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ADJOINT SYSTEM FOR A MAGNETO-CONVECTIVE FLOW IN AN ACTIVE MUSHY LAYER

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Abstract. Here we consider magneto-convection in a mushy layer which is formed during solidification of binary alloys. The mushy layer is treated as an active porous media with variable permeability. The equations governing the layer are conservation of mass, conservation of heat, conservation of solute, magnetic induction equation, momentum equation governed by the Darcy's law and Maxwell's equations for the magnetic field. To study the second order effects on the flow without solving the second order system, we need to obtain the adjoint system for the flow. This motivates the authors we derive the adjoint system analytically for the mushy layer case. Numerical results of the adjoint system are presented for passive and active mushy layers at the onset of the motion using a set of parameters experimentalists use.

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1. Introduction

Analysis of stability of flows has been an important active research area in many applied fields including applied mathematics, various sciences and many branches of engineering. Instability leads to a common phenomena known as turbulence in many applied problems. Stability related to hydrodynamics has been studied in the nineteenth century by various famous scientists including Helmholtz, Kelvin, Rayleigh and Reynolds.

Experimentalists ([1], [2], [3], [4], [5]) observed a horizontal dendritic layer during alloy solidification. This partially solidified horizontal region is known as the mushy layer. Convective flows in the mushy layer are known to produce undesirable effects on the solid in the final form ([6], [7], [8]). During the last two decades, various theoretical and experimental studies have been performed to

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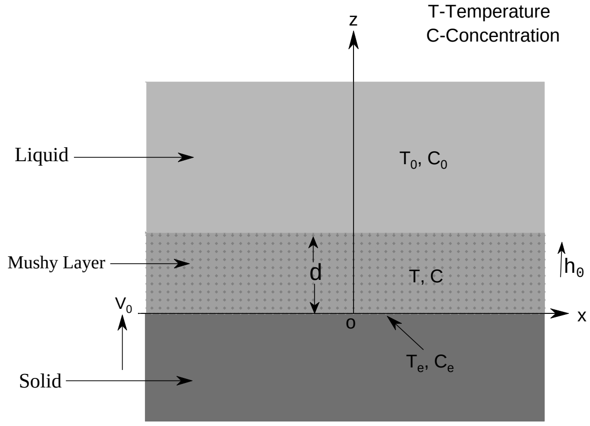
study the effect of convective flow in chimney formation. The fluid flow within the mushy layer can cause impurity in the final form of solidified alloy. During solidification, experimentalists observed vertical chimneys or channels void of solid that are typically oriented in the direction of gravity which can generate imperfections on the final product. It is accepted by various researcher in this field that convection in the chimneys causes a thin hair like structure called freckles. The phenomenon of natural convection arises in a fluid when the temperature change causes density variations resulting in buoyancy forces. This kind of heat transfer can be seen in solidification of binary alloys. Various studies ([5], [6], [7], [8]) have been carried out in details to analyze the mechanism of freckle formation during the solidification of alloys. A set of governing equations for a mushy layer, and performed stability analysis for linear case was proposed in ([7], [8]) by Worster. He observed two modes of compositional convection which are primarily responsible for the instability at the onset of motion and concluded that freckle formation was due to the mushy layer mode. By assuming small growth Peclet number and infinite Lewis number, a simplified single layer model was introduced by Amberg and Homsy ([9]). They performed weakly nonlinear analysis and calculated a critical value of the combined parameter (mush permeability and solid fraction variations) for the transition from supercritical to subcritical rolls. A weakly nonlinear analysis of simplified mushy layer model that was proposed in ([9]) was carried out by Anderson and Worster ([10]). A near eutectic approximation was applied and the limit of large far-field temperature was considered. Such asymptotic limits allowed them to examine the dynamics of mushy layer. They also considered the limit of large Stefan number, which enabled them to reach a domain for the existence of the oscillatory mode of convection. Okhuysen and Riahi ([11], [12]) analyzed a weakly nonlinear buoyant convection for a mushy layer with permeable mush-liquid interface. They generalized a number of assumptions made in the previous theoretical studies by other researchers. They concluded a subcritical down-hexagonal pattern for variable permeability case that corresponds to the smallest value of the Rayleigh number. Results using perturbation and marginal stability analysis of magneto-convection in an active mushy layer have been presented in ([13], [14]).

