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SOME IDENTITIES OF THE GENOCCHI NUMBERS AND POLYNOMIALS ASSOCIATED WITH BERNSTEIN POLYNOMIALS

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Abstract. Recently, several mathematicians have studied some interesting relations between extended q-Euler number and Bernstein polynomials(see [3, 5, 7, 8, 10]). In this paper, we give some interesting identities on the Genocchi polynomials and Bernstein polynomials.

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1. Introduction

Throughout this paper, let p be a fixed odd prime number. The symbol, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. As well known definition, the p-adic absolute value is given by $|x|_p = p^{-r}$ where $x = p^r \frac{t}{s}$ with $(t,p) = (s,p) = (t,s) = 1$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. In this paper we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$.

We assume that $UD(\mathbb{Z}_p)$ is the space of the uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the fermionic p-adic invariant integral on $\mathbb{Z}_p$ is defined as follows:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \text{ see [1, 2, 3, 4]} . \qquad (1.1)$$

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For $n \in \mathbb{N}$, let $f_n(x) = f(x+n)$ be translation. As well known equation, by (1.1), we have

$$\int_{\mathbb{Z}_p} f(x+n)d\mu_{-1}(x) = (-1)^n \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) + 2\sum_{l=0}^{n-1}(-1)^{n-1-l}f(l). \quad (1.2)$$

The Genocchi numbers are defined by the generating function as follows:

$$\frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \text{ see [3, 4, 11, 12]}. \quad (1.3)$$

The Genocchi numbers are defined by the generating function as follows:

$$\frac{2t}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \text{ see [3, 4, 11, 12]}. \quad (1.4)$$

From (1.4), we note that

$$G_n(x) = \sum_{l=0}^{n} \binom{n}{l} G_l x^{n-l}. \quad (1.5)$$

From (1.2) and (1.4), for $n = 1$, we have

$$t\int_{\mathbb{Z}_p} e^{(x+y)t}d\mu_{-1}(y) = \frac{2t}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}. \quad (1.6)$$

By (1.6), we obtain

$$G_0(x) = 0, \quad \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = \frac{G_{n+1}(x)}{n+1}, \text{ for } n \in \mathbb{N}. \quad (1.7)$$

As well known definition, Bernstein polynomials of degree n are given by

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \text{ where } x \in [0,1], n, k \in \mathbb{Z}_+. \quad (1.8)$$

In [1], Kim introduced p-adic extension of Bernstein polynomials as follows:

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \text{ where } x \in \mathbb{Z}_p \text{ and } n, k \in \mathbb{Z}_+. \quad (1.9)$$

In this paper, we investigate some properties for Genocchi numbers and polynomials. By using these properties, we give some interesting identities on Genocchi polynomials and Bernstein polynomials.

2. Some identities on the Bernstein and Genocchi polynomials

From (1.6), we can derive the following recurrence formula for the Genocchi numbers:

$$G_0 = 0, \text{ and } (G+1)^n + G_n = \begin{cases} 2, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases} \quad (2.1)$$

with usual convention about replacing G^n by G_n.

By (1.4), we easily get

$$\sum_{n=0}^{\infty} G_n(1-x)(-1)^n \frac{t^n}{n!} = (-1)\frac{2t}{e^t+1}e^{xt} = (-1)\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}. \tag{2.2}$$

By (2.2), we obtain the following theorem.

Theorem 2.1. *Let $n \in \mathbb{Z}_+$. Then we have*

$$G_n(x) = (-1)^{n-1} G_n(1-x).$$

From (1.7), we note that

$$G_0 = 0, \quad \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = \frac{G_{n+1}}{n+1}, \text{ for } n \in \mathbb{N}. \tag{2.3}$$

By (2.1), for $n \in \mathbb{N}$ with $n > 1$, we have

$$\begin{aligned} G_n(2) = (G+1+1)^n &= \sum_{l=0}^{n} \binom{n}{l} G_l(1) \\ &= \sum_{l=1}^{n} \binom{n}{l} G_l(1) \\ &= (nG_1(1)) + \sum_{l=2}^{n} \binom{n}{l} G_l(1) \\ &= 2n - G_n(1) \\ &= 2n + G_n. \end{aligned} \tag{2.4}$$

Therefore, by (2.4), we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{N}$ with $n > 1$, we have*

$$G_n(2) = 2n + G_n.$$

By (2.3) and Theorem 2.2, we obtain the following corollary.

Corollary 2.3. *For $n \in \mathbb{N}$ with $n > 1$, we have*

$$\int_{\mathbb{Z}_p} (x+2)^n d\mu_{-1}(x) = 2 + \frac{G_{n+1}}{n+1}.$$

By (1.7), (2.3) and Corollary 2.3, we know that

$$\begin{aligned}\int_{\mathbb{Z}_p}(1-x)^n d\mu_{-1}(x) &= (-1)^n\int_{\mathbb{Z}_p}(x-1)^n d\mu_{-1}(x)\\ &= (-1)^n\frac{G_{n+1}(-1)}{n+1}\\ &= \frac{G_{n+1}(2)}{n+1}\\ &= \int_{\mathbb{Z}_p}(x+2)^n d\mu_{-1}(x)\\ &= 2+\frac{G_{n+1}}{n+1}\\ &= 2+\int_{\mathbb{Z}_p}x^n d\mu_{-1}(x).\end{aligned}$$

Therefore, we have the following theorem.

