J. Appl. Math. & Informatics Vol. 29(2011), No. 5 - 6, pp. 1205 - 1219
Website: http://www.kcam.biz

A STABILIZED CHARACTERISTIC FINITE VOLUME METHOD FOR TRANSIENT NAVIER-STOKES EQUATIONS

TONG ZHANG

Abstract. In this work, a stabilized characteristic finite volume method for the time-dependent Navier-Stokes equations is investigated based on the lowest equal-order finite element pair. The temporal differentiation and advection term are dealt with by characteristic scheme. Stability of the numerical solution is derived under some regularity assumptions. Optimal error estimates of the velocity and pressure are obtained by using the relationship between the finite volume and finite element methods.

AMS Mathematics Subject Classification : 35Q10, 65N30, 76D05.
Key words and phrases : Navier-Stokes equations, Finite volume method, Stabilized method, Characteristic, Error estimate.

1. Introduction

In this paper, we consider the accuracy of stabilized characteristic finite volume method for the Navier-Stokes problem: Find (u,p) such that

$$\begin{cases} u_t + (u\cdot\nabla)u - \nu\Delta u + \nabla p = f, \quad \text{div } u = 0 & \text{in } \Omega\times(0,T], \\ u = 0 & \text{on } \partial\Omega\times(0,T], \\ u = u_0 & \text{on } \Omega\times\{0\}, \end{cases} \tag{1.1}$$

where $u=(u_1,u_2)^T$ is the velocity and $p=p(x,t)$ the pressure, $f=f(x,t)$ the prescribed body force, $\nu>0$ the viscosity, u_0 the initial velocity, T the given final time and $u_t=\frac{\partial u}{\partial t}$.

Finite volume method (FVM) has a long history as a kinds of numerical methods for the differential equations, this method has been also termed as box scheme [2], generalized finite difference method [19]. FVM is a numerical technique that lies somewhere between the finite element and finite difference methods. It has a flexibility similar to that of the finite element method for handling complicated geometries, and its implementation is comparable to that

Received September 11, 2010. Revised November 8, 2010. Accepted November 16, 2010.

of the finite difference method. Therefore, FVM has been widely used in computational fluid mechanics [5, 6]. However, the theoretical analysis of FVM lags far behind that of finite element and finite difference methods, we can refer to [11, 12, 15] for some recent developments.

On the other hand, the non-stationary Navier-Stokes problem has some hyperbolic nature at high Reynolds numbers, therefore, constructing an appropriate numerical method to solve such problem is very important in mathematics and mechanics. Characteristic method is an efficient scheme for the convection dominate problem. Many researchers have studied the transient Navier-Stokes equations with characteristic method and obtained some important results. For example, Pironneau investigated the Navier-Stokes problem by applying the characteristic scheme in [23] and provided the suboptimal convergence analysis. After then, Süli [24] improved the results of [23] and presented the optimal error estimates for the approximate solutions. Under some restriction about time step, Boukir et al. gave the stability and error estimates for the velocity and pressure by combining the characteristic method and second-order time scheme in [4]. For more literature about the characteristic method, we can refer to [1, 22, 26] and the references therein.

In this work, we combine the characteristic finite volume method with the unstable P_1-P_1 element to solve the transient Navier-Stokes problem, optimal error estimates for the numerical solutions are established. This paper is organized as follows. In Section 2, we introduce some notations and formulate the stabilized FVM approximations for the problem (1.1). Stability of the approximate solutions is presented in Sections 3. Section 4 is devoted to establish the optimal order estimates for the numerical solutions.

2. Preliminaries

2.1. Basic notation. Standard notations for the Sobolev spaces $W^{s,p}(\Omega)^r, (r = 1, 2)$ and the associated norms and seminorms are adopted in this paper. We set $(\cdot,\cdot)$ and $\|\cdot\|_0$ are the inner product and norm on $L^2(\Omega)$ or $L^2(\Omega)^2$. The space $H_0^1(\Omega)^i (i = 1, 2)$ is equipped with the scalar product and norm: $(\nabla u, \nabla v)$ and $|u|_1^2 = (\nabla u, \nabla u),\ \forall\ u, v \in H_0^1(\Omega)^i (i = 1, 2)$.

In order to present the variational formulation for (1.1), let

$$X = H_0^1(\Omega)^2,\quad Y = L^2(\Omega)^2,\quad D(A) = H^2(\Omega)^2 \cap X,$$
$$M = L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q dx = 0\}.$$

Set $Au = -\Delta u$, which is a positive self-adjoint operator from $D(A)$ onto Y. In particular, $D(A^{\frac{1}{2}}) = X,\ D(A^0) = Y$. If $\partial\Omega$ is of C^2 or Ω is a two-dimensional convex polygon, there hold [7]:

$$\|\ v\ \|_0 \le \gamma_0 \|v\|_1,\quad \forall\ v \in X;\quad \|v\|_1 \le \gamma_0 \|Av\|_0,\quad \forall\ v \in D(A), \tag{2.1}$$

where γ_0 is a positive constant only depending on Ω.

Let H^{-1} be a dual, with respect to L^2-duality, space to H_0^1 with the corresponding norm:

$$\|f\|_{-1} = \sup_{0\neq u\in H_0^1} \frac{(f,u)}{|u|_1}, \quad f \in H^{-1}.$$

We assume that the data u_0, f satisfy the following assumption [13]:

(**A1**) $u_0 \in D(A)$ with div$u_0 = 0$ and $f, f_t \in L^2(0,T;Y)$. Moreover

$$\|u_0\|_2 + \sup_{t\in[0,T]} \{\|f\|_0 + \|f_t\|_0\} \leq C.$$

Here and hereafter, the letter C denotes a generic positive constant which is independent of the parameters h and Δt, and maybe different at its different occurrences.

Let

$$\psi(x,t) = (1+|u|^2)^{\frac{1}{2}},$$

where $|u|^2 = u_1^2 + u_2^2$. The characteristic direction corresponding to the hyperbolic part of (1.1), $u_t + (u\cdot\nabla)u$, be denoted by τ, so

$$\frac{\partial}{\partial\tau} = \frac{1}{\psi(x,t)}\frac{\partial}{\partial t} + \frac{1}{\psi(x,t)} u\cdot\nabla.$$

With this definition, the equations (1.1) can be put in the form

$$\begin{cases} \psi(x,t)\frac{\partial u}{\partial\tau} - \nu\Delta u + \nabla p = f, \quad \text{div } u = 0 & \text{in } \Omega\times(0,T], \\ u = 0 & \text{on } \partial\Omega\times(0,T], \\ u = u_0 & \text{on } \Omega\times\{0\}. \end{cases} \tag{2.2}$$

The continuous bilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ on $X\times X$ and $X\times M$ are, respectively, defined by

$$a(u,v) = \nu(\nabla u, \nabla v), \quad d(v,q) = -(\nabla q, v) = (q, \text{div} v).$$

Moreover, the generalized bilinear form on $(X,M)\times(X,M)$ is given by

$$B_0((u,p);(v,q)) = a(u,v) - d(v,p) + d(u,q).$$

With notations above, the variational formulation of problem (2.2) is to seek $(u,p)\in(X,M)$, for all $t\in(0,T]$, such that

$$\begin{cases} (\psi(x,t)\frac{\partial u}{\partial\tau}, v) + B_0((u,p);(v,q)) = (f,v), \quad \forall\ (v,q)\in(X,M), \\ u(0) = u_0. \end{cases} \tag{2.3}$$

