J. Appl. Math. & Informatics Vol. **29***(2011), No. 5 - 6, pp. 1245 - 1256*
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A REGULARIZATION INTERIOR POINT METHOD FOR SEMIDEFINITE PROGRAMMING WITH FREE VARIABLES†

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ABSTRACT. In this paper, we proposed a regularization interior point method for semidefinite programming with free variables which can be taken as an extension of the algorithm for standard semidefinite programming. Since an inexact search direction at each iteration is used, the computation of the designed algorithm is much less compared with the existing solution methods. The convergence analysis of the method is established under weak conditions.

AMS Mathematics Subject Classification : 65H10, 90C22, 90C31.
Key words and phrases : semidefinite optimization, free variables, regularization, global convergence.

1. Introduction

Semidefinite program (SDP) is a kind of convex optimization problem which minimizes the inner product of two matrices over an intersection of an affine set and the cone of positive semidefinite matrices, which admits the following standard form

$$\begin{array}{ll} \min & C \bullet X \\ \text{s.t.} & \mathcal{A} \bullet X = b \\ & X \succeq 0 \end{array}$$

where $\mathcal{A}$ is a linear map from $R^{n\times n} \to R^m$, $X \succeq 0$ means that X is an $n \times n$ positive semi-definite matrix, $C \in R^{n\times n}, b \in R^m$, and $C \bullet X = \mathrm{tr}(C^\top X)$. Throughout this paper, we assume that $\mathcal{A}^\top$ can be written as $(\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_m)$ such that $\mathcal{A}_i \bullet X = b_i, \quad i = 1, 2, \cdots, m$ and matrices $\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_m$ are linearly independent.

Semi-definite programming frequently arises in control theory, structural optimization, graph theory and combinatorial optimization [1, 16]. In past decades,

Received November 20, 2010. Revised January 31, 2011. Accepted February 7, 2011. *Corresponding author. †This work was supported by Shandong Provincial Postgraduate Education Program (SDYY10067).

SDP was a popular research topic in mathematical programming. Obviously, SDP is a direct generalization of linear programming and the interior point method was successfully extended to SDP whose high efficiency further stimulates the researchers' interest for SDP. A rather comprehensive list of references dealing with feasible interior-point methods for SDP are referred to [15, 17], and that deals with the infeasible interior-point methods for SDP are referred to [6, 7, 12, 13].

Notwithstanding the substantial progress made in recent years, work continues on the numerical methods for the SDP, and one outstanding issue is the case with free variables which usually appears in such as quantum chemistry [18], polynomial optimization [10, 11], and among others. In this paper, we consider the semidefinite programming problem with free variables assuming the following form

$$\begin{aligned} \min \quad & C \bullet X + g^\top z \\ \text{s.t.} \quad & \mathcal{A} \bullet X + Gz = b \\ & X \succeq 0 \end{aligned} \tag{1}$$

where $G \in \mathcal{R}^{m\times p}$, $g \in \mathcal{R}^p$, and z is a free variable.

For this kind of SDP, we can not directly apply the existing solution methods to solve the problem, and one natural way is to split the free variable vector $z \in \mathcal{R}^p$ into the difference of two nonnegative variable vectors to convert problem (1) into a standard SDP as was done in [5]. However, this conversion not only makes the scale of the SDP doubled but also yields some difficulties in computing as the converted SDP may have a continuum of optimal solutions and its dual may have no interior feasible solution. An alternative approach to the problem is to transform it into a standard SDP via eliminating the free variables [5]. Although the scale of the problem is reduced by doing so, the operation is not easy to implement and it may make a SDP problem from sparse to denser and hence may affect the efficiency of the solution method.

In this paper, motivated by the successful application of the regularization technique of Mészáros to linear conic optimization with free variables [9], we design a regularization interior point method for SDP with free variables. Since there are no relax variables are involved in the new reformulation and no tedious preprocess needed in the new algorithm, the designed method is more practical compared with the existing solution methods for SDP with free variables. The convergence of the method is also established under weak conditions.

To end this section, we give some of notations used in the paper. Let $\mathcal{S}^n_+(\mathcal{S}^n_{++})$ denote the set of the $n\times n$ positive semidefinite (definite) symmetric matrices. The abbreviation s.p.d. is the shorthand for symmetric positive definite. For $A \in \mathcal{R}^{m\times n}, B \in \mathcal{R}^{k\times l}$, its Kronecker product, denoted by $A \otimes B$, is defined as

$$\begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}.$$

For $A \in \mathcal{R}^{n\times n}$, $\lambda_i(A)$ denotes eigenvalues of A which are arranged in nonincreasing order w.r.t. $i = 1, 2, \cdots, n$, and $\rho(A)$ denotes the spectrum radius of matrix A. For $A \in \mathcal{R}^{m\times n}$ with rank r, its i-th singular value is denoted by $\sigma_i(A)$, and its Euclidean norm and the Frobenius norm are denoted by $\|A\|$ and $\|A\|_F$, respectively. Furthermore, we let $\mathrm{vec}(A)$ denote the mn-vector obtained by stacking the columns of A from the first to the last.

