongly to $Pr_C(x^0)$ as $k \to \infty$.

Now, we prove the main convergence theorem.

Theorem 3.3. *Suppose that Assumptions 2.2-2.4 are satisfied. Then the sequences* $\{x^k\}, \{y^k\}$ *and* $\{z^k\}$ *generated by Algorithm 2.5 converge strongly to the same point* $x^* \in \cap_{i=1}^p Fix(S_i) \cap Sol(f, C)$, *where*

$$x^* = Pr_{\cap_{i=1}^p Fix(S_i) \cap Sol(f,C)}(x^0).$$

The proof of this theorem is divided into several steps.

Step 1. Suppose that $x^* \in \cap_{i=1}^p Fix(S_i) \cap Sol(f, C)$. We show that

$$\|t^k - x^*\|^2 \leq \|x^k - x^*\|^2 - (1 - 2\lambda_k c_2)\|t^k - y^k\|^2 - (1 - 2\lambda_k c_1)\|x^k - y^k\|^2 \ \forall k \geq 0. \quad (7)$$

Proof of Step 1. Since $f(x, \cdot)$ is convex on C for each $x \in C$, applying Lemma 3.1, we see that $t^k = \operatorname{argmin}\{\frac{1}{2}\|t - x^k\|^2 + \lambda_k f(y^k, t) \mid t \in C\}$ if and only if

$$0 \in \partial_2\big(\lambda_k f(y^k, y) + \frac{1}{2}\|y - x^k\|^2\big)(t^k) + N_C(t^k), \quad (8)$$

where $N_C(x)$ is the (outward) normal cone of C at $x \in C$. Thus, since $f(y^k, \cdot)$ is subdifferentiable on C, by the well-known Moreau-Rockafellar theorem (see [17]), there exists $w \in \partial_2 f(y^k, t^k)$ such that

$$f(y^k, t) - f(y^k, t^k) \geq \langle w, t - t^k\rangle \ \forall t \in C.$$

Substituting $t = x^*$ into this inequality to obtain

$$f(y^k, x^*) - f(y^k, t^k) \geq \langle w, x^* - t^k\rangle. \quad (9)$$

On the other hand, it follows from (8) that $0 = \lambda_k w + t^k - x^k + \bar{w}$, where $w \in \partial_2 f(y^k, t^k)$ and $\bar{w} \in N_C(t^k)$. By the definition of the normal cone N_C we have, from this relation that

$$\langle t^k - x^k, t - t^k\rangle \geq \lambda_k \langle w, t^k - t\rangle \ \forall t \in C. \quad (10)$$

Substituting $t = x^* \in C$ into the last inequality, we obtain

$$\langle t^k - x^k, x^* - t^k\rangle \geq \lambda_k \langle w, t^k - x^*\rangle. \quad (11)$$

Now, we combine (9) and (11) to obtain

$$\langle t^k - x^k, x^* - t^k\rangle \geq \lambda_k\big(f(y^k, t^k) - f(y^k, x^*)\big). \quad (12)$$

Furthermore, since $x^* \in \mathrm{Sol}(f, C), f(x^*, y) \geq 0$ for all $y \in C$, and f is pseudomonotone on C, we have $f(y^k, x^*) \leq 0$. Hence, (12) implies that

$$\langle t^k - x^k, x^* - t^k\rangle \geq \lambda_k f(y^k, t^k). \quad (13)$$

Applying the Lipschitz condition (2) of f with $x = x^k$, $y = y^k$ and $z = t^k$, it follows from (13) that

$$f(y^k, t^k) \geq f(x^k, t^k) - f(x^k, y^k) - c_1\|y^k - x^k\|^2 - c_2\|t^k - y^k\|^2. \quad (14)$$

Combining (13) and (14), we get

$$\langle t^k - x^k, x^* - t^k\rangle \geq \lambda_k\big(f(x^k, t^k) - f(x^k, y^k) - c_1\|y^k - x^k\|^2 - c_2\|t^k - y^k\|^2\big). \quad (15)$$

Similarly, since y^k is the unique solution to the strongly convex program: $\min\{\frac{1}{2}\|y - x^k\|^2 + \lambda_k f(x^k, y)|\ y \in C\}$, we have

$$\lambda_k\big(f(x^k, y) - f(x^k, y^k)\big) \geq \langle y^k - x^k, y^k - y\rangle\ \forall y \in C.$$