As for the existence uniqueness and regularity of the global solution to the transient Navier-Stokes problem, we have

Lemma 2.1 ([17]). *Assume that $\partial\Omega$ is of C^2 or Ω is a two-dimensional convex polygon and* (**A1**) *holds. Then, for any given $f \in L^2(0,T;L^2(\Omega)^2)$, problem (1.1) admits a unique solution (u,p) satisfying the following regularities:*

$$\sup_{0\leq t\leq T} (\|u_t(t)\|_0^2 + \|Au(t)\|_0^2 + \|p(t)\|_1^2) \leq C,$$

$$\sup_{0\le t\le T}\sigma(t)\|u_t(t)\|_1^2+\int_0^T\sigma(t)(\|u_{tt}(t)\|_0^2+\|Au_t(t)\|_0^2+\|p_t(t)\|_1^2)dt\le C,$$

where $\sigma(t)=\min\{1,t\}$.

2.2. The stabilized finite volume method. Let $\mathcal{T}_h=\{K\}$ be a regular trianglations (see [9]) of Ω with mesh size $h>0$. P is the set containing all the interior nodes associated with the triangulation $\mathcal{T}_h$. N be the total number of the nodes. To define the finite volume method, a dual mesh $\widetilde{\mathcal{T}}_h$ is introduced based on $\mathcal{T}_h$, and the elements in $\widetilde{\mathcal{T}}_h$ are called control volumes. The dual mesh can be constructed in this way: for each element $K\in\mathcal{T}_h$ with vertices $P_j(j=1,2,3)$, select its barycenter O and the midpoint Q_j on each of the edges of K and construct the control volumes in $\widetilde{\mathcal{T}}_h$ by connecting O to Q_j. This work focuses on the analysis of the lowest equal-order mixed finite element pair

$$X_h=\{v\in X: v_i|_K\in P_1(K),\forall K\in\mathcal{T}_h, i=1,2\},$$

and

$$M_h=\{q\in M: q|_K\in P_1(K),\forall K\in\mathcal{T}_h\}.$$

where $P_1(K)$ represents the set of all linear polynomials on K. The dual finite element space is defined by

$$\widetilde{X}_h=\{\widetilde{v}\in L^2(\Omega)^2:\widetilde{v}_i|_{\widetilde{K}}\in P_0(\widetilde{K}),\forall\widetilde{K}\in\widetilde{\mathcal{T}}_h;\ \widetilde{v}|_{\widetilde{K}}=0.\ \text{on any boundary dual element } \widetilde{K}\},$$

Obviously, the dimensions of X_h and $\widetilde{X}_h$ are the same. There exists an invertible linear mapping $\Gamma_h: X_h\to\widetilde{X}_h$ satisfying the following lemma.

Lemma 2.2 ([8, 21])**.** *Let* $K\in\mathcal{T}_h$, *if* $v_h\in X_h$ *and* $1\le r\le\infty$. *Then*

$$\int_K(v_h-\Gamma_h v_h)dx=0,\quad \|v_h-\Gamma_h v_h\|_{L^r(K)}\le Ch_K\|v_h\|_{W^{1,r}(K)},$$

where h_K *is the diameter of the element* K. *It is well known that the choice of the* P_1-P_1 *element violates the so-called Inf-Sup condition* [25]. *However, we can overcome this restriction by adding a simple and effective stabilization term* $G(\cdot,\cdot)$, *which is defined as follows. Let* $\Pi: M\to R_0$ *be the standard* L^2*-projection with the following properties* [20]

$$(p-\Pi p,q_h)=0,\quad \|\Pi p\|_0\le C\|p\|_0,\quad \forall\ p\in M, q_h\in R_0;$$
$$\|p-\Pi p\|_0\le Ch\|p\|_1,\quad \forall\ p\in H^1(\Omega)\cap M,$$

where $R_0=\{q_h\in M, q_h|_K$ *is a constant,* $\forall\ K\in\mathcal{T}_h\}$. *Then, we can define the stabilization term* $G(\cdot,\cdot)$ *by* [3]

$$G(p,q)=((I-\Pi)p,(I-\Pi)q)\quad\forall\ p,q\in L^2(\Omega).$$

We denote the discrete norm $|||u_h|||_0^2 = (u_h, \Gamma_h u_h),\ \forall\ u_h, v_h \in X_h$, *and this discrete norm is equivalent to the standard* L^2*-norm* [19]*: there exist two positive constants* C_*, C^*, *such that*

$$C_*\|u_h\|_0 \le |||u_h|||_0 \le C^*\|u_h\|_0. \tag{2.4}$$

Lemma 2.3 ([12])**.** *For all* $u_h, v_h \in X_h$, *it holds that*

$$(u_h, \Gamma_h v_h) = (v_h, \Gamma_h u_h).$$

With the help of the Green's formula, the stabilized finite volume method for problem (1.1) reads: find $(u_h, p_h) \in (X_h, M_h)$, $\forall (v_h, q_h) \in (X_h, M_h)$ *such that*

$$\begin{cases} (u_{ht}, \Gamma_h v_h) + A(u_h, \Gamma_h v_h) + D(\Gamma_h v_h, p_h) + d(u_h, q_h) \\ \qquad\qquad +b(u_h, u_h, \Gamma_h v_h) + G(p_h, q_h) = (f, \Gamma_h v_h), \\ u_h(0) = u_{0h}, \end{cases} \tag{2.5}$$

where

$$\begin{aligned} A(u_h, \Gamma_h v_h) &= -\sum_{j=1}^{N} v_h(P_j) \int_{\partial \widetilde{K}_j} \frac{\partial u_h}{\partial n} dx, \quad u_h, v_h \in X_h, \\ D(\Gamma_h v_h, p_h) &= \sum_{j=1}^{N} v_h(P_j) \int_{\partial \widetilde{K}_j} p_h n dx, \quad p_h \in M_h, \\ (f, \Gamma_h v_h) &= \sum_{j=1}^{N} v_h(P_j) \int_{\widetilde{K}_j} f dx, \quad v_h \in X_h. \end{aligned}$$

The following lemma establishes the relationship between the finite element and finite volume methods for the Navier-Stokes equations.

Lemma 2.4 ([18, 27])**.** *It holds that*

$$A(u_h, \Gamma_h v_h) = a(u_h, v_h), \qquad \forall\ u_h, v_h \in X_h.$$

Moreover, the bilinear form $D(\cdot,\cdot)$ *satisfies*

$$D(q_h, \Gamma_h v_h) = -d(q_h, v_h), \qquad \forall\ (v_h, q_h) \in (X_h, M_h).$$

We denote $B_h((u_h, p_h), (\Gamma_h v_h, q_h)) = A(u_h, \Gamma_h v_h) + D(\Gamma_h v_h, p_h) + d(u_h, q_h) + G(p_h, q_h)$, *then, the following lemma establishes the continuity and weak coercivity for* $B_h((u_h, p_h), (v_h, q_h))$.