Nonlinear studies are gaining importance in many branches of applied sciences and engineering. In 1944, Lev Landau ([15]) proposed a nonlinear equation to analyze hydrodynamic stability. The Landau equation has been derived for various cases ([16], [17]). To derive Landau equation, the adjoint operator is required so that one can avoid solving the second order problem which is more complicated. Our aim here is to derive the adjoint operator for a horizontal mushy layer. Here we consider the convective flow in a horizontal mushy layer which has an impermeable mush-liquid interface. The mushy layer obeys Darcy's law. We solve the system by adding perturbations to the basic state solutions. Then we derive the linear system and we obtain the adjoint system analytically using the linear system by introducing an inner product.

2. Governing system for the mushy layer

Here we consider a horizontal mushy layer of thickness d which is cooled from below during solidification of binary alloys as shown in the figure 1.

FIGURE 1. Geometry for our system



We add magnetic component following Chandrasekhar [17] to the model proposed by Worster ([7], [8]). This system is expressed as

$$
\begin{aligned}
\frac{\partial T}{\partial t} + \overrightarrow{U}.\nabla T &= \kappa \nabla^2 T + \frac{L}{\gamma}\frac{\partial \Phi}{\partial t} \\
\chi \frac{\partial C}{\partial t} + \overrightarrow{U}.\nabla C &= (C - C_s)\frac{\partial \Phi}{\partial t} \\
\frac{\mu}{\Pi}\overrightarrow{U} &= -\nabla p - (\rho - \rho_0)\, g\, \overrightarrow{k} + \frac{\alpha}{4\pi}\left(\nabla \times \overrightarrow{H}\right) \times \overrightarrow{H} \qquad (1) \\
\frac{\partial \overrightarrow{H}}{\partial t} + \overrightarrow{U}.\nabla \overrightarrow{H} &= \eta \nabla^2 \overrightarrow{H} + \overrightarrow{H}.\nabla \overrightarrow{U} \\
\nabla.\overrightarrow{U} &= 0.
\end{aligned}
$$

The equations mentioned in (1) represent conservation of heat, and conservation of solute, Darcy's equation, Magnetic Induction equation, conservation of mass, divergence free magnetic field, respectively. Here $\overrightarrow{U} = U\vec{i} + V\vec{j} + W\vec{k}$ is the liquid flux where U, V are used to denote horizontal components, W denotes the vertical component of $\overrightarrow{U}$ and $\vec{i}$, $\vec{j}$, $\vec{k}$ are the unit vectors along x, y, z directions. Here $\mu, p, \rho, \rho_0, g, t, T, \kappa, \gamma, \eta, \alpha, L$ are used to denote the dynamic viscosity of the liquid, the dynamic pressure, the density of the liquid, a reference density, the acceleration due to gravity, time, temperature, thermal diffusivity of the liquid, specific heat of the liquid, magnetic diffusivity, magnetic permeability, latent heat per unit mass respectively. Also Φ stands for the local solid volume fraction, i.e., $\Phi = 1 - \chi$ with χ is the local liquid volume fraction, C is the composition of the liquid and C_s is the composition of the solid. Permeability $\Pi = \Pi(\chi)$ is a function of the local liquid volume fraction, χ.

The boundary conditions are

$$\begin{aligned} T = T_e, \quad W = 0, \quad \overrightarrow{H} = \overrightarrow{k} \quad \text{at} \quad z = 0 \\ T = T_0, \quad \Phi = W = 0, \quad \overrightarrow{H} = \overrightarrow{k} \quad \text{at} \quad z = d. \end{aligned}$$

Here T_e and C_e represent eutectic temperature and eutectic concentration (at the solid-mush interface, $z = 0$) respectively and T_0 denotes the temperature at the mush-liquid interface (at $z = d$).