Substituting $y = t^k \in C$ into the last inequality, we obtain

$$\lambda_k\big(f(x^k, t^k) - f(x^k, y^k)\big) \geq \langle y^k - x^k, y^k - t^k\rangle. \tag{16}$$

From (15), (16) and the relation $2\langle t^k - x^k, x^* - t^k\rangle = \|x^k - x^*\|^2 - \|t^k - x^k\|^2 - \|t^k - x^*\|^2$, it follows that

$$\begin{aligned}\|x^k - x^*\|^2 - \|t^k - x^k\|^2 - \|t^k - x^*\|^2 \geq & 2\langle y^k - x^k, y^k - t^k\rangle - 2\lambda_k c_1\|x^k - y^k\|^2 \\ & - 2\lambda_k c_2\|t^k - y^k\|^2.\end{aligned}$$

Hence,

$$\begin{aligned}\|t^k - x^*\|^2 \leq & \|x^k - x^*\|^2 - \|t^k - x^k\|^2 - 2\langle y^k - x^k, y^k - t^k\rangle + 2\lambda_k c_1\|x^k - y^k\|^2 \\ & + 2\lambda_k c_2\|t^k - y^k\|^2 \\ = & \|x^k - x^*\|^2 - \|(t^k - y^k) + (y^k - x^k)\|^2 - 2\langle y^k - x^k, y^k - t^k\rangle + 2\lambda_k c_1\|x^k - y^k\|^2 \\ & + 2\lambda_k c_2\|t^k - y^k\|^2 \\ \leq & \|x^k - x^*\|^2 - \|t^k - y^k\|^2 - \|x^k - y^k\|^2 + 2\lambda_k c_1\|x^k - y^k\|^2 + 2\lambda_k c_2\|t^k - y^k\|^2 \\ = & \|x^k - x^*\|^2 - (1 - 2\lambda_k c_1)\|x^k - y^k\|^2 - (1 - 2\lambda_k c_2)\|y^k - t^k\|^2.\end{aligned}$$

The last inequality is indeed (7).

Step 2. Claim that $\cap_{i=1}^p Fix(S_i) \cap Sol(f, C) \subseteq P_k \cap Q_k$ for all $k \geq 0$.
Proof of Step 2. Set

$$\bar{S}_k := \sum_{i=1}^p \lambda_{k,i} S_i.$$

For each $x^* \in \cap_{i=1}^p Fix(S_i) \cap Sol(f, C)$, using Proposition 2.1($d$) and the relation

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2\quad \forall \lambda \in [0, 1]\ \forall x, y \in \mathcal{H},$$

we have $z^k = \alpha_k t^k + (1 - \alpha_k)\bar{S}_k(t^k)$ and

$$\begin{aligned}\|z^k - x^*\|^2 = & \|\alpha_k t^k + (1 - \alpha_k)\bar{S}_k(t^k) - x^*\|^2 \\ = & \|\alpha_k(t^k - x^*) + (1 - \alpha_k)\{\bar{S}_k(t^k) - x^*\}\|^2 \\ = & \alpha_k\|t^k - x^*\|^2 + (1 - \alpha_k)\|\bar{S}_k(t^k) - x^*\|^2 - \alpha_k(1 - \alpha_k)\|\bar{S}_k(t^k) - t^k\|^2 \\ = & \alpha_k\|t^k - x^*\|^2 + (1 - \alpha_k)\|\bar{S}_k(t^k) - \bar{S}_k(x^*)\|^2 - \alpha_k(1 - \alpha_k)\|\bar{S}_k(t^k) - t^k\|^2 \\ \leq & \alpha_k\|t^k - x^*\|^2 + (1 - \alpha_k)\{\|t^k - x^*\|^2 + \bar{L}\|(I - \bar{S}_k)(t^k) - (I - \bar{S}_k)(x^*)\|^2\} \\ & - \alpha_k(1 - \alpha_k)\|\bar{S}_k(t^k) - t^k\|^2 \\ = & \|t^k - x^*\|^2 - (1 - \alpha_k)(\alpha_k - \bar{L})\|\bar{S}_k(t^k) - t^k\|^2.\end{aligned}$$