2. Direction Searching

For problem (1), we can readily establish its Lagrangian dual program [9]

$$\begin{array}{ll} \max & b^\top y \\ \text{s.t.} & \sum_{i=1}^m y_i \mathcal{A}_i + S = C \\ & G^\top y = g \\ & S \succeq 0 \end{array} \tag{2}$$

and its optimal condition

$$\begin{array}{rcll} \mathcal{A} \bullet X + Gz &=& b, & X \succeq 0 \\ \sum_{i=1}^m y_i \mathcal{A}_i + S &=& C, & S \succeq 0 \\ G^\top y &=& g, & \\ XS &=& 0. & \end{array} \tag{3}$$

which is equivalent to the original problem (1).

It is well known that Newton method is an efficient solution method for solving system of equations. However, the classical Newton method can not be directly applied to solve system (3) due to the nonnegative definiteness constraint of variables X, S. Similar to the interior algorithm for standard SDP [3, 4], we adapt a stepsize rule in the Newton iterative procedure to guarantee that all the iterates remain strictly feasible and meanwhile they are not too close to the boundary. Furthermore, to obtain a longstep in the iterative procedure, we make a bias of the Newton direction to the interior of the feasible region. To be precise, we replace the fourth equation in system (3) by a parameterized equation to obtain the following system of equation

$$\begin{cases} \mathcal{A} \bullet X + Gz = b, \quad X \succeq 0 \\ \sum_{i=1}^m y_i \mathcal{A}_i + S = C, \quad S \succeq 0 \\ G^\top y = g \\ XS = \sigma\mu I \end{cases} \tag{4}$$

For this system, it can be shown that the parameterized system (4) has a unique solution, denoted by $(X(\mu), z(\mu), y(\mu), S(\mu))$, for each $\mu > 0$ under the condition that matrices $\mathcal{A}_i$ are linearly independent for $i = 1, 2, \cdots, m$ and SDP (1) has a strictly feasible interior point [4]. In this sense, we call

$(X(\mu), z(\mu), y(\mu), S(\mu))$ a central path of (1) and (2). Note that for each $\mu > 0$, we derive the duality gap

$$\mu = X(\mu) \bullet S(\mu)/n.$$

Certainly, if $\mu \to 0$, then the limit of the central path yields a solution for (1) and (2).

For point (X, z, y, S) with $X,\, S \succ 0$, the Newton direction $(\Delta X, \Delta z, \Delta y, \Delta S)$ of (4) at the current point satisfies

$$\begin{cases} \mathcal{A} \bullet \Delta X + G\Delta z = -\Big(\mathcal{A} \bullet X + Gz - b\Big) \\ \sum\limits_{i=1}^{m} \Delta y_i \mathcal{A}_i + \Delta S = -\Big(\sum\limits_{i=1}^{m} y_i \mathcal{A}_i + S - C\Big) \\ G^\top \Delta y = -(G^\top y - g) \\ X\Delta S + \Delta X S = \sigma\mu I - XS \end{cases} \tag{5}$$

Obviously, ΔS is symmetric due to the second equation in (5). However, a crucial observation is that ΔX is not necessarily symmetric since the product of two symmetric matrices $X, \Delta S$ may not be symmetric. To make ΔX to be symmetric and hence make $X + \alpha\Delta X$ symmetric for any $\alpha > 0$, many researchers have proposed various techniques to symmetrize the fourth equation in the preceding Newton system [4, 6, 14, 15] and a popular one is that proposed by Zhang in [17]

$$\mathcal{H}_P(A) = \frac{1}{2}(PAP^{-1} + (PAP^{-1})^\top), \quad \forall A \in \mathcal{R}^{n\times n},$$

where $P \in \mathcal{R}^{n\times n}$ is a nonsingular matrix. In our algorithm designed below, we take the matrix $P = S^{\frac{1}{2}}$. Then Newton equation (5) becomes

$$\begin{cases} \mathcal{A} \bullet \Delta X + G\Delta z = -\Big(\mathcal{A} \bullet X + Gz - b\Big) \\ \sum\limits_{i=1}^{m} \Delta y_i \mathcal{A}_i + \Delta S = -\Big(\sum\limits_{i=1}^{m} y_i \mathcal{A}_i + S - C\Big) \\ G^\top \Delta y = -(G^\top y - g) \\ \mathcal{H}_{S^{\frac{1}{2}}}(X\Delta S + \Delta X S) = \sigma\mu I - \mathcal{H}_{S^{\frac{1}{2}}}(XS) \end{cases} \tag{6}$$