2.1.Nondimensionalization. The solidification front is moving upward in the vertical direction at a constant speed V_0. We nondimensionalize the system in a frame moving with the speed V_0 and use the following scalings: velocity scale is V_0, i.e., $\overrightarrow{\mathcal{U}} = \frac{\overrightarrow{U}}{V_0}$, length scale is $\frac{\kappa}{V_0}$, time scale is $\frac{\kappa}{V_0^2}$, pressure scale is $\frac{\kappa\mu}{\Pi_0}$, $\Theta = \frac{T-T_0}{\Delta T}$, $\mathcal{K} = \frac{\Pi_0}{\Pi}$ where $\Delta T = T_0 - T_e$,$\Delta C = C_0 - C_e$. We have three nondimensional constants appearing in the derivation, and these are Rayleigh number, $\mathcal{R} = \frac{\beta g \Pi_0 \Delta C}{V_0 \mu}$, Stefan number, $\mathcal{S} = \frac{L}{\gamma \Delta T}$, concentration ratio, $\mathcal{C} = \frac{C_s - C_0}{\Delta C}$, Robert's number,$\tau = \frac{\kappa}{\eta}$, Chandrasekhar number, $Q = \frac{\alpha \tilde{h}^2 \Pi_0}{4\pi\rho_0\nu\eta}$ with uniform magetic strength, $\tilde{h}$.

Dimensionless system can be expressed as

$$\begin{aligned} &\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)[\Theta - \mathcal{S}\Phi] + \left(\overrightarrow{\mathcal{U}}.\nabla\right)\Theta = \nabla^2\Theta \\ &\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)[(1-\Phi)\,\Theta + \mathcal{C}\Phi] + \left(\overrightarrow{\mathcal{U}}.\nabla\right)\Theta = 0 \\ &\mathcal{K}\overrightarrow{\mathcal{U}} + \nabla\mathcal{P} + \mathcal{R}\Theta\overrightarrow{k} = \frac{Q}{\tau}\left(\frac{\partial}{\partial z} + \overrightarrow{\mathcal{H}}.\nabla\right)\overrightarrow{\mathcal{H}} \\ &\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)\overrightarrow{\mathcal{H}} + \left(\overrightarrow{\mathcal{U}}.\nabla\right)\overrightarrow{\mathcal{H}} = \left(\frac{\partial}{\partial z} + \overrightarrow{\mathcal{H}}.\nabla\right)\overrightarrow{\mathcal{U}} + \frac{1}{\tau}\nabla^2\overrightarrow{\mathcal{H}} \\ &\nabla.\overrightarrow{\mathcal{U}} = 0 \end{aligned} \tag{2}$$

and the boundary conditions are

$$\begin{aligned} \Theta = -1,\ \mathcal{W} = 0,\ \overrightarrow{\mathcal{H}} = \hat{k} \quad \text{at} \quad z = 0 \\ \Theta = \Phi = \mathcal{W} = 0,\ \overrightarrow{\mathcal{H}} = \hat{k} \quad \text{at} \quad z = \delta \end{aligned}$$

Here $\mathcal{W}$ is the vertical component of $\overrightarrow{\mathcal{U}}$.

3. Basic State and Perturbed Systems

Using perturbation, we assume the solutions of the form

$$\begin{aligned}
\Theta(x,y,z,t) &= \theta_b(z) + \epsilon\theta\,(x,y,z,t) \\
\Phi(x,y,z,t) &= \phi_b(z) + \epsilon\phi\,(x,y,z,t) \\
\overrightarrow{\mathcal{U}}\,(x,y,z,t) &= \overrightarrow{0} + \epsilon\,\overrightarrow{u}\,(x,y,z,t) \\
\mathcal{P}(x,y,z,t) &= p_b(z) + \epsilon p\,(x,y,z,t) \\
\mathcal{K}(x,y,z,t) &= k_b(z) + \epsilon K\,(x,y,z,t) \\
\overrightarrow{\mathcal{H}}(x,y,z,t) &= \hat{k} + \epsilon\,\overrightarrow{h}\,(x,y,z,t)
\end{aligned} \tag{3}$$

where θ_b, ϕ_b, p_b, k_b, $\hat{k}$ are solutions to the steady basic state system (system with no flow) and $\overrightarrow{u}$, θ, ϕ, p, K, $\overrightarrow{\mathcal{H}}$ are perturbed solutions and ϵ is the perturbation parameter.