Lemma 2.5 ([18])**.** *It holds that for all* $(u_h, p_h) \in (X_h, M_h)$

$$|B_h((u_h, p_h), (\Gamma_h v_h, q_h))| \le C(|u_h|_1 + \|p_h\|_0)(|v_h|_1 + \|q_h\|_0) \qquad \forall\ (v_h, q_h) \in (X_h, M_h).$$

Moreover,

$$\sup_{0\neq(v_h,q_h)\in(X_h,M_h)}\frac{|B_h((u_h,p_h);(\Gamma_h v_h,q_h))|}{|v_h|_1+\|q_h\|_0}\geq\beta(|u_h|_1+\|p_h\|_0).$$

Let (u,p) and (u_h,p_h) be the solutions of problems (1.1) and (2.5). By using the techniques to one used in [15, 16]*, we can obtain the regularity results.*

Lemma 2.6. *Under the assumptions of Lemma 2.1, for all $t\in[0,T]$, the numerical solution (u_h,p_h) of (2.5) satisfies*

$$\|u_h(t)\|_0^2\leq C\Big(\|u_0\|_0^2+\delta_0^{-2}C_f^2\Big),\qquad e^{-\delta_0 t}\int_0^t e^{\delta_0 s}\|u_{ht}\|_0^2ds\leq C.$$

$$\sigma(t)\|\nabla(u-u_h)\|_0^2+e^{-\delta_0 t}\int_0^t e^{\delta_0 s}\Big(\sigma(s)\|u_t-u_{ht}\|_0^2+\nu\|u-u_h\|_1^2\Big)ds\leq Ch^2.$$

where $\delta_0=\frac{\nu}{2\gamma_0}$, $C_f=\sup_{t\geq0}|f(t)|$.

3. Stabilized characteristic finite volume method

We consider a time step Δt and approximate the solution at $t^n=n\Delta t, n=1,2,...,N,\Delta t=\frac{T}{N}$. The characteristic derivative is approximated in the following way at $t=t^n$

$$(\psi(x,t)\frac{\partial u}{\partial\tau})^n\approx\psi(x,t)\frac{u(x,t^n)-u(\overline{x},t^{n-1})}{\sqrt{(x-\overline{x})^2+\Delta t^2}}=\frac{u^n-\overline{u}^{n-1}}{\Delta t}.$$

Namely, a backtracking algorithm is used to approximate the characteristic derivative. $\overline{x}=x-u(x,t^n)\Delta t$ is the foot (at level t^{n-1}) of the characteristic corresponding to x at the head (at level t^n).

The stabilized characteristic finite volume method for problem (1.1) at $t=t^n$ reads: Find $(u_h^n,p_h^n)\in X_h\times M_h$, for all $(v_h,q_h)\in X_h\times M_h$ such that

$$\begin{cases}(\frac{u_h^n-\overline{u}_h^{n-1}}{\Delta t},\Gamma_h v_h)+B_h((u_h^n,p_h^n),(\Gamma_h v_h,q_h))=(f^n,\Gamma_h v_h),\\ u_h^0=u_{0h}.\end{cases}\tag{3.1}$$

We define a projection operator $(R_h,Q_h):(X,M)\to(X_h,M_h)$ by

$$\begin{aligned}&B_h\Big((R_h(v,q),Q_h(v,q));(\Gamma_h v_h,q_h)\Big)=B_0\Big((v,q);(v_h,q_h)\Big),\\ &\forall(v,q)\in(X,M),(v_h,q_h)\in(X_h,M_h),\end{aligned}\tag{3.2}$$

Noting that due to Lemmas 2.4 and 2.5, (R_h,Q_h) is well defined and satisfies the following approximate properties.

Lemma 3.1. *Under the assumptions of Lemma 2.1, the projection operator $(R_h(u,p),Q_h(u,p))$ satisfies*

$$\|u-R_h(u,p)\|_1+\|p-Q_h(u,p)\|_0\leq C(\|u\|_1+\|p\|_0).$$

for all $(v,q) \in H_0^1(\Omega)^2 \times L_0^2(\Omega)$ *and*

$$\|u - R_h(u,p)\|_0 + h(\|u - R_h(u,p)\|_1 + \|p - Q_h(u,p)\|_0) \le Ch^2(\|Au\|_0 + \|p\|_1).$$

for all $(u,p) \in D(A) \times (H^1(\Omega) \cap M)$.

Proof. The proof can be completed by using the techniques to one used in [14, 16]. Here, we omit it.

Owing to $u_0 \in D(A)$, we can define $p_0 \in H^1(\Omega) \cap L_0^2(\Omega)$ [17], and denote $(u_{0h}, p_{0h}) = (R_h(u_0,p_0), Q_h(u_0,p_0))$. Furthermore, we set $(e_h^n, \eta_h^n) = (R_h(u^n,p^n) - u_h^n, Q_h(u^n,p^n) - p_h^n)$.

The following lemma can be found in reference [28]. □

Lemma 3.2. *It holds that*

$$(\overline{u}, \overline{u}) - (u,u) \le C\Delta t(u,u) \quad \forall\ u \in X,$$

where $\overline{u} = u(x - u(x,t)\Delta t)$.

Now, we present the stability of the numerical solutions for problem (3.1).

Theorem 3.3. *Under the assumptions of Lemma 2.1, for* $1 \le n \le N$, *the solution* (u_h^n, p_h^n) *of (3.1) satisfies*

$$\|u_h^N\|_0^2 + \nu \sum_{n=1}^N \|\nabla u_h^n\|_0^2 \Delta t + \sum_{n=1}^N \|p_h^n\|_0^2 \Delta t \le C,$$

Proof. At $t = t^n$, choosing $(v_h, q_h) = (u_h^n, p_h^n)$ in (3.1) and using the Lemma 2.4, we get

$$\Big(\frac{u_h^n - \overline{u}_h^{n-1}}{\Delta t}, \Gamma_h u_h^n\Big) + \nu\|\nabla u_h^n\|_0^2 + G(p_h^n, p_h^n) = (f^n, \Gamma_h u_h^n). \tag{3.3}$$

Noting the definition of $|||\cdot|||$ and (2.4), applying Lemma 2.3, we have

$$\begin{aligned} & \frac{1}{\Delta t}\Big(u_h^n - \overline{u}_h^{n-1}, \Gamma_h u_h^n\Big) \\ = & \frac{1}{2\Delta t}\bigg(u_h^n - \overline{u}_h^{n-1}, \Gamma_h\Big[(u_h^n + \overline{u}_h^{n-1}) + (u_h^n - \overline{u}_h^{n-1})\Big]\bigg) \\ \ge & \frac{1}{2\Delta t}\Big[(u_h^n, u_h^n) - (\overline{u}_h^{n-1}, \overline{u}_h^{n-1})\Big] \\ = & \frac{1}{2\Delta t}\bigg\{\Big[(u_h^n, u_h^n) - (u_h^{n-1}, u_h^{n-1})\Big] + \Big[(u_h^{n-1}, u_h^{n-1}) - (\overline{u}_h^{n-1}, \overline{u}_h^{n-1})\Big]\bigg\}. \end{aligned} \tag{3.4}$$

For the right terms of (3.3), using the Young inequality, we have

$$\Big|(f^n, \Gamma_h u_h^n)\Big| \le C\|f^n\|_0\|u_h^n\|_0 \le \frac{\nu}{2}\|\nabla u_h^n\|_0^2 + \frac{1}{2\nu}\|f^n\|_0^2. \tag{3.5}$$