To solve this system, we transformed it into the vector version to obtain

$$\begin{cases} \mathcal{A}\mathrm{vec}(\Delta X) + G\Delta z = r_p \\ \mathcal{A}^\top \Delta y + \mathrm{vec}(\Delta S) = \mathrm{vec}(R_d) \\ G^\top \Delta y = r_d \\ E\mathrm{vec}(\Delta X) + F\mathrm{vec}(\Delta S) = \mathrm{vec}(R_c) \end{cases} \tag{7}$$

where

$$\begin{aligned} &\mathcal{A}^\top = (\mathrm{vec}\mathcal{A}_1, \mathrm{vec}\mathcal{A}_2, \cdots, \mathrm{vec}\mathcal{A}_m), \\ &r_p = -(\mathcal{A}\mathrm{vec}X + Gz - b), \\ &\mathrm{vec}(R_d) = -(\mathcal{A}^\top y + \mathrm{vec}(S) - \mathrm{vec}(C)), \\ &r_d = -(G^\top y - g), \\ &R_c = 2(\sigma\mu S - SXS), \\ &E = 2S \otimes S, \quad F = SX \otimes I + I \otimes SX. \end{aligned}$$

For simplicity, we drop the subscript $S^{\frac{1}{2}}$ from $\mathcal{H}_{S^{\frac{1}{2}}}$ from now on. For the current interior iterate (X_k, z_k, y_k, S_k) of (3), the new iterate $(X_{k+1}, z_{k+1}, y_{k+1}, S_{k+1}) \in \mathcal{S}^n_{++} \times \mathcal{R}^p \times \mathcal{R}^m \times \mathcal{S}^n_{++}$ is generated as follows

$$\begin{cases} X_{k+1} = X_k + \alpha_k \Delta X_k, \\ z_{k+1} = z_k + \alpha_k \Delta z_k, \\ y_{k+1} = y_k + \alpha_k \Delta y_k, \\ S_{k+1} = S_k + \alpha_k \Delta S_k, \end{cases} \tag{8}$$

where $\alpha_k \in (0, 1]$ is carefully chosen in order that the new iterate satisfies the centrality conditions

$$\begin{aligned} &\rho(\mathcal{H}(X_{k+1}S_{k+1})) \geq \gamma_1 \mu_{k+1}, \\ &X_{k+1} \bullet S_{k+1} \geq \max\{\gamma_2 \|r_p^{k+1}\|, \gamma_3 \|\mathrm{vec} R_d^{(k+1)}\|, \gamma_4 \|r_d^{(k+1)}\|\} \end{aligned} \tag{9}$$

and the decrease condition of the merit function $X \bullet S$:

$$X_{k+1} \bullet S_{k+1} \leq (1 - \alpha_k(1 - \beta)) X_k \bullet S_k, \tag{10}$$

where $\hat{\gamma} \in (0, 1)$, $\beta \in (0, 1)$, and constants $\gamma_1, \gamma_2, \gamma_3, \mathrm{and} \gamma_4$ are defined by

$$\begin{aligned} &\gamma_1 = \min\left\{\hat{\gamma}, \tfrac{\lambda_n(\mathcal{H}(X_0S_0))}{\mu_0}\right\}, \qquad &&\gamma_2 = \min\left\{\hat{\gamma}, \tfrac{X_0 \bullet S_0}{\|r_p^{(0)}\|}\right\}, \\ &\gamma_3 = \tfrac{X_0 \bullet S_0}{\|\mathrm{vec} R_d^{(0)}\|}, &&\gamma_4 = \min\left\{\hat{\gamma}, \tfrac{X_0 \bullet S_0}{\|r_d^{(0)}\|}\right\}. \end{aligned} \tag{11}$$

This choice of constants guarantees that the quantity $X_k \bullet S_k$ is driven to zero, and $\|\mathrm{vec}(R_d^{(k)})\|$, $\|r_p^{(k)}\|, \|r_d^{(k)}\|$ are all pushed to zero due to condition (9).

It should be noted that condition (9) and decrease condition (10) are the generalization of that for standard SDP [2], which are actually the generalization of that for standard linear programming [8].

To solve linear system (7), we reduce system (7) to the following system

$$\begin{aligned} \mathcal{A}E_k^{-1}F_k\mathcal{A}^\top \Delta y_k + G\Delta z_k &= r_p^{(k)} + \mathcal{A}E_k^{-1}F_k \mathrm{vec}(R_d^{(k)}) - \mathcal{A}E_k^{-1}\mathrm{vec}(R_c^{(k)}), \\ G^\top \Delta y_k &= r_d^{(k)} \end{aligned} \tag{12}$$

For simplicity, set $M_k = \mathcal{A}E_k^{-1}F_k\mathcal{A}^\top$. Now, if the coefficient matrix of the system above is nonsingular, then the linear system (12) can be solved via Cholesky factorization to obtain $\Delta y_k, \Delta z_k$, and then ΔX_k, and ΔS_k can be computed via

$$\begin{aligned} \mathrm{vec}(\Delta S_k) &= \mathrm{vec}(R_d^{(k)}) - \mathcal{A}^\top \Delta y \\ \mathrm{vec}(\Delta X_k) &= E_k^{-1}(\mathrm{vec}(R_c^{(k)}) - F_k \mathrm{vec}(\Delta S_k)). \end{aligned} \tag{13}$$