3.1.Basic State System. Using (3) in the system (2), the basic state system (i.e., system with no flow) can be expressed as

$$\frac{d^2\theta_b}{dz^2} + \frac{d\theta_b}{dz} - \mathcal{S}\frac{d\phi_b}{dz} = 0 \tag{4}$$

$$(1-\phi_b)\,\frac{d\theta_b}{dz} + (\mathcal{C}-\theta_b)\,\frac{d\phi_b}{dz} = 0 \tag{5}$$

$$\frac{dp_b}{dz} + \mathcal{R}\theta_b = 0\,. \tag{6}$$

with the boundary conditions

$$\begin{aligned}
\theta_b &= -1 \qquad && \text{at} \quad z = 0 \\
\theta_b &= \phi_b = 0 \qquad && \text{at} \quad z = \delta.
\end{aligned}$$

Solutions to the basic state system had been presented previously by various authors. The basic state solution θ_b can be expressed implicitly by the equation

$$z = \frac{\alpha-\mathcal{C}}{\alpha-\beta}\ln\left[\frac{1+\alpha}{\alpha-\theta_b}\right] + \frac{\mathcal{C}-\beta}{\alpha-\beta}\ln\left[\frac{1+\beta}{\beta-\theta_b}\right] \tag{7}$$

where α, β are given by

$$\alpha,\ \beta = \frac{\mathcal{C}+\mathcal{S}+\theta_\infty \pm \sqrt{(\mathcal{C}+\mathcal{S}+\theta_\infty)^2 - 4\mathcal{C}\theta_\infty}}{2}.$$

The solution ϕ_b can be obtained from θ_b via

$$\phi_b = \frac{\theta_b}{\theta_b - \mathcal{C}} \tag{8}$$

The thickness of the layer can be derived from (7) by using the boundary condition $\theta_b = 0$ at $z = \delta$.

3.2.Perturbed System. Using (3) in the system (2), the perturbed system can be obtained as

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z} - \nabla^2\right)\theta - \mathcal{S}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)\phi + w\frac{d\theta_b}{dz} = -\epsilon\left(\overrightarrow{u}.\nabla\theta\right) \tag{9}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)\{(1-\phi_b)\,\theta + (\mathcal{C}-\theta_b)\,\phi\} + w\frac{d\theta_b}{dz}$$
$$= \epsilon\left\{-\overrightarrow{u}.\nabla\theta + \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)(\theta\phi)\right\} \tag{10}$$

$$k_b\overrightarrow{u} + \nabla p + \mathcal{R}\theta\hat{k} - \frac{Q}{\tau}\frac{\partial\overrightarrow{h}}{\partial z} = \epsilon\left(\frac{Q}{\tau}\overrightarrow{h}.\nabla\overrightarrow{h} - K\overrightarrow{u}\right) \tag{11}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)\overrightarrow{h} - \frac{\partial\overrightarrow{u}}{\partial z} - \frac{1}{\tau}\left(\nabla^2\overrightarrow{h}\right) = \epsilon\left(\overrightarrow{h}.\nabla\overrightarrow{u} - \overrightarrow{u}.\nabla\overrightarrow{h}\right) \tag{12}$$

$$\nabla.\overrightarrow{u} = 0 \tag{13}$$

with $\theta = w = h_3 = 0$ at $z = 0$ and $\theta = \phi = w = h_3 = 0$ at $z = \delta$. Here w and h_3 are vertical components of $\overrightarrow{u}$ and $\overrightarrow{h}$ respectively.

3.3.Linear Perturbed System. Linear perturbed system can be obtained using equations (3) in the system (2) by comparing the coefficients of ϵ^1 as

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z} - \nabla^2\right)\theta - \mathcal{S}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)\phi + w\frac{d\theta_b}{dz} = 0 \tag{14}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)\{(1-\phi_b)\,\theta + (\mathcal{C}-\theta_b)\,\phi\} + w\frac{d\theta_b}{dz} = 0 \tag{15}$$

$$k_b\overrightarrow{u} + \nabla p + \mathcal{R}\theta\hat{k} - \frac{Q}{\tau}\frac{\partial\overrightarrow{h}}{\partial z} = 0 \tag{16}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)\overrightarrow{h} - \frac{\partial\overrightarrow{u}}{\partial z} - \frac{1}{\tau}\left(\nabla^2\overrightarrow{h}\right) = 0 \tag{17}$$