Substituting (3.4)-(3.5) into (3.3), using Lemma 3.2, multiplying by $2\Delta t$ and summing n from 1 to N, one gets

$$\|u_h^N\|_0^2+\sum_{n=1}^N\Big[\nu\|\nabla u_h^n\|_0^2+G(p_h^n,p_h^n)\Big]\Delta t\le\frac{1}{\nu}\sum_{n=1}^N\|f^n\|_0^2\Delta t+\|u_{0h}\|_0^2+\sum_{n=1}^N\|u_h^{n-1}\|_0^2\Delta t.$$

Note that

$$\|u_{0h}\|_0=\|R_h(u_0,p_0)\|_0\le\|u_0\|_0+\|u_0-R_h(u_0,p_0)\|_0\le C(\|u_0\|_1+\|p_0\|_0).$$

Applying the discrete Gronwall inequality, we arrive at

$$\|u_h^N\|_0^2+\nu\sum_{n=1}^N\|\nabla u_h^n\|_0^2\Delta t\le C\Big(\sum_{n=1}^N\|f^n\|_0^2\Delta t+\|u_0\|_1^2+\|p_0\|_0^2\Big). \tag{3.6}$$

Combining (3.1) with Lemma 2.5, we get

$$\begin{aligned}\beta\Big(|u_h^n|_1+\|p_h^n\|_0\Big) &\le \sup_{(v_h,q_h)\in(X_h,M_h)}\frac{\Big|B\Big((u_h^n,p_h^n);(\Gamma_h v_h,q_h)\Big)\Big|}{|v_h|_1+\|q_h\|_0}\\ &\le \sup_{(v_h,q_h)\in(X_h,M_h)}\frac{\Big|(f^n,\Gamma_h v_h)\Big|+\Big|\Big(\frac{u_h^n-\overline{u}_h^{n-1}}{\Delta t},\Gamma_h v_h\Big)\Big|}{|v_h|_1+\|q_h\|_0}. \end{aligned}\tag{3.7}$$

Writing $u_h^n-\overline{u}_h^{n-1}$ as a sum of two terms $(u_h^n-u_h^{n-1})+(u_h^{n-1}-\overline{u}_h^{n-1})$, thanks to the results provided in [10] and Lemma 2.6, we have

$$\begin{aligned}&\Big(\frac{u_h^n-\overline{u}_h^{n-1}}{\Delta t},\Gamma_h v_h\Big)=\Big(\frac{u_h^n-u_h^{n-1}}{\Delta t},\Gamma_h v_h\Big)+\Big(\frac{u_h^{n-1}-\overline{u}_h^{n-1}}{\Delta t},\Gamma_h v_h\Big)\\ \le\;& \frac{C_1}{\Delta t}\|v_h\|_0\|u_h^n-u_h^{n-1}\|_0+C_2\|v_h\|_0\|\nabla u_h^{n-1}\|_0\\ \le\;& \frac{C_1}{\Delta t}e^{\frac{-\delta_0 t^n}{2}}\|v_h\|_0\|e^{\frac{\delta_0 t^n}{2}}u_h^n-e^{\frac{\delta_0 t^{n-1}}{2}}u_h^{n-1}\|_0+C_2\|v_h\|_0\|\nabla u_h^{n-1}\|_0\\ \le\;& \frac{C_1}{\Delta t}e^{\frac{-\delta_0 t^n}{2}}\|v_h\|_0\int_{t^{n-1}}^{t^n}\|\frac{\partial(e^{\frac{\delta_0 t}{2}}u_h)}{\partial t}\|_0 ds+C_2\|v_h\|_0\|\nabla u_h^{n-1}\|_0\\ \le\;& \frac{C_1}{\Delta t}e^{\frac{-\delta_0 t^n}{2}}\|v_h\|_0\int_{t^{n-1}}^{t^n}\Big[e^{\frac{\delta_0 t}{2}}\|u_{ht}\|_0+\frac{\delta_0}{2}e^{\frac{\delta_0 t}{2}}\|u_h\|_0\Big]ds+C_2\|v_h\|_0\|\nabla u_h^{n-1}\|_0\\ \le\;& \frac{C_1}{\Delta t^{\frac{1}{2}}}e^{\frac{-\delta_0 t^n}{2}}\|v_h\|_0\Bigg[\Big(\int_{t^{n-1}}^{t^n}e^{\delta_0 t}\|u_{ht}\|_0^2 ds\Big)^{\frac{1}{2}}+\frac{\delta_0}{2}\Big(\int_{t^{n-1}}^{t^n}e^{\delta_0 t}\|u_h\|_0^2 ds\Big)^{\frac{1}{2}}\Bigg]\\ &+C_2\|v_h\|_0\|\nabla u_h^{n-1}\|_0.\end{aligned}$$

Combining above estimate with (3.7), squaring, multiplying by Δt, summing n from 1 to N, using (2.1) and Lemma 2.6, we get

$$\sum_{n=1}^N\|p_h^n\|_0^2\Delta t\le C\Big(\sum_{n=1}^N\|f^n\|_0^2\Delta t+\|u_0\|_0^2+\delta_0^{-2}C_f^2+\sum_{n=1}^N\|\nabla u_h^n\|_0^2\Delta t\Big).$$

Thanks to (3.6), we finish the proof. □

4. Error estimates

This section is devoted to present the convergence analysis for the numerical solution (u_h^n, p_h^n) of problem (3.1).

Theorem 4.1. *Let (u,p) and (u_h^n, p_h^n) be the solutions of (2.3) and (3.1). Under the assumptions of Lemma 3.1, for all $1 \le n \le N$ we have*

$$\sigma(t)\Big(\sum_{n=1}^{N}\|\nabla(u-u_h^n)\|_0^2\Delta t+\sigma(t)\sum_{n=1}^{N}\|p-p_h^n\|_0^2\Delta t\Big)\le C\Big(\Delta t+h\Big).$$

Proof. This proof of Theorem 4.1 is consisted of Lemmas 3.1 and 4.2-4.3. □

Lemma 4.2. *Under the assumptions of Theorem 4.1, the following error estimate holds for $1 \le n \le N$.*

$$\begin{aligned}&\sigma^2(t)\Big(\|e_h^N\|_0^2+\sum_{n=1}^{N}\|\nabla e_h^n\|_0^2\Delta t\Big)\\ \le\ & C\Big[\Delta t^2\int_0^T\|\frac{\partial^2 u}{\partial\tau^2}\|_0^2ds+h^2\Big(\sum_{n=1}^{N}\|f^n\|_0^2\Delta t+\delta_0^{-2}C_f^2+\sum_{n=1}^{N}\|\nabla u_h^{n-1}\|_0^2\Delta t\Big)\\ &+h^2\|u_0\|_0^2+h^4\int_0^T(\sigma(s)\|Au_t\|_0^2+\|Au\|_0^2+\|p\|_1^2)ds+h^4(\|Au\|_0^2+\|p\|_1^2)\Big].\end{aligned}$$