Theorem 2.4. *For $n \in \mathbb{N}$ with $n > 1$, we have*

$$\int_{\mathbb{Z}_p}(1-x)^n d\mu_{-1}(x) = 2+\int_{\mathbb{Z}_p}x^n d\mu_{-1}(x).$$

In (1.9), we take the twisted fermionic p-adic invariant integral on $\mathbb{Z}_p$ for one Bernstein polynomials as follows:

$$\begin{aligned}\int_{\mathbb{Z}_p}B_{k,n}(x)d\mu_{-1}(x) &= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^{n-k-l}\int_{\mathbb{Z}_p}x^{n-l}d\mu_{-1}(x)\\ &= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^{n-k-l}\frac{G_{n-l+1}}{n-l+1}\\ &= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^{l}\frac{G_{k+l+1}}{k+l+1},\end{aligned} \quad (2.5)$$

where $n, k \in \mathbb{Z}_+$.

From the reflection symmetric properties of Bernstein polynomials, we note that

$$B_{k,n}(x) = B_{n-k,n}(1-x), \text{ where } n,k \in \mathbb{Z}_+ \text{ and } x \in \mathbb{Z}_p. \quad (2.6)$$

For $n, k \in \mathbb{Z}_+$ with $n > k+1$, we have

$$\begin{aligned}\int_{\mathbb{Z}_p}B_{k,n}(x)d\mu_{-1}(x) &= \int_{\mathbb{Z}_p}B_{n-k,n}(1-x)d\mu_{-1}(x)\\ &= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\int_{\mathbb{Z}_p}(1-x)^{n-l}d\mu_{-1}(x)\\ &= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\left(2+\int_{\mathbb{Z}_p}x^{n-l}d\mu_{-1}(x)\right).\end{aligned}$$

Therefore, we have the following theorem.

Theorem 2.5. *For $n, k \in \mathbb{Z}_+$ with $n > k+1$, we have*

$$\int_{\mathbb{Z}_p} B_{k,n}(x)d\mu_{-1}(x) = \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\left(2+\frac{G_{n-l+1}}{n-l+1}\right).$$

By (2.5) and Theorem 2.5, we have the following theorem.

Theorem 2.6. *Let $n, k \in \mathbb{Z}_+$ with $n > k+1$. Then we have*

$$\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\frac{G_{k+l+1}}{k+l+1} = \sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\left(2+\frac{G_{n-l+1}}{n-l+1}\right).$$

Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1+n_2 > 2k+1$. Then we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n_1}(x)B_{k,n_2}(x)d\mu_{-1}(x)\\
&= \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\int_{\mathbb{Z}_p}(1-x)^{n_1+n_2-l}d\mu_{-1}(x)\\
&= \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\int_{\mathbb{Z}_p}(x+2)^{n_1+n_2-l}d\mu_{-1}(x)\\
&= \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\left(2+\int_{\mathbb{Z}_p}x^{n_1+n_2-l}d\mu_{-1}(x)\right).
\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 2.7. *For $n_1, n_2\, k \in \mathbb{Z}_+$ with $n_1+n_2 > 2k+1$, we have*

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n_1}(x)B_{k,n_2}(x)d\mu_{-1}(x)\\
&= \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\left(2+\frac{G_{n_1+n_2-l+1}}{n_1+n_2-l+1}\right).
\end{aligned}$$

By simple calculation, we easily see that

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n_1}(x)B_{k,n_2}(x)d\mu_{-1}(x)\\
&= \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{n_1+n_2-2k}(-1)^l\binom{n_1+n_2-2k}{l}\int_{\mathbb{Z}_p}x^{l+2k}d\mu_{-1}(x) \qquad (2.7)\\
&= \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{n_1+n_2-2k}(-1)^l\binom{n_1+n_2-2k}{l}\frac{G_{l+2k+1}}{l+2k+1},
\end{aligned}$$

where $n_1, n_2, k \in \mathbb{Z}_+$. Therefore, by (2.7) and Theorem 2.7, we obtain the following theorem.