However, the coefficient matrix is not generally nonsingular due to the existence of the free variable z. To handle this case, we apply the Mészáros regularization technique to the system. That is, we introduce a specified $\delta_k > 0$ into the second equation of system (12) to write it as

$$\begin{aligned} \mathcal{A}E_k^{-1}F_k\mathcal{A}^\top \Delta y_k + G\Delta z_k &= r_p^{(k)} + \mathcal{A}E_k^{-1}F_k \mathrm{vec}(R_d^{(k)}) - \mathcal{A}E_k^{-1}\mathrm{vec}(R_c^{(k)}), \\ G^\top \Delta y_k - \delta_k \Delta z_k &= r_d^{(k)} \end{aligned} \tag{14}$$

To decrease the computation quantity, we only need to calculate an approximated solution to system (14). Here, we let the residual vectors $\bar{r}_p^{(k)}, \bar{r}_d^{(k)}$ with respect to the two equations satisfy

$$\begin{aligned} \|\bar{r}_p^{(k)}\| &\le \eta_{1k} X_k \bullet S_k, \\ \|\bar{r}_d^{(k)}\| &\le \eta_{2k} X_k \bullet S_k, \end{aligned} \tag{15}$$

where $\eta_{1k}, \eta_{2k} \in (0,1)$.

3. Algorithm and Convergence

Based on the analysis in the previous section, we give the description of our designed method.

Algorithm 3.1

Initial Step: Let $\sigma_0 \in (0,1), \varepsilon > 0, \hat{\gamma} \in (0,1), \beta \in (0,1), \eta_{10}, \eta_{20} \in (0,1), \delta_0 > 0, X_0, S_0 \in \mathcal{S}_{++}^n, z_0 \in \mathcal{R}^p, y_0 \in \mathcal{R}^m$ be given, compute $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ from (11) and let $k = 0$.

Iterative Step: If $X_k \bullet S_k \le \varepsilon$, stop; otherwise, set *Accept* $= 0$ and go to the following while loop.
while *Accept* $= 0$ do
solve system (14) approximately with condition (15) to obtain $(\Delta y_k, \Delta z_k)$. Then compute $(\Delta X_k, \Delta S_k)$ from (13).
if $\delta_k \|\Delta z_k\| \le \frac{1}{2}\frac{\sigma_k}{\gamma_4} X_k \bullet S_k$, then set Accept =1; otherwise set $\delta_k = \frac{1}{2}\frac{\sigma_k}{\gamma_4}\frac{X_k \bullet S_k}{2\|\Delta z_k\|}$
end if
end while
Take step $\alpha_k > 0$ such that $(X_{k+1}, z_{k+1}, y_{k+1}, S_{k+1})$ satisfy conditions (9)and (10).
Set $\delta_{k+1} = \frac{1}{2}\frac{\sigma_k}{\gamma_4}\frac{X_{k+1} \bullet S_{k+1}}{\|\Delta z_k\|}$.
Let $k = k + 1$ and goto the next iteration.

Remark 3.1. The main difference of the designed algorithm with the interior algorithm for standard SDP designed in [2] lies in that we take an δ_k-update strategy into the iterative procedure to make $\delta_k \|\Delta z_k\| \le \frac{1}{2}\frac{\sigma_k}{\gamma_4} X_k \bullet S_k$ at each step which guarantees the generated sequence converges to the solution of our concerned problem. In the while-loop procedure, δ_k is monotonically decreased by at least factor 0.5 to make the while-loop terminate in finite steps.

To establish the convergence of the algorithm, we first explore some properties of method for SDP.

Lemma 3.1 ([2], Lemma 4.1)**.** *Let matrix P be nonsingular, and matrix A, $B \in \mathcal{S}^n$, then* $|A \bullet B| \le n\|\mathcal{H}_p(AB)\|_F$.

The following lemma tells us that if (X_k, z_k, y_k, S_k) satisfy (9)-(10), then there exists $\alpha_k \in (0,1]$ such that, for all $\alpha \in [0, \alpha_k]$, $(X_k(\alpha), z_k(\alpha), y_k(\alpha), S_k(\alpha))$

satisfy

$$\begin{cases} \rho(\mathcal{H}(X_k(\alpha)S_k(\alpha))) \geq \gamma_1\mu_k(\alpha) \\ X_k(\alpha)\bullet S_k(\alpha) \leq (1-\alpha(1-\beta))X_k\bullet S_k \\ X_k(\alpha)\bullet S_k(\alpha) \geq \max\{\gamma_2\|r^k_{(p)}(\alpha)\|, \gamma_3\|\mathrm{vec}(R_d^{(k)}(\alpha))\|\} \\ X_k(\alpha)\bullet S_k(\alpha) \geq \gamma_4\|r_d^{(k)}(\alpha)\| \end{cases} \tag{16}$$