Proof. Subtracting (3.1) from (2.3) at $t=t^n$, thank to Lemma 2.4 and (3.2), we get

$$\begin{aligned}&(\frac{e_h^n-\overline{e}_h^{n-1}}{\Delta t},v_h)+B_h((e_h^n,\eta_h^n),(v_h,q_h))\\ =\ &-(\psi(x,t^n)\frac{\partial u^n}{\partial\tau}-\frac{u^n-\overline{u}^{n-1}}{\Delta t},v_h)+(\frac{u_h^n-\overline{u}_h^{n-1}}{\Delta t},v_h-\Gamma_h v_h)\\ &+(\frac{(u^{n-1}-R_h(u^{n-1},p^{n-1}))-(u^n-R_h(u^n,p^n))}{\Delta t},v_h)\\ &+(\frac{(\overline{u}^{n-1}-R_h(\overline{u}^{n-1},p^{n-1}))-(u^{n-1}-R_h(u^{n-1},p^{n-1}))}{\Delta t},v_h)\\ &+(f,v_h-\Gamma_h v_h)\equiv\sum_{i=1}^{5}T_i.\end{aligned}\tag{4.1}$$

Choosing $(v_h,q_h)=(e_h^n,\eta_h^n)$ and estimating the left terms of (4.1) by $a(a-b)\ge\frac{1}{2}(a^2-b^2)$, we have

$$\begin{aligned}(\frac{e_h^n-\overline{e}_h^{n-1}}{\Delta t},e_h^n)&\ge\frac{1}{2\Delta t}\Big[\|e_h^n\|_0^2-\|\overline{e}_h^{n-1}\|_0^2\Big]\\ &=\frac{1}{2\Delta t}\Big[\|e_h^n\|_0^2-\|e_h^{n-1}\|_0^2+\|e_h^{n-1}\|_0^2-\|\overline{e}_h^{n-1}\|_0^2\Big],\qquad(4.2)\\ B_h((e_h^n,\eta_h^n),(e_h^n,\eta_h^n))&=\nu\|\nabla e_h^n\|_0^2+G(\eta_h^n,\eta_h^n).\qquad(4.3)\end{aligned}$$

Using the results provided in [10, 24] and Lemma 2.2, for the right terms T_1-T_5 of (4.1), we have

$$
\begin{aligned}
&|T_1| \le C\|\psi(x,t^n)\frac{\partial u^n}{\partial \tau} - \frac{u^n - \overline{u}^{n-1}}{\Delta t}\|_0\|e_h^n\|_0 \\
\le\ & C\Delta t\int_{t^{n-1}}^{t^n}\|\frac{\partial^2 u}{\partial \tau^2}\|_0^2 ds + \frac{\nu}{8}\|\nabla e_h^n\|_0^2, \\
&|T_2| \le \left|\Big(\frac{u_h^n - u_h^{n-1}}{\Delta t}, e_h^n - \Gamma_h e_h^n\Big)\right| + \left|\Big(\frac{u_h^{n-1} - \overline{u}_h^{n-1}}{\Delta t}, e_h^n - \Gamma_h e_h^n\Big)\right| \\
\le\ & \frac{C_1 h}{\Delta t}\|\nabla e_h^n\|_0\|u_h^n - u_h^{n-1}\|_0 + C_2 h\|\nabla e_h^n\|_0\|\nabla u_h^{n-1}\|_0 \\
\le\ & \frac{C_1 h}{\Delta t}e^{-\frac{1}{2}\delta_0 t^n}\|\nabla e_h^n\|_0\|e^{\frac{1}{2}\delta_0 t^n}u_h^n - e^{\frac{1}{2}\delta_0 t^{n-1}}u_h^{n-1}\|_0 + C_2 h\|\nabla e_h^n\|_0\|\nabla u_h^{n-1}\|_0 \\
\le\ & \frac{C_1 h}{\Delta t}e^{-\frac{1}{2}\delta_0 t^n}\|\nabla e_h^n\|_0\int_{t^{n-1}}^{t^n}\|\frac{\partial(e^{\frac{1}{2}\delta_0 t}u_h)}{\partial t}\|_0 ds + C_2 h\|\nabla e_h^n\|_0\|\nabla u_h^{n-1}\|_0 \\
\le\ & \frac{C_1 h}{\Delta t}e^{-\frac{1}{2}\delta_0 t^n}\|\nabla e_h^n\|_0\int_{t^{n-1}}^{t^n}\Big[e^{\frac{1}{2}\delta_0 t}\|u_{ht}\|_0 + \frac{\delta_0}{2}e^{\frac{1}{2}\delta_0 t}\|u_h\|_0\Big]ds \\
&+ C_2 h\|\nabla e_h^n\|_0\|\nabla u_h^{n-1}\|_0 \\
\le\ & \frac{C_1 h}{\Delta t^{\frac{1}{2}}}\|\nabla e_h^n\|_0 e^{-\frac{1}{2}\delta_0 t^n}\Big(\int_{t^{n-1}}^{t^n} e^{\delta_0 t}\|u_{ht}\|_0^2 ds\Big)^{\frac{1}{2}} + C_2 h\|\nabla e_h^n\|_0\|\nabla u_h^{n-1}\|_0 \\
&+ \frac{C_2 h}{\Delta t^{\frac{1}{2}}}\|\nabla e_h^n\|_0 e^{-\frac{1}{2}\delta_0 t^n}\Big(\int_{t^{n-1}}^{t^n} e^{\delta_0 t}\|u_h\|_0^2 ds\Big)^{\frac{1}{2}}, \\
&|T_3| \le \frac{C}{\Delta t}\|(u^n - R_h(u^n,p^n)) - (u^{n-1} - R_h(u^{n-1},p^{n-1}))\|_0\|e_h^n\|_0 \\
\le\ & \frac{C\cdot\|\nabla e_h^n\|_0}{\Delta t\sigma(t^n)}\|\sigma(t^n)(u^n - R_h(u^n,p^n)) - \sigma(t^{n-1})(u^{n-1} - R_h(u^{n-1},p^{n-1}))\|_0 \\
\le\ & \frac{C}{\Delta t\sigma(t^n)}\int_{t^{n-1}}^{t^n}\|\frac{\partial(\sigma(s)(u - R_h(u,p)))}{\partial t}\|_0 ds\cdot\|\nabla e_h^n\|_0 \\
\le\ & \frac{C}{\Delta t\sigma(t^n)}\int_{t^{n-1}}^{t^n}\Big(\sigma(s)\|u_t - R_{ht}(u,p)\|_0 + \|u - R_h(u,p)\|_0\frac{d\sigma(s)}{dt}\Big)ds\cdot\|\nabla e_h^n\|_0.
\end{aligned}
$$