Then from $\alpha_k \geq \alpha > \bar{L}$, it follows that

$$\|z^k - x^*\| \leq \|t^k - x^*\|. \tag{17}$$

Using this and Step 1, we have $\|z^k - x^*\| \leq \|x^k - x^*\|$ for all $k \geq 0$. By the definition of P_k, we get

$$\cap_{i=1}^p Fix(S_i) \cap Sol(f, C) \subseteq P_k \quad \forall k \geq 0. \tag{18}$$

Next, we show by mathematical induction that

$$\cap_{i=1}^p Fix(S_i) \cap Sol(f, C) \subseteq Q_k \quad \forall k \geq 0.$$

For $k = 0$ we have $Q_0 = C$, hence we have $\cap_{i=1}^p Fix(S_i) \cap Sol(f, C) \subseteq Q_0$. Now we suppose that $\cap_{i=1}^p Fix(S_i) \cap Sol(f, C) \subseteq Q_k$ for some $k \geq 0$. From $x^{k+1} = Pr_{P_k \cap Q_k}(x^0)$, it follows that

$$\langle x^{k+1} - x, x^0 - x^{k+1} \rangle \geq 0 \quad \forall x \in P_k \cap Q_k.$$

Using this and $\cap_{i=1}^p Fix(S_i) \cap Sol(f, C) \subseteq Q_k$, we have

$$\langle x^{k+1} - x, x^0 - x^{k+1} \rangle \geq 0 \quad \forall x \in \cap_{i=1}^p Fix(S_i) \cap Sol(f, C),$$

and hence $\cap_{i=1}^p Fix(S_i) \cap Sol(f, C) \subseteq Q_{k+1}$. By the mathematical induction, we have

$$\cap_{i=1}^p Fix(S_i) \cap Sol(f, C) \subseteq Q_k \quad \forall k \geq 0.$$

This and (18) prove Step 2.

Step 3. Claim that

$$\begin{aligned} \lim_{k\to\infty} \|x^{k+1} - x^k\| &= \lim_{k\to\infty} \|x^k - z^k\| \\ &= \lim_{k\to\infty} \|x^k - y^k\| \\ &= \lim_{k\to\infty} \|x^k - t^k\| \\ &= \lim_{k\to\infty} \|\bar{S}_k(t^k) - t^k\| \\ &= 0. \end{aligned}$$

Proof of Step 3. It follows from Step 2 and $x^{k+1} = Pr_{P_k \cap Q_k}(x^0)$ that

$$\|x^{k+1} - x^0\| \leq \|Pr_{\cap_{i=1}^p Fix(S_i) \cap Sol(f,C)}(x^0) - x^0\| \quad \forall k \geq 0. \tag{19}$$

Hence, the sequence $\{x^k\}$ is bounded. By Step 1, also the sequences $\{t^k\}$ and $\{z^k\}$. Otherwise, we have

$$\langle x^k - x, x^0 - x^k \rangle \geq 0 \quad \forall x \in Q_k,$$

and hence $x^k = Pr_{Q_k}(x^0)$. Using this and $x^{k+1} \in P_k \cap Q_k \subseteq Q_k$, we have

$$\|x^k - x^0\| \leq \|x^{k+1} - x^0\| \quad \forall k \geq 0.$$

Therefore, there exists

$$A = \lim_{k\to\infty} \|x^k - x^0\|. \tag{20}$$

Using $x^k = Pr_{Q_k}(x^0)$, $x^{k+1} \in Q_k$ and the property of $Pr_{Q_k}(\cdot)$

$$\|Pr_{Q_k}(x) - x\|^2 \leq \|x - y\|^2 - \|Pr_{Q_k}(x) - y\|^2 \quad \forall x \in \mathcal{H}, y \in Q_k,$$

we have

$$\|x^{k+1}-x^k\|^2 \le \|x^{k+1}-x^0\|^2 - \|x^k-x^0\|^2 \quad \forall k \ge 0.$$

Combinating this and (20), we get

$$\lim_{k\to\infty} \|x^{k+1}-x^k\| = 0. \tag{21}$$

It follows from $x^{k+1} = Pr_{P_k\cap Q_k}(x^0)$ that $x^{k+1} \in P_k$, i.e.,

$$\|z^k - x^{k+1}\| \le \|x^k - x^{k+1}\|.$$

Hence

$$\|x^k - z^k\| \le \|x^k - x^{k+1}\| + \|x^{k+1} - z^k\| \le 2\|x^k - x^{k+1}\| \quad \forall k \ge 0.$$