$$\nabla.\overrightarrow{u} = 0 \tag{18}$$

Here the boundary conditions at the solidifying front are

$$\theta = w = h_3 = 0$$

and at the mush-liquid interface are

$$\theta = \phi = w = h_3 = 0 \quad .$$

To eliminate the pressure from the equation (16), we take the double curl of the equation (16). To do that, we use

$$\nabla\times\nabla\times\overrightarrow{f} = \left(\frac{\partial^2 f_3}{\partial x\partial z}, \frac{\partial^2 f_3}{\partial y\partial z}, -\frac{\partial^2 f_3}{\partial x^2} - \frac{\partial^2 f_3}{\partial y^2}\right)$$

and

$$\nabla \times \overrightarrow{f} = \left(\frac{\partial f_3}{\partial y},\ -\frac{\partial f_3}{\partial x},\ 0 \right)$$

with $\overrightarrow{f} = (0,\ 0,\ f_3)$. Now writing $\overrightarrow{u} = (u,\ v,\ w)$, we have

$$\begin{aligned} \nabla \times \nabla \times (k_b \overrightarrow{u}) \quad = \quad & \left(\frac{\partial^2 (k_b v)}{\partial x \partial y} + \frac{\partial^2 (k_b w)}{\partial x \partial z} - \frac{\partial^2 (k_b u)}{\partial y^2} - \frac{\partial^2 (k_b u)}{\partial z^2}, \right. \\ & \frac{\partial^2 (k_b w)}{\partial y \partial z} + \frac{\partial^2 (k_b u)}{\partial x \partial y} - \frac{\partial^2 (k_b v)}{\partial z^2} - \frac{\partial^2 (k_b v)}{\partial x^2}, \\ & \left. \frac{\partial^2 (k_b u)}{\partial x \partial z} + \frac{\partial^2 (k_b v)}{\partial y \partial z} - \frac{\partial^2 (k_b w)}{\partial x^2} - \frac{\partial^2 (k_b w)}{\partial y^2} \right). \end{aligned}$$

The continuity equation implies $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}$, which yields

$$\begin{aligned} & k_b \frac{\partial u}{\partial x} + k_b \frac{\partial v}{\partial y} = -k_b \frac{\partial w}{\partial z} \\ & \frac{\partial}{\partial z} \left(k_b \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial z} \left(k_b \frac{\partial v}{\partial y} \right) = -\frac{\partial}{\partial z} \left(k_b \frac{\partial w}{\partial z} \right) \\ & \frac{\partial^2 (k_b u)}{\partial x \partial z} + \frac{\partial^2 (k_b v)}{\partial y \partial z} = -\frac{\partial k_b}{\partial z} \frac{\partial w}{\partial z} - k_b \frac{\partial^2 w}{\partial z^2}. \end{aligned}$$

Thus the third component of $\nabla \times \nabla \times (k_b \overrightarrow{u})$ becomes

$$\begin{aligned} & \frac{\partial^2 (k_b u)}{\partial x \partial z} + \frac{\partial^2 (k_b v)}{\partial y \partial z} - \frac{\partial^2 (k_b w)}{\partial x^2} - \frac{\partial^2 (k_b w)}{\partial y^2} \\ & = -\frac{\partial k_b}{\partial z} \frac{\partial w}{\partial z} - k_b \frac{\partial^2 w}{\partial z^2} - k_b \frac{\partial^2 w}{\partial x^2} - k_b \frac{\partial^2 w}{\partial y^2} \\ & = -\frac{\partial k_b}{\partial z} \frac{\partial w}{\partial z} - k_b \nabla^2 w. \end{aligned}$$

Also writing $\overrightarrow{u} = (u,\ v,\ w) = \nabla \times \nabla \times \left(u_P \overrightarrow{k} \right) + \nabla \times \left(u_T \overrightarrow{k} \right)$(as $\nabla . \overrightarrow{u} = 0$, Chandrasekhar 1961), where u_P and u_T represent poloidal and toroidal components of $\overrightarrow{u}$, we obtain

$$\begin{aligned} u &= \frac{\partial^2 u_P}{\partial x \partial z} + \frac{\partial u_T}{\partial y} \\ v &= \frac{\partial^2 u_P}{\partial y \partial z} - \frac{\partial u_T}{\partial x} \\ w &= -\Delta_2 u_P \end{aligned}$$