where

$$\begin{aligned} &X_k(\alpha) = X_k + \alpha\Delta X, \\ &z_k(\alpha) = z_k + \alpha\Delta z, \\ &S_k(\alpha) = X_k + \alpha\Delta S, \\ &y_k(\alpha) = y_k + \alpha\Delta y, \\ &r_p^{(k)}(\alpha) = b - \mathcal{A}\mathrm{vec}(X_k(\alpha)) - Gz_k(\alpha), \\ &\mathrm{vec}(R_d^{(k)}) = \mathrm{vec}(C) - \mathrm{vec}(S_k(\alpha)) - \mathcal{A}^\top y_k(\alpha), \\ &r_d^{(k)}(\alpha) = g - G^\top y_k(\alpha). \end{aligned}$$

Lemma 3.2. *Assume iterate (X_k, z_k, y_k, S_k) satisfies condition (9)-(10) and $(\Delta X_k, \Delta z_k,\ \Delta y_k, \Delta S_k)$ is the searching directed generated by Algorithm 3.1 at each iteration. Then, under condition (15) and*

$$\sigma_k - \gamma_2\eta_{1k} > 0, \qquad \frac{\sigma_k}{2} - \gamma_4\eta_{2k} > 0, \qquad \beta - \sigma_k > 0, \tag{17}$$

there exists $\hat{\alpha_k} \in (0, 1]$ such that, for all $\alpha \in (0, \hat{\alpha_k}]$, (16) holds.

Proof. For simplicity, we omit the iteration index k in the proof. Since the proof of first three inequalities in (16) can follow from that of Proposition 4.1 in [2], therefore, we only show that there exists $\hat{\alpha_4}$ such that for all $\alpha \in (0, \hat{\alpha_4}]$ the last inequality of (16) holds.

Taking into account that $\|\bar{r}_d\| \leq \eta_2 X\bullet S$, $X\bullet S \geq \gamma_4\|r_d\|$ and the regularization condition $\delta\|\Delta z\| \leq \frac{1}{2}\frac{\sigma}{\gamma_4}X\bullet S$, we obtain

$$\begin{aligned} &X(\alpha)\bullet S(\alpha) - \gamma_4\|g - G^\top y(\alpha)\| \\ &= (1-\alpha+\alpha\sigma)X\bullet S + \alpha^2\Delta X\bullet\Delta S - \gamma_4\|g - G^\top y - \alpha G^\top\Delta y\| \\ &= (1-\alpha+\alpha\sigma)X\bullet S + \alpha^2\Delta X\bullet\Delta S - \gamma_4\|r_d - \alpha(\delta\Delta z + r_d + \bar{r}_d\| \\ &= (1-\alpha+\alpha\sigma)X\bullet S + \alpha^2\Delta X\bullet\Delta S - \gamma_4\|(1-\alpha)r_d - \alpha\bar{r}_d - \alpha\delta\Delta z\| \\ &\geq (1-\alpha+\alpha\sigma)X\bullet S + \alpha^2\Delta X\bullet\Delta S - (1-\alpha)\gamma_4\|r_d\| - \alpha\gamma_4\|\bar{r}_d\| - \gamma_4\alpha\delta\|\Delta z\| \\ &\geq \alpha\sigma X\bullet S + \alpha^2\Delta X\bullet\Delta S - \alpha\gamma_4\eta_2 X\bullet S - \frac{\alpha\sigma}{2}X\bullet S \\ &= \alpha(\sigma/2 - \gamma_4\eta_2)X\bullet S + \alpha^2\Delta X\bullet\Delta S \\ &\geq \alpha(\sigma/2 - \gamma_4\eta_2)X\bullet S - \alpha^2|\Delta X\bullet\Delta S|. \end{aligned}$$

Since $\frac{\sigma}{2} - \gamma_4\eta_2 > 0$ by the hypothesis, it follows that the fourth inequality of (16) holds for $\alpha \in (0, \hat{\alpha_4}]$ with $\hat{\alpha_4} = \frac{(\sigma/2-\gamma_4\eta_2)X\bullet S}{|\Delta X\Delta S|}$.

Consequently, the desired result follows if we take

$$\hat{\alpha}_k = \min\{1, \kappa X \bullet S\}, \tag{18}$$

where $\kappa = \min\left\{\frac{1-\gamma_1}{1+\gamma_1}, \frac{\sigma}{n\|\mathcal{H}(\Delta X \Delta S)\|_F}, \frac{\sigma-\gamma_2\eta_4}{|\Delta X \Delta S|}, \frac{\sigma}{|\Delta X \Delta S|}, \frac{\sigma/2-\gamma_4\eta_2}{|\Delta X \Delta S|}, \frac{\beta-\sigma}{|\Delta X \Delta S|}\right\}$. □

The following lemma is taken from Proposition 4.2 in [2], which ensures that the matrices $X_k(\alpha)$ and $S_k(\alpha)$ with $\alpha \in [0, \hat{\alpha}_k]$ are s.p.d.