Note that $0 \le \sigma(t) \le t$, $\frac{d\sigma(t)}{dt} \le 1 (\forall\ t \ge 0)$, the estimate of $|T_3|$ can be rewritten as

$$
\begin{aligned}
|T_3| \quad \le\ & \frac{C\cdot\|\nabla e_h^n\|_0}{\Delta t\sigma(t^n)}\int_{t^{n-1}}^{t^n}\Big(\sigma(s)\|u_t - R_{ht}(u,p)\|_0 ds + \|u - R_h(u,p)\|_0\Big)ds \\
\le\ & \frac{C}{\Delta t\sigma(t^n)}\Bigg[\Big(\int_{t^{n-1}}^{t^n}\sigma(s)\|u_t - R_{ht}(u,p)\|_0^2 ds\Big)^{\frac{1}{2}}\Big(\int_{t^{n-1}}^{t^n} s ds\Big)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\quad +\Big(\int_{t^{n-1}}^{t^n}\|u-R_h(u,p)\|_0^2ds\Big)^{\frac{1}{2}}\Big(\int_{t^{n-1}}^{t^n}1ds\Big)^{\frac{1}{2}}\Bigg]\cdot\|\nabla e_h^n\|_0\\
&\le \frac{C}{\Delta t^{\frac{1}{2}}\sigma(t^n)}\Bigg[\Big(\int_{t^{n-1}}^{t^n}\sigma(s)\|u_t-R_{ht}(u,p)\|_0^2ds\Big)^{\frac{1}{2}}\\
&\quad +\Big(\int_{t^{n-1}}^{t^n}\|u-R_h(u,p)\|_0^2ds\Big)^{\frac{1}{2}}\Bigg]\cdot\|\nabla e_h^n\|_0\\
&\le \frac{C}{\Delta t\sigma^2(t^n)}\Bigg[\Big(\int_{t^{n-1}}^{t^n}\sigma(s)\|u_t-R_{ht}(u,p)\|_0^2ds\Big)\\
&\quad +\Big(\int_{t^{n-1}}^{t^n}\|u-R_h(u,p)\|_0^2ds\Big)\Bigg]+\frac{\nu}{8}\|\nabla e_h^n\|_0^2,\\
|T_4| &\le \frac{C\cdot\|\nabla e_h^n\|_0}{\Delta t}\Bigg\|\Big(\overline{u}^{n-1}-R_h(\overline{u}^{n-1},p^{n-1})\Big)-\Big(u^{n-1}-R_h(u^{n-1},p^{n-1})\Big)\Bigg\|_{-1}\\
&\le C\|u^{n-1}-R_h(u^{n-1},p^{n-1})\|_0\|\nabla e_h^n\|_0\\
&\le C\|u^{n-1}-R_h(u^{n-1},p^{n-1})\|_0^2+\frac{\nu}{8}\|\nabla e_h^n\|_0^2,\\
|T_5| &\le \|f\|_0\|e_h^n-\Gamma_h e_h^n\|_0\le Ch\|f\|_0\|\nabla e_h^n\|_0.
\end{aligned}
$$

Combining the above estimates with (4.1)-(4.3) and applying Lemma 3.2 yields

$$
\begin{aligned}
&\frac{1}{2\Delta t}\Bigg[\|e_h^n\|_0^2-\|e_h^{n-1}\|_0^2\Bigg]+\frac{\nu}{2}\|\nabla e_h^n\|_0^2+G(\eta_h^n,\eta_h^n)\\
\le\;& C\Bigg[\Delta t\int_{t^{n-1}}^{t^n}\|\frac{\partial^2 u}{\partial\tau^2}\|_0^2ds+\frac{1}{\Delta t\sigma^2(t^n)}\int_{t^{n-1}}^{t^n}\sigma(s)\|u_t-R_{ht}(u,p)\|_0^2ds+\frac{\|e_h^{n-1}\|_0^2}{2}\\
&+\frac{1}{\Delta t\sigma^2(t^n)}\int_{t^{n-1}}^{t^n}\|u-R_h(u,p)\|_0^2ds+\|u^{n-1}-R_h(u^{n-1},p^{n-1})\|_0^2\\
&+h^2\|f\|_0^2+\frac{h^2}{\Delta t}e^{-\delta_0 t^n}\int_{t^{n-1}}^{t^n}e^{\delta_0 t}\Big(\|u_{ht}\|_0^2+\|u_h\|_0^2\Big)ds+h^2\|\nabla u_h^{n-1}\|_0^2\Bigg].
\end{aligned}
\tag{4.4}
$$

It follows from the definition $(u_h^0,p_h^0)=(R_h(u_0,p_0),Q_h(u_0,p_0))$ that $e_h^0=0$. Multiplying (4.4) by $2\Delta t\sigma^2(t^n)$, summing over n and applying the discrete Gronwall lemma, we complete the proof. □

Lemma 4.3. *Under the assumptions of Lemma 4.2, it holds that*

$$
\sigma^2(t)\Big(\sum_{n=1}^{N}\|p-p_h^n\|_0^2\Delta t\Big)^{\frac{1}{2}}\le C(\Delta t+h)\quad\forall\ 1\le n\le N.
$$

Proof. From Lemma 2.5 and (4.1), we find that

$$\begin{aligned}
&\beta\sigma(t^n)\Big(|e_h^n|_1+\|\eta_h^n\|_0\Big)\\
\le\ & \sup_{(v_h,q_h)\in(X_h,M_h)}\frac{\Big|\sigma(t^n)B\Big((e_h^n,\eta_h^n);(v_h,q_h)\Big)\Big|}{|v_h|_1+\|q_h\|_0}\\
\le\ & \sup_{(v_h,q_h)\in(X_h,M_h)}\frac{1}{|v_h|_1+\|q_h\|_0}\Bigg[\Big|\Big(\frac{\sigma(t^n)[(u^n-u_h^n)-(\overline{u}^{n-1}-\overline{u}_h^{n-1})]}{\Delta t},v_h\Big)\Big|\\
&+\Big|\Big(f,v_h-\Gamma_h v_h\Big)\Big|+\Big|\Big(\psi(x,t^n)\frac{\partial u^n}{\partial\tau}-\frac{u^n-\overline{u}^{n-1}}{\Delta t},v_h\Big)\Big|\\
&+\Big|\Big(\frac{u_h^n-\overline{u}_h^{n-1}}{\Delta t},v_h-\Gamma_h v_h\Big)\Big|\Bigg]. \qquad (4.5)
\end{aligned}$$

Now, we estimate the right side terms of (4.5). Using the results provided in [24], we arrive at

$$\Big|(f,v_h^n-\Gamma_h v_h^n)\Big|\le\|f\|_0\|v_h^n-\Gamma_h v_h^n\|_0\le Ch\|f\|_0\|\nabla v_h^n\|_0, \qquad (4.6)$$

$$\begin{aligned}
&\Big|\sigma(t^n)\Big(\psi(x,t^n)\frac{\partial u^n}{\partial\tau}-\frac{u^n-\overline{u}^{n-1}}{\Delta t},v_h\Big)\Big|\\
\le\ & \sigma(t^n)\|\psi(x,t^n)\frac{\partial u^n}{\partial\tau}-\frac{u^n-\overline{u}^{n-1}}{\Delta t}\|_0\|v_h\|_0\\
\le\ & C\sigma(t^n)\Delta t^{\frac12}\Big(\int_{t^{n-1}}^{t^n}\|\frac{\partial^2 u}{\partial\tau^2}\|_0^2ds\Big)^{\frac12}\|\nabla v_h\|_0, \qquad (4.7)
\end{aligned}$$