Then, by (21), we have

$$\lim_{k\to\infty} \|x^k - z^k\| = 0. \tag{22}$$

Using Step 1, (17) and $\{\lambda_k\} \subset [a,b]$ for some $a, b \in (0, \frac{1}{L})$, for each $x^* \in \cap_{i=1}^p Fix(S_i) \cap Sol(f,C)$ we have

$$\begin{aligned}
\|z^k - x^*\|^2 &\le \|t^k - x^*\|^2 \\
&\le \|x^k - x^*\|^2 - (1-2\lambda_k c_2)\|t^k - y^k\|^2 - (1-2\lambda_k c_1)\|x^k - y^k\|^2 \\
&\le \|x^k - x^*\|^2 - (1-2\lambda_k c_1)\|x^k - y^k\|^2 \\
&\le \|x^k - x^*\|^2 - (1-bL)\|x^k - y^k\|^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\|x^k - y^k\|^2 &\le \frac{1}{1-bL}(\|x^k - x^*\|^2 - \|z^k - x^*\|^2) \\
&= \frac{1}{1-bL}(\|x^k - x^*\| - \|z^k - x^*\|)(\|x^k - x^*\| + \|z^k - x^*\|) \\
&\le \frac{1}{1-bL}\|x^k - z^k\|(\|x^k - x^*\| + \|z^k - x^*\|).
\end{aligned}$$

Since this, (22) and the sequences $\{x^k\}, \{z^k\}$ are bounded, we obtain

$$\lim_{k\to\infty} \|x^k - y^k\| = 0. \tag{23}$$

By the similar way, we also have

$$\lim_{k\to\infty} \|t^k - y^k\| = 0. \tag{24}$$

Combining (23), (24) and $\|x^k - t^k\| \le \|x^k - y^k\| + \|y^k - t^k\|$, we have

$$\lim_{k\to\infty} \|x^k - t^k\| = 0. \tag{25}$$

Using (22), (25) and $z^k = \alpha_k t^k + (1-\alpha_k)\bar{S}_k(t^k)$, we have

$$\begin{aligned}
(1-\beta)\|\bar{S}_k(t^k) - t^k\| \le &(1-\alpha_k)\|\bar{S}_k(t^k) - t^k\| \\
= &\|z^k - t^k\| \\
\le &\|z^k - x^k\| + \|t^k - x^k\|,
\end{aligned}$$

and hence $\lim_{k\to\infty} \|t^k - \bar{S}_k(t^k)\| = 0$. Then the results (21)-(25) prove Step 3.

In Step 4 and Step 5 of this theorem, we will consider weakly clusters of $\{x^k\}$. It follows from (19) that the sequence $\{x^k\}$ is bounded and hence there exists a subsequence $\{x^{k_j}\}$ converges weakly to $\bar{x}$ as $j \to \infty$. By Step 3, the sequences $\{y^{k_j}\}, \{t^{k_j}\}$ and $\{z^{k_j}\}$ converge weakly to $\bar{x}$.

Step 4. Claim that $\bar{x} \in \cap_{i=1}^p Fix(S_i)$.
Proof of Step 4. For each $i = 1, \cdots, p$, we suppose that $\lambda_{k_j,i}$ converges $\bar{\lambda}_i$ as $j \to \infty$ such that $\sum\limits_{i=1}^p \bar{\lambda}_i = 1$. Then we have

$$S_{k_j}(x) \to S(x) := \sum_{i=1}^p \bar{\lambda}_i S_i(x) \quad (\text{as } j \to \infty) \quad \forall x \in C.$$

Since $\sum\limits_{i=1}^p \bar{\lambda}_i = 1$, from Step 3 and

$$\begin{aligned}
\|t^{k_j} - S(t^{k_j})\| &\le \|t^{k_j} - \bar{S}_{k_j}(t^{k_j})\| + \|\bar{S}_{k_j}(t^{k_j}) - S(t^{k_j})\| \\
&= \|t^{k_j} - \bar{S}_{k_j}(t^{k_j})\| + \|\sum_{i=1}^p \lambda_{k_j,i} S_i(t^{k_j}) - \sum_{i=1}^p \bar{\lambda}_i S_i(t^{k_j})\| \\
&= \|t^{k_j} - \bar{S}_{k_j}(t^{k_j})\| + \|\sum_{i=1}^p (\lambda_{k_j,i} - \bar{\lambda}_i) S_i(t^{k_j})\| \\
&\le \|t^{k_j} - \bar{S}_{k_j}(t^{k_j})\| + \sum_{i=1}^p |\lambda_{k_j,i} - \bar{\lambda}_i| \|S_i(t^{k_j})\|,
\end{aligned}$$