Lemma 3.3. *Assume that the hypotheses of Lemma 3.1 are satisfied and let $\hat{\alpha}_k$ be defined as in (18). Then for all $\alpha \in [0, \hat{\alpha_k}]$ matrices $X_k(\alpha)$ and $S_k(\alpha)$ are all s.p.d., unless $\hat{\alpha_k} = 1$ and $X_k(1) \bullet S_k(1) = 0$ in which case $(X_k(1), z_k(1), y_k(1), S_k(1))$ is a solution to (3).*

Lemma 3.4 ([17], Lemma 4.2)**.** *If S_k and S_{k+1} are s.p.d., then*

$$\lambda_n(\mathcal{H}_{S_k^{1/2}}(X_{k+1}S_{k+1})) \le \rho(\mathcal{H}_{S_{k+1}^{1/2}}(X_{k+1}S_{k+1})).$$

Based on the previous conclusions, we deduce that the new iterate $(X_{k+1}, z_{k+1}, y_{k+1}, S_{k+1})$ satisfies conditions (9) and (10). Hence Algorithm 3.1 is well defined.

Before proceeding on the analysis of the behavior of the sequence $\{X_k \bullet S_k\}$, we give some observations whose proofs can be found in [2].

Lemma 3.5 ([17], Proposition 2.3)**.** *The matrix $\hat{S}_k = F_k E_k^\top$ is s.p.d. and can be written as*

$$\hat{S}_k = E_k^{1/2} \hat{F}_k E_k^{1/2},$$

where

$$\hat{F}_k = E_k^{-1/2} F_k E_k^{1/2} = S_k^{1/2} X_k S_k^{1/2} \otimes I + I \otimes S_k^{1/2} X_k S_k^{1/2},$$

which is s.p.d.

Remark 3.2. Matrices $X_k S_k$, $S_k X_k$, $X_k^{1/2} S_k X_k^{1/2}$ and $S_k^{1/2} X_k S_k^{1/2}$ are all similar. Moreover, since

$$\mathcal{H}(X_k S_k) = \frac{1}{2}(S_k^{1/2} X_k S_k S_k^{-1/2} + (S_k^{1/2} X_k S_k S_k^{-1/2})^\top) = S_k^{1/2} X_k S_k^{1/2},$$

matrices $\mathcal{H}(X_k S_k)$ and $X_k S_k$ are similar. If we denote the eigenvalues of these matrices as λ_i, $i = 1, \cdots, n$, then from (g) of Lemma 6.1 in [2], the eigenvalues of F_k and $\hat{F}_k$ are given by $\lambda_i^k + \lambda_j^k$, for $i, j = 1, \cdots, n$.

Set $D_k = \hat{S}_k^{-1/2} F_k = \hat{S}_k^{1/2} E_k^{-\top}$. Then $D_k^{-\top} = \hat{S}_k^{-1/2} E_k = \hat{S}_k^{1/2} F_k^{-\top}$ and

$$D_k^\top D_k = (\hat{S}_k^{-1/2} F_k)^\top \hat{S}_k^{1/2} E_k^{-\top} = F_k^\top (\hat{S}_k^{-1/2})^\top \hat{S}_k^{1/2} E_k^{-\top} = F_k^\top E_k^{-\top}.$$

The symmetry of $E_k^{-1} F_k$ yields

$$D_k^\top D_k = E_k^{-1} F_k.$$

Furthermore, from Lemma 3.3 in [17], one has

$$\|\mathcal{H}(\Delta X_k \Delta S_k)\|_F \le \frac{1}{2}\sqrt{\frac{\lambda_1^k}{\lambda_n^k}}(\|D_k^{-\top}\mathrm{vec}(\Delta X_k)\|^2 + \|D_k \mathrm{vec}(\Delta S_k)\|^2), \tag{19}$$

and

$$|\Delta X_k \bullet \Delta S_k| \le \|D_k^{-\top}\mathrm{vec}(\Delta X_k)\|\|D_k\mathrm{vec}(\Delta S_k)\| \le \frac{1}{2}(\|D_k^{-\top}\mathrm{vec}(\Delta X_k)\|^2 + \|D_k\mathrm{vec}(\Delta S_k)\|^2). \tag{20}$$

In order to prove the global convergence of the method, we further need the following assumption.

Assumption 3.1. The sequence $\{X_k \bullet S_k\}$ is bounded, i.e. there exists a constant c_1 such that $\max\{\|X_k\|, \|S_k\|\} \le c_1$ for every $k > 0$.

Next, we will show the boundedness of $\|E_k^{-1}F_k\|$, $\|M_k\|$ and $\|\mathcal{H}(\Delta X_k \Delta S_k)\|_F$, and the existence of $\bar{\alpha} \in (0,1)$ such that for every k, $\hat{\alpha_k} \ge \bar{\alpha}$.