Theorem 2.8. *Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k + 1$. Then we have*

$$\sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \left(2 + \frac{G_{n_1+n_2-l+1}}{n_1 + n_2 - l + 1}\right)$$
$$= \sum_{l=0}^{n_1+n_2-2k} (-1)^l \binom{n_1 + n_2 - 2k}{l} \frac{G_{l+2k+1}}{l + 2k + 1}.$$

For $n_1, n_2, n_3, k \in \mathbb{Z}_+$ with $n_1+n_2+n_3 > 3k+1$, by the symmetry of Bernstein polynomials, we see that

$$\int_{\mathbb{Z}_p} B_{k,n_1}(x) B_{k,n_2}(x) B_{k,n_3}(x) d\mu_{-1}(x)$$
$$= \binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k} \sum_{l=0}^{3k} \binom{3k}{l} (-1)^{l+3k} \int_{\mathbb{Z}_p} (1 - x)^{n_1+n_2+n_3-l} d\mu_{-1}(x)$$
$$= \binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k} \sum_{l=0}^{3k} \binom{3k}{l} (-1)^{l+3k} \int_{\mathbb{Z}_p} (x + 2)^{n_1+n_2+n_3-l} d\mu_{-1}(x)$$
$$= \binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k} \sum_{l=0}^{3k} \binom{3k}{l} (-1)^{l+3k} \left(2 + \int_{\mathbb{Z}_p} x^{n_1+n_2+n_3-l} d\mu_{-1}(x)\right).$$

Therefore, we have the following theorem.

Theorem 2.9. *For $n_1, n_2, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 + n_3 > 3k + 1$, we have*

$$\int_{\mathbb{Z}_p} B_{k,n_1}(x) B_{k,n_2}(x) B_{k,n_3}(x) d\mu_{-1}(x)$$
$$= \binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k} \sum_{l=0}^{3k} \binom{3k}{l} (-1)^{l+3k} \left(2 + \frac{G_{n_1+n_2+n_3-l+1}}{n_1 + n_2 + n_3 - l + 1}\right).$$

In the same manner, multiplication of three Bernstein polynomials can be given by the following relation:

$$\int_{\mathbb{Z}_p} B_{k,n_1}(x) B_{k,n_2}(x) B_{k,n_3}(x) d\mu_{-1}(x)$$
$$= \binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k} \sum_{l=0}^{n_1+n_2+n_3-3k} (-1)^l \binom{n_1 + n_2 + n_3 - 3k}{l} \int_{\mathbb{Z}_p} x^{l+3k} d\mu_{-1}(x)$$
$$= \binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k} \sum_{l=0}^{n_1+n_2+n_3-3k} (-1)^l \binom{n_1 + n_2 + n_3 - 3k}{l} \frac{G_{l+3k+1}}{l + 3k + 1},$$

where $n_1, n_2, n_3, k \in \mathbb{Z}_+$ with $n_1 + n_2 + n_3 > 3k + 1$. Therefore, by Theorem 2.9 we obtain the following theorem.

Theorem 2.10. *Let $n_1, n_2, n_3, k \in \mathbb{Z}_+$ with $n_1 + n_2 + n_3 > 3k + 1$. Then we have*

$$\sum_{l=0}^{3k} \binom{3k}{l} (-1)^{l+3k} \left(2 + \frac{G_{n_1+n_2+n_3-l+1}}{n_1+n_2+n_3-l+1}\right)$$
$$= \sum_{l=0}^{n_1+n_2+n_3-3k} (-1)^l \binom{n_1+n_2+n_3-3k}{l} \frac{G_{l+3k+1}}{l+3k+1}.$$

Using the above theorem and mathematical induction, we have the following theorem.

Theorem 2.11. *Let $m \in \mathbb{N}$. For $n_1, n_2, \ldots, n_m, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_m > mk + 1$, the multiplication of the sequence of Bernstein polynomials $B_{k,n_1}(x), \ldots, B_{k,n_m}(x)$ with different degrees under fermionic p-adic invariant integral on $\mathbb{Z}_p$ can be given as*

$$\int_{\mathbb{Z}_p} \left(\prod_{i=1}^{m} B_{k,n_i}(x)\right) d\mu_{-1}(x)$$
$$= \left(\prod_{i=1}^{m} \binom{n_i}{k}\right) \sum_{l=0}^{mk} \binom{mk}{l} (-1)^{l+mk} \left(2 + \frac{G_{n_1+\cdots+n_m-l+1}}{n_1+\cdots+n_m-l+1}\right).$$

We also easily see that

$$\int_{\mathbb{Z}_p} \left(\prod_{i=1}^{m} B_{k,n_i}(x)\right) d\mu_{-1}(x)$$
$$= \left(\prod_{i=1}^{m} \binom{n_i}{k}\right) \sum_{l=0}^{n_1+\cdots+n_m-mk} \binom{n_1+\cdots+n_m-mk}{l} (-1)^l \frac{G_{l+mk+1}}{l+mk+1}. \tag{2.8}$$

By Theorem 2.11 and (2.8), we have the following corollary.

Corollary 2.12. *Let $m \in \mathbb{N}$. For $n_1, n_2, \ldots, n_m, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_m > mk + 1$, we have*

$$\sum_{l=0}^{mk} \binom{mk}{l} (-1)^{l+mk} \left(2 + \frac{G_{n_1+\cdots+n_m-l+1}}{n_1+\cdots+n_m-l+1}\right)$$
$$= \sum_{l=0}^{n_1+\cdots+n_m-mk} \binom{n_1+\cdots+n_m-mk}{l} (-1)^l \frac{G_{l+mk+1}}{l+mk+1}.$$

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