$$\begin{aligned}
&\Big|\Big(\frac{u_h^n-\overline{u}_h^{n-1}}{\Delta t},v_h-\Gamma_h v_h\Big)\Big|\\
\le\ & \Big|\Big(\frac{u_h^n-u_h^{n-1}}{\Delta t},v_h-\Gamma_h v_h\Big)\Big|+\Big|\Big(\frac{u_h^{n-1}-\overline{u}_h^{n-1}}{\Delta t},v_h-\Gamma_h v_h\Big)\Big|\\
\le\ & \frac{C_1h}{\Delta t}e^{-\frac12\delta_0t^n}\|\nabla v_h\|_0\|e^{\frac12\delta_0t^n}u_h^n-e^{\frac12\delta_0t^{n-1}}u_h^{n-1}\|_0+C_2h\|\nabla v_h\|_0\|\nabla u_h^{n-1}\|_0\\
\le\ & \frac{C_1h}{\Delta t}e^{-\frac12\delta_0t^n}\|\nabla v_h\|_0\int_{t^{n-1}}^{t^n}\|\frac{\partial(e^{\frac12\delta_0t}u_h)}{\partial t}\|_0ds+C_2h\|\nabla v_h\|_0\|\nabla u_h^{n-1}\|_0\\
\le\ & \frac{C_1h}{\Delta t}\|\nabla v_h\|_0e^{-\frac12\delta_0t^n}\int_{t^{n-1}}^{t^n}\Big(e^{\frac12\delta_0t}\|u_{ht}\|_0ds+\frac12\delta_0e^{\frac12\delta_0t}\|u_h\|_0\Big)ds\\
&+C_2h\|\nabla v_h\|_0\|\nabla u_h^{n-1}\|_0\\
\le\ & \frac{C_1h}{\Delta t^{\frac12}}\|\nabla v_h\|_0e^{-\frac12\delta_0t^n}\Bigg[\Big(\int_{t^{n-1}}^{t^n}e^{\delta_0t}\|u_{ht}\|_0^2ds\Big)^{\frac12}+\frac{\delta_0}{2}\Big(\int_{t^{n-1}}^{t^n}e^{\delta_0t}\|u_h\|_0^2ds\Big)^{\frac12}\Bigg]\\
&+C_2h\|\nabla v_h\|_0\|\nabla u_h^{n-1}\|_0. \qquad (4.8)
\end{aligned}$$

From the definition of $\sigma(t)$ and t^n, we know that $t^{n-1} \le t^n \le t^{n-1} + \Delta t$, with this relationship in mind, we have

$$\begin{aligned}
&\Big|\Big(\frac{\sigma(t^n)[(u^n - u_h^n) - (\overline{u}^{n-1} - \overline{u}_h^{n-1})]}{\Delta t}, v_h\Big)\Big| \\
\le\ &\Big|\Big(\frac{\sigma(t^n)[(u^n - u_h^n) - (u^{n-1} - u_h^{n-1})]}{\Delta t}, v_h\Big)\Big| \\
&+\Big|\Big(\frac{\sigma(t^n)[(u^{n-1} - u_h^{n-1}) - (\overline{u}^{n-1} - \overline{u}_h^{n-1})]}{\Delta t}, v_h\Big)\Big| \\
\le\ &\Big|\Big(\frac{\sigma(t^n)(u^n - u_h^n) - \sigma(t^{n-1})(u^{n-1} - u_h^{n-1})}{\Delta t}, v_h\Big)\Big| \\
&+\Big|\Big((u^{n-1} - u_h^{n-1}) - (\overline{u}^{n-1} - \overline{u}_h^{n-1}), v_h\Big)\Big| \\
&+\Big|\Big(\frac{\sigma(t^{n-1})[(u^{n-1} - u_h^{n-1}) - (\overline{u}^{n-1} - \overline{u}_h^{n-1})]}{\Delta t}, v_h\Big)\Big| \triangleq \sum_{i=1}^{3} T_i. \quad (4.9)
\end{aligned}$$

For T_1, due to $0 \le \sigma(t) \le t$, $\frac{d\sigma(t)}{dt} \le 1 (\forall\ t \ge 0)$, we have

$$\begin{aligned}
T_1 = &\Big|\Big(\frac{\sigma(t^n)(u^n - u_h^n) - \sigma(t^{n-1})(u^{n-1} - u_h^{n-1})}{\Delta t}, v_h\Big)\Big| \\
\le\ &\frac{\sigma(t^n)^{\frac{1}{2}} e^{\frac{-\delta_0 t^n}{2}}}{\Delta t}\Big|\Big(\sigma(t^n)^{\frac{1}{2}} e^{\frac{\delta_0 t^n}{2}}(u^n - u_h^n) - \sigma(t^{n-1})^{\frac{1}{2}} e^{\frac{\delta_0 t^{n-1}}{2}}(u^{n-1} - u_h^{n-1}), v_h\Big)\Big| \\
\le\ &\frac{\sigma(t^n)^{\frac{1}{2}} e^{\frac{-\delta_0 t^n}{2}}}{\Delta t}\int_{t^{n-1}}^{t^n}\|\frac{\partial((\sigma(t)e^{\delta_0 t})^{\frac{1}{2}}(u - u_h))}{\partial t}\|_0 ds \cdot \|v_h\|_0 \\
\le\ &\frac{\sigma(t^n)^{\frac{1}{2}} e^{\frac{-\delta_0 t^n}{2}}}{\Delta t}\int_{t^{n-1}}^{t^n}\Big(\sigma(t)^{\frac{1}{2}} e^{\frac{\delta_0 t}{2}}\|u_t - u_{ht}\|_0 + e^{\frac{\delta_0 t}{2}}\|u - u_h\|_0\Big) ds \cdot \|v_h\|_0 \\
&+\frac{\delta_0 \sigma(t^n)^{\frac{1}{2}} e^{\frac{-\delta_0 t^n}{2}}}{2\Delta t}\int_{t^{n-1}}^{t^n}\sigma(t)^{\frac{1}{2}} e^{\frac{\delta_0 t}{2}}\|u - u_h\|_0 ds \cdot \|v_h\|_0 \\
\le\ &\frac{\sigma(t^n)^{\frac{1}{2}} e^{\frac{-\delta_0 t^n}{2}}}{\Delta t^{\frac{1}{2}}}\Big[\Big(\int_{t^{n-1}}^{t^n}\sigma(t)e^{\delta_0 t}\|u_t - u_{ht}\|_0^2 ds\Big)^{\frac{1}{2}} + \Big(\int_{t^{n-1}}^{t^n} e^{\delta_0 t}\|u - u_h\|_0^2 ds\Big)^{\frac{1}{2}}\Big] \cdot \|v_h\|_0 \\
&+\frac{\delta_0 \sigma(t^n)^{\frac{1}{2}} e^{-\frac{1}{2}\delta_0 t^n}}{2\Delta t^{\frac{1}{2}}}\Big(\int_{t^{n-1}}^{t^n}\sigma(t)e^{\delta_0 t}\|u - u_h\|_0^2 ds\Big)^{\frac{1}{2}} \cdot \|v_h\|_0.
\end{aligned}$$