Here Δ_2 is used to denote the horizontal Laplacian. Thus the third component of $\nabla \times \nabla \times (k_b \overrightarrow{u})$ is given by

$$\frac{\partial k_b}{\partial z}\frac{\partial}{\partial z}\left(\Delta_2 u_P\right) + k_b \nabla^2 \left(\Delta_2 u_P\right)$$

Similarly, for the third component of $\nabla \times \nabla \times (\mathcal{R}\theta \overrightarrow{k})$, we have

$$-\mathcal{R}\left[\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right] = -\mathcal{R}(\Delta_2 \theta).$$

Writing $\overrightarrow{h} = (h_1, h_2, h_3) = \nabla \times \nabla \times \left(h_P \overrightarrow{k}\right) + \nabla \times \left(h_T \overrightarrow{k}\right)$ (as $\nabla . \overrightarrow{h} = 0$), we can express

$$\overrightarrow{h} = (h_1, h_2, h_3) \quad = \quad \left(\frac{\partial^2 h_P}{\partial x \partial z} + \frac{\partial h_T}{\partial y}, \frac{\partial^2 h_P}{\partial y \partial z} - \frac{\partial h_T}{\partial x}, -\Delta_2 h_P\right)$$

Now the third component of $\nabla \times \nabla \times \left(\frac{\partial \overrightarrow{h}}{\partial z}\right)$ as

$$\begin{aligned} &\frac{\partial^2}{\partial x \partial z}\left(\frac{\partial h_1}{\partial z}\right) + \frac{\partial^2}{\partial y \partial z}\left(\frac{\partial h_2}{\partial z}\right) - \frac{\partial^2}{\partial x^2}\left(\frac{\partial h_3}{\partial z}\right) - \frac{\partial^2}{\partial y^2}\left(\frac{\partial h_3}{\partial z}\right) \\ &= -\frac{\partial^2}{\partial x^2}\left(\frac{\partial h_3}{\partial z}\right) - \frac{\partial^2}{\partial y^2}\left(\frac{\partial h_3}{\partial z}\right) - \frac{\partial^2}{\partial z^2}\left(\frac{\partial h_3}{\partial z}\right) \\ &= -\nabla^2\left(\frac{\partial h_3}{\partial z}\right) = \nabla^2\left\{\frac{\partial}{\partial z}\left(\Delta_2 h_P\right)\right\} \end{aligned}$$

Thus, the third component of $\nabla \times \nabla \times \left(\frac{Q}{\tau}\frac{\partial \overrightarrow{h}}{\partial z}\right)$ is

$$\frac{Q}{\tau}\nabla^2\left\{\frac{\partial}{\partial z}\left(\Delta_2 h_P\right)\right\}$$

Now eliminating the pressure term from equation (16), the linear system, i.e., system with equations (16), (14), (15) and (17) respectively, becomes

$$k_b' \frac{\partial}{\partial z}\left(\Delta_2 u_P\right) + k_b \nabla^2\left(\Delta_2 u_P\right) - \mathcal{R}\left(\Delta_2 \theta\right) - \frac{Q}{\tau}\nabla^2\left\{\frac{\partial}{\partial z}\left(\Delta_2 h_P\right)\right\} = 0 \qquad (19)$$

$$\left(\nabla^2 + \frac{\partial}{\partial z} - \frac{\partial}{\partial t}\right)\theta - \mathcal{S}\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial t}\right)\phi + \left(\Delta_2 u_P\right)\frac{d\theta_b}{dz} = 0 \qquad (20)$$

$$\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial t}\right)\left[\left(1 - \phi_b\right)\theta + \left(\mathcal{C} - \theta_b\right)\phi\right] + \left(\Delta_2 u_P\right)\frac{d\theta_b}{dz} = 0 \qquad (21)$$

$$\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial t}\right)\left(\Delta_2 h_P\right) + \frac{\partial}{\partial z}\left(\Delta_2 u_P\right) + \frac{1}{\tau}\nabla^2\left(\Delta_2 h_P\right) = 0 \qquad (22)$$

with boundary conditions

$$\begin{aligned} u_P = \theta = h_P = 0 \qquad &\text{at} \quad z = 0 \\ u_P = \theta = \phi = h_P = 0 \qquad &\text{at} \quad z = \delta. \end{aligned}$$