In fact, from (a) and (f) of Lemma 6.1 in [2], one has

$$\|E_k\| = 2\|S_k \otimes S_k\| = 2\|S_k\|^2 \le 2c_1^2,$$

$$\|E_k^{-1}\| = \frac{1}{2}\|S_k^{-1} \otimes S_k^{-1}\| = \frac{1}{2}\|S_k^{-1}\|^2.$$

Thus, from Propositions 4.3, 4.4, 4.5 in [2], we obtain the following conclusion.

Proposition 3.1. *Let Assumption 3.1 hold and (X_k, z_k, y_k, S_k) be the generated sequence by Algorithm 3.1. Then the followings hold:*

(1) $\|E_k^{-1}F_k\| \le \frac{c_1^2 n}{\gamma_1 X_k \bullet S_k}$,

(2) $\|(E_k^{-1}F_k)^{-1}\| \le \frac{c_1^2 n}{\gamma_1 X_k \bullet S_k}$,

(3) $\|M_k^{-1}\| \le \frac{c_1^2 n}{\gamma_1 X_k \bullet S_k \sigma_m^2(A)}$,

(4) $\|S_k^{-1}\| \le \frac{c_1^2 n\sqrt{n}}{\gamma_1 X_k \bullet S_k}$.

Proposition 3.2. *Let Assumption 3.1 hold and $\{X_k, z_k, y_k, S_k\}$ be the sequence generated by Algorithm 3.1. If $X_k \bullet S_k \ge \tilde{\varepsilon}$ for some $\tilde{\varepsilon} > 0$ and all k, then there exists a constant $\omega > 0$ such that $\|\mathcal{H}(\Delta X_k \Delta S_k)\|_F \le \omega$ for all k.*

Proof. Here we omit the iteration index k in the following proof.

From the proof of Proposition 4.6 in [2], we have

$$\|D^{-\top}\mathrm{vec}(\Delta X)\| \le \|D\mathcal{A}^\top \Delta y\| + \|D\mathrm{vec}(R_d)\| + 2\sigma\mu\|\hat{S}^{-1/2}\mathrm{vec}(S)\| + \|D^{-\top}\mathrm{vec}(X)\| \tag{21}$$

$$\|D\mathrm{vec}(\Delta S)\| \le \|D\mathrm{vec}(R_d)\| + \|D\mathcal{A}^\top \Delta y\|, \tag{22}$$

$$2\sigma\mu\|\hat{S}^{-1/2}\mathrm{vec}(S)\| \le \frac{\sigma c_1 n^{3/4}}{\gamma_1}, \tag{23}$$

$$\|D\mathrm{vec}(R_d)\| \le \frac{c_1\sqrt{nX \bullet S}}{\sqrt{\gamma_1}X_0 \bullet S_0}\|\mathrm{vec}(R_d^{(0)})\|, \tag{24}$$

$$\|D^{-\top}\mathrm{vec}(X)\| \le \frac{c_1^2 n}{\sqrt{\gamma_1 X \bullet S}}. \tag{25}$$

As for term $\|D\mathcal{A}^\top \Delta y\|$, it holds that

$$\|D\mathcal{A}^\top \Delta y\|^2 = \Delta y^\top \mathcal{A}D^\top D\mathcal{A}^\top \Delta y = \Delta y^\top \mathcal{A}E^{-1}F\mathcal{A}^\top \Delta y \le \|\Delta y\|\|M\Delta y\|. \tag{26}$$

Since the term $r_p = b - \mathcal{A}\mathrm{vec}(X) - Gz$ and iterate $(X+\Delta X, z+\Delta z, y+\Delta y, S+\Delta S)$ is feasible, it holds that

$$\begin{aligned} M\Delta y &= r_p - \mathcal{A}E^{-1}\mathrm{vec}(R_c) + \mathcal{A}E^{-1}F\mathrm{vec}(R_d) + \bar{r}_p - G\Delta z \\ &= b - \mathcal{A}\mathrm{vec}(X) - Gz - \mathcal{A}E^{-1}\mathrm{vec}(2(\sigma\mu S - SXS)) \\ &\quad + \mathcal{A}E^{-1}F\mathrm{vec}(R_d) + \bar{r}_p - G\Delta z \\ &= b - \mathcal{A}\mathrm{vec}(X) - G(z+\Delta z) - \sigma\mu\mathcal{A}\mathrm{vec}(S^{-1}) \\ &\quad + \mathcal{A}\mathrm{vec}(X) + \mathcal{A}E^{-1}F\mathrm{vec}(R_d) + \bar{r}_p \\ &= b - G(z+\Delta z) - \sigma\mu\mathcal{A}\mathrm{vec}(S^{-1}) + \mathcal{A}E^{-1}F\mathrm{vec}(R_d) + \bar{r}_p \\ &= \mathcal{A}\mathrm{vec}(X+\Delta x) - \sigma\mu\mathcal{A}\mathrm{vec}(S^{-1}) + \mathcal{A}E^{-1}F\mathrm{vec}(R_d) + \bar{r}_p. \end{aligned}$$