For T_2 and T_3, we get

$$\begin{aligned}
T_2 = &\Big|\Big((u^{n-1} - u_h^{n-1}) - (\overline{u}^{n-1} - \overline{u}_h^{n-1}), v_h\Big)\Big| \\
\le\ &C\|(u^{n-1} - u_h^{n-1}) - (\overline{u}^{n-1} - \overline{u}_h^{n-1})\|_{H^{-1}(\Omega)}\|v_h\|_1 \\
\le\ &C\Delta t\|u^{n-1} - u_h^{n-1}\|_0\|v_h\|_1, \\
T_3 = &\Big|\Big(\frac{\sigma(t^{n-1})[(u^{n-1} - u_h^{n-1}) - (\overline{u}^{n-1} - \overline{u}_h^{n-1})]}{\Delta t}, v_h\Big)\Big| \\
\le\ &C\sigma(t^{n-1})\|\frac{(u^{n-1} - u_h^{n-1}) - (\overline{u}^{n-1} - \overline{u}_h^{n-1})}{\Delta t}\|_{H^{-1}(\Omega)}\|v_h\|_1 \\
\le\ &C\|u^{n-1} - u_h^{n-1}\|_0\|v_h\|_1.
\end{aligned}$$

Combining above estimates with (4.5)-(4.9), multiplying by $\Delta t\sigma^2(t)$, summing n from 1 to N, using Lemmas 2.1, 3.1, 3.2 and 4.2, we get

$$\begin{aligned}
&\sigma^4(t)\Big(\sum_{n=1}^{N}\|\nabla e_h^n\|_0^2\Delta t+\sum_{n=1}^{N}\|\eta_h^i\|_0^2\Delta t\Big)\\
\leq\ & C\Big(e^{-\delta_0 t}\int_0^T(e^{\delta_0 t}\sigma(s)\|u_t-u_{ht}\|_0^2+e^{\delta_0 t}\|u-u_h\|_0^2)ds\Big)\\
&+Ch^2\Big(\sum_{n=1}^{N}\|f^n\|_0^2\Delta t+\sum_{n=1}^{N}\|\nabla u_h^{n-1}\|_0^2\Delta t+e^{-\delta_0 t}\int_0^T e^{\delta_0 t}\|u_{ht}\|_0^2 ds\Big)\\
&+\Delta t^2\int_0^T\|\frac{\partial^2 u}{\partial\tau^2}\|_0^2 ds+C\sigma^2(t)\sum_{n=1}^{N}\|u-u_h^n\|_0^2\Delta t\\
\leq\ & C\Big(\Delta t^2+h^2\Big).
\end{aligned}$$

Combining with Lemma 3.1, we complete the proof. □

References

1. A.Allievi, R.Bermejo, Finite element modified method of characteristics for the Navier-Stokes equations, Int. J. Numer. Meth. Fluids, 32 (2000) 439-464.
2. R.Bank, D.Rose, Some error estimates for the Box method, SIAM J. Numer. Anal., 24 (1987) 777-787.
3. P.Bochev, C.Dohrmann, M.Gunzburger, Stabilization of low-order mixed finite elements for the Stokes equations, SIAM J. Numer. Anal. 44 (2006) 82-101.
4. K.Boukir, Y.Maday, B.MÉtivet, E.Razafindrakoto, A high-order characteristic/finite element method for the incompressible Navier-Stokes equations, Int. J. Numer. Meth. Fluids, 25 (1997) 1421-1454.
5. Z.Cai, S.McCormick, On the accuracy of the finite volume element method for diffusion equations on composite grids, SIAM J. Numer. Anal., 27 (1990) 635-655.
6. Z.Cai, J.Mandel, S.McCormick, The finite volume element method for diffusion equations on general triangulations, SIAM J. Numer. Anal., 28 (1991) 392-402.
7. Z.Chen, Finite element methods and their applications, Spring-Verlag, Heidelberg, 2005.
8. Z.Chen, R.Li, A.Zhou, A note on the optimal L^2-estimate of the finite volume element method, Adv. Comput. Math., 16 (2002) 291-303.
9. P.Ciarlet, The finite element method for elliptic problems, North-Holland Publishing Company, 1978.
10. J.Douglas, T.Russell, Numerical methods for convection-dominated diffusion problems based on combining the method of characteristics with finite element or finite difference procedures, SIAM J. Numer. Anal., 19 (1982) 871-885.
11. R.Ewing, T.Lin, Y.Lin, On the accuracy of the finite volume element method based on piecewise linear polynomials, SIAM J.Numer. Anal., 39 (2002) 1865-1888.
12. F.Gao, Y.Yuan, The characteristic finite volume element method for the nonlinear convection-dominated diffusion problem, Comput. Math. Appl., 56 (2008) 71-81.
13. V.Girault, P.A.Raviart, Finite element method for Navier-Stokes equations: theory and algorithms, Springer-Verlag, Berlin, Herdelberg, 1987.
14. G.He, Y.He, The finite volume method based on stabilized finite element for the stationary Navier-Stokes problem, J. Comput. Appl. Math., 205 (2007) 651-665.

15. G.He, Y.He, Z.Chen, A penalty finite volume method for the transient Navier-Stokes problem, Appl. Numer. Math., 58 (2008) 1583-1613.
16. G.He, Y.He, X.Feng, Finite volume method based on stabilized finite elements for the nonstationary Navier-Stokes problem, Numer. Methods Partial Differential Eq., 23 (2007) 1167-1191.
17. J.Heywood, R.Rannacher, Finite element approximation of the nonstationary Navier-Stokes problem I: regularity of solutions and second-order error estimates for spatial discretization, SIAM J. Numer. Anal. 19 (1982) 275-311.
18. J.Li, Z.Chen, A new stabilized finite volume method for the stationary Stokes equations, Adv. Comput. Math., 30 (2009) 141-152.
19. R.Li, Z.Chen, W.Wu, Generalized difference methods for differential equations, Marcel Dekker, New York, 2000.
20. J.Li, Y.He, A stabilized finite element method based on two local Gauss integrations for the Stokes equations, J. Comput. Appl. Math. 214 (2008) 58-65.
21. H.Wu, R.Li, Error estimates for finite volume element methods for general second-order elliptic problems, Numer. Methods Partial Differential Eq., 19 (2003) 693-708.
22. H.Notsu, M.Tabata, A single-step characteristic-curve finite element scheme of second order in time for the incompressible Navier-Stokes equations, J. Sci. Comput., 38 (2009) 1-14.
23. O.Pironneau, On the transport-diffusion algorithm and its applications to the Navier-Stokes equations, Numer. Math., 38 (1982) 309-332.
24. E.Süli, Convergence and nonlinear stability of the Lagrange-Galerkin method for the Navier-Stokes equations, Numer. Math., 53 (1988) 459-483.
25. R. Temam, Navier-Stokes equation: Theory and numerical analysis (Third edition), North-Holland, Amsterdam, New York, Oxford, 1984.
26. C.Wang, Characteristic finite analytic method (CFAM) for incompressible Navier-Stokes equations, ACTA Mech., 143 (2000) 57-66.
27. X.Ye, On the relationship between finite volume and finite element methods qpplied to the Stokes equations, Numer. Methods Partial Differential Eq., 5 (2001) 440-453.
28. T.Zhang, Z.Y.Si, Y.N.He, A stabilized characteristic finite element method for the transient Navier-Stokes equations, Int. J. Comput. Fluid Dyn., on line.

Tong Zhang received M.Sc. from Yunnan University and Ph.D at Xi'an Jiaotong University. Since 2011 he has been at Henan Polytechnic University. His research interests include numerical analysis and computational method.

School of Mathematics & Information Science, Henan Polytechnic University, Jiaozuo, 454003, P.R. China.
`e-mail: zhangtong0616@163.com`