We assume that normal mode solutions ([16], [17]) in two dimensions, i.e.,

$$\begin{aligned} u_p &= u_{P_0}(z) e^{\sigma t + i\alpha x} + \text{C.C.} \\ \theta &= \theta_0(z) e^{\sigma t + i\alpha x} + \text{C.C.} \end{aligned}$$

$$\begin{aligned}\phi &= \phi_0(z)e^{\sigma t+i\alpha x} + \text{C.C.}\\ h_P &= h_{P_0}(z)e^{\sigma t+i\alpha x} + \text{C.C.}\end{aligned}$$

Here σ and α are real numbers, and C.C. stands for complex conjugate. The equations (19) to (22) becomes

$$\begin{aligned}&k_b\left[D^2-\alpha^2\right]u_{P_0} - \mathcal{R}\theta_0 + k_b'\left(Du_{P_0}\right) - \frac{Q}{\tau}D\left[D^2-\alpha^2\right]h_{P_0} = 0\\ &\left[D^2+D-\alpha^2-\sigma\right]\theta_0 - \mathcal{S}\left(D-\sigma\right)\phi_0 - \alpha^2\theta_b' u_{P_0} = 0\\ &\left(D-\sigma\right)\left[\left(1-\phi_b\right)\theta_0 + \left(\mathcal{C}-\theta_b\right)\phi_0\right] - \alpha^2\theta_b' u_{P_0} = 0\\ &Du_{P_0} + \left[D + \frac{1}{\tau}\left(D^2-\alpha^2\right) - \sigma\right]h_{P_0} = 0\end{aligned}$$

where $D = \frac{\partial}{\partial z}$ and the boundary conditions are given by

$$\begin{aligned}&u_{P_0} = \theta_0 = h_{P_0} = 0 \qquad \text{at} \quad z = 0\\ &u_{P_0} = \theta_0 = \phi_0 = h_{P_0} = 0 \qquad \text{at} \quad z = \delta.\end{aligned}$$

Substituting $\frac{1}{\tau}\left(D^2-\alpha^2\right)h_{P_0}$ from the last equation into the first equation, we obtain the perturbed linear system as

$$L_0 q_0 = 0 \tag{23}$$

with

$$L_0 = \begin{bmatrix} \begin{matrix}(k_b+Q)D^2\\ +k_b'D-\alpha^2 k_b\end{matrix} & -\mathcal{R} & 0 & Q\left(D^2-\sigma D\right)\\ -\alpha^2\theta_b' & D^2+D-\alpha^2-\sigma & -\mathcal{S}\left(D-\sigma\right) & 0\\ \alpha^2\theta_b' & \phi_b' - \left(1-\phi_b\right)\left(D-\sigma\right) & \theta_b' + \left(\theta_b-\mathcal{C}\right)\left(D-\sigma\right) & 0\\ \tau D & 0 & 0 & \begin{matrix}D^2+\tau D\\ -\alpha^2-\tau\sigma\end{matrix}\end{bmatrix} \tag{24}$$

and $q_0 = \begin{bmatrix} u_{P_0} & \theta_0 & \phi_0 & h_{p_0}\end{bmatrix}^{T_r}$. Here T_r is used to denote the transpose.

4. Adjoint System

Now we define the adjoint operator L_a of the linear operator L_0 as

$$< L_0 q_0,\ q_a > \; = \; < q_0,\ L_a q_a > \tag{25}$$

where L_0 is given by (24), $q_a = \begin{bmatrix} u_{P_a} & \theta_a & \phi_a & h_{P_a}\end{bmatrix}^{T_r}$. Here the sub-index a is used to denote the quantities belonging to the adjoint system and T_r represents the transpose. The inner product mentioned in (25) is defined by

$$< \overrightarrow{f},\ \overrightarrow{g} > = \int_0^\delta \overrightarrow{f} \cdot \overrightarrow{g}^* dz = \sum_{i=1}^{3} \int_0^\delta f_i g_i^* dz \tag{26}$$

with $\overrightarrow{f} = (f_1,\ f_2,\ f_3)$ and $\overrightarrow