Thus, combining (15) with inequalities

$$\mu_{k+1} \geq \prod_{i=1}^{k}(1-\alpha_i)\mu_k, \quad R_d^{(k+1)} = (1-\alpha_k)R_d^{(k-1)} = \prod_{i=1}^{k}(1-\alpha_i)R_d^{(0)},$$

and (2),(4) in Proposition 3.1 yields

$$\begin{aligned} \|M\Delta y\| &= \|\mathcal{A}\mathrm{vec}(X+\Delta x) - \sigma\mu\mathcal{A}\mathrm{vec}(S^{-1}) + \mathcal{A}E^{-1}F\mathrm{vec}(R_d) + \bar{r}_p\| \\ &\leq \sigma_1(\mathcal{A})\|\mathrm{vec}(X+\Delta x)\| + \sigma\mu\sigma_1(\mathcal{A})\|\mathrm{vec}(S^{-1})\| \\ &\quad + \sigma_1(\mathcal{A})\|E^{-1}F\|\|\mathrm{vec}(R_d)\| + \|\bar{r}_p\| \\ &\leq \sigma_1(\mathcal{A})\sqrt{n}c_1 + \sigma_1(\mathcal{A})\frac{c_1^2 n}{\gamma_1}\frac{\|\mathrm{vec}(R_d^{(0)})\|}{X_0 \bullet S_0} + \sigma\sigma_1(\mathcal{A})\frac{c_1\sqrt{n}}{\gamma_1} + \eta X \bullet S. \end{aligned} \tag{27}$$

Thus, by (3) of Proposition 3.1, we get

$$\|\Delta y\| \leq \frac{c_1^2 n\sigma_1(A)}{\gamma_1 X \bullet S\sigma_m(A)}\left(\sqrt{n}c_1 + \frac{c_1^2 n}{\gamma_1}\frac{\|\mathrm{vec}(R_d^{(0)})\|}{X_0 \bullet S_0} + \sigma\frac{c_1\sqrt{n}}{\gamma_1}\right) + \eta\frac{c_1^2 n}{\gamma_1\sigma_m^2(\mathcal{A})}. \tag{28}$$

Summarizing (19), inequality $\sqrt{\frac{\lambda_1}{\lambda_n}} \leq \sqrt{\frac{n}{\gamma_1}}$ and (21)-(28) yields that there exists constant $\omega > 0$ such that $\|\mathcal{H}(\Delta X\Delta S)\|_F \leq \omega$. □

Remark 3.3. From (18),(19), (20) and Proposition 3.2, we can show that there exists constant $\bar{\alpha} \in (0,1)$ independent of k, such that $\hat{\alpha_k} \geq \bar{\alpha}$.

With these conclusions at hand, we are now at the position to state our main result in this section.

Theorem 3.1. *Let Assumption 3.1 hold and constants $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ be defined by in (11). Assume that constant β and the parameters η_{1k}, η_{2k} and σ_k are such that $\beta - \sigma_k > \theta_1 > 0$ and $\sigma_k - \gamma_2\eta_{1k} > \theta_2 > 0$, $\frac{\sigma_k}{2} - \gamma_4\eta_{2k} > \theta_3 > 0$, $\forall\ k > 0$, and assume σ_k is bounded away from zero whenever $X_k \bullet S_k \to 0$. Then the sequence $\{X_k \bullet S_k\}$ generated by Algorithm 3.1 with $\varepsilon = 0$ converges to 0.*

Proof. Since sequence $\{X_k \bullet S_k\}$ is monotonically decreasing and bounded below from zero, therefore, it is convergent.

Now, we prove the conclusion by reductio ad absurdum. Presuppose that $X_k \bullet S_k \to \bar{\varepsilon} > 0$. From assumption, it follows that there exists $\bar{\sigma}$ such that $\sigma_k \geq \bar{\sigma}$ for all k. From Lemma 3.2, we have

$$X_{k+1} \bullet S_{k+1} \leq (1 - \hat{\alpha_k}(1-\beta))X_k \bullet S_k, \tag{29}$$

with $\hat{\alpha_k}$ being given by (18). Furthermore, by the assumption and Remark 3.2, it follows that $\hat{\alpha_k} \geq \bar{\alpha}$, where

$$\bar{\alpha} = \min\left\{1, \frac{\bar{\varepsilon}}{n\omega} \min\left\{\frac{1-\gamma_1}{1+\gamma_1}\bar{\sigma}, \theta_1, \theta_2, \theta_3\right\}\right\}.$$

Then $\hat{\alpha_k}$ is bounded away from zero, and this along with (29) implies that $X_k \bullet S_k \to 0$, as $k \to \infty$. Thus, we arrive at a contradiction, this completes the proof. □

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