we obtain that $\lim_{k\to\infty} \|t^{k_j} - S(t^{k_j})\| = 0$. By Proposition 2.1($b$), we have $\bar{x} \in Fix(S) = \mathrm{Fix}(\sum\limits_{i=1}^p \bar{\lambda}_i S_i)$. Then, it implies from Proposition 2.1(e) that $\bar{x} \in \cap_{i=1}^p Fix(S_i)$.

Step 5. When $x^{k_j} \rightharpoonup \bar{x}$ as $j \to \infty$, we show that $\bar{x} \in \mathrm{Sol}(f, C)$.
Proof of Step 5. Since y^k is the unique strongly convex problem

$$\min\{\frac{1}{2}\|x - x^k\|^2 + f(x^k, y) \mid y \in C\}$$

and Lemma 3.1, we have

$$0 \in \partial_2\big(\lambda_k f(x^k, y) + \frac{1}{2}\|y - x^k\|^2\big)(y^k) + N_C(y^k).$$

This follows that

$$0 = \lambda_k w + y^k - x^k + \bar{w},$$

where $w \in \partial_2 f(x^k, y^k)$ and $\bar{w} \in N_C(y^k)$. By the definition of the normal cone N_C we imply that

$$\langle y^k - x^k, y - y^k \rangle \geq \lambda_k \langle w, y^k - y \rangle \quad \forall y \in C. \tag{26}$$

On the other hand, since $f(x^k, \cdot)$ is subdifferentiable on C, by the well known Moreau-Rockafellar theorem, there exists $w \in \partial_2 f(x^k, y^k)$ such that

$$f(x^k, y) - f(x^k, y^k) \geq \langle w, y - y^k \rangle \quad \forall y \in C.$$

Combining this with (26), we have

$$\lambda_k \big(f(x^k, y) - f(x^k, y^k) \big) \geq \langle y^k - x^k, y^k - y \rangle \quad \forall y \in C.$$

Hence

$$\lambda_{k_j} \big(f(x^{k_j}, y) - f(x^{k_j}, y^{k_j}) \big) \geq \langle y^{k_j} - x^{k_j}, y^{k_j} - y \rangle \quad \forall y \in C.$$

Then, using $\{\lambda_k\} \subset [a, b] \subset (0, \frac{1}{L})$, Step 2, $x^{k_j} \rightharpoonup \bar{x}, y^{k_j} \rightharpoonup \bar{x}$ as $j \to \infty$ and weakly continuity of f, we have

$$f(\bar{x}, y) \geq 0 \quad \forall y \in C.$$

This means that $\bar{x} \in \mathrm{Sol}(f, C)$.

Step 6. Claim that the sequences $\{x^k\}, \{y^k\}, \{z^k\}$ and $\{t^k\}$ converge strongly to the same point x^*, where

$$x^* = Pr_{\cap_{i=1}^p Fix(S_i) \cap Sol(f,C)}(x^0).$$

Proof of Step 6. It follows from Step 4 and Step 5 that for every weakly cluster point $\bar{x}$ of the sequence $\{x^k\}$ satisfies $\bar{x} \in \cap_{i=1}^p \mathrm{Fix}(S_i, C) \cap \mathrm{Sol}(f, C)$. On the other hand, using the definition of Q_k, we have

$$x^k = Pr_{Q_k}(x^0).$$

Combining this with Step 2, we obtain

$$\|x^0 - x^k\| \leq \|x^0 - x\| \quad \forall x \in \cap_{i=1}^p \mathrm{Fix}(S_i, C) \cap \mathrm{Sol}(f, C).$$

With $x = x^*$, we have

$$\|x^0 - x^k\| \leq \|x^0 - x^*\|.$$

By Lemma 3.2, we claim that the sequence $\{x^k\}$ converges strongly to x^* as $k \to \infty$, where $x^* = Pr_{\cap_{i=1}^p \mathrm{Fix}(S_i,C) \cap \mathrm{Sol}(f,C)}(x^0)$. By Step 3, we also have $y^k, z^k, t^k \to x^*$ as $k \to \infty$.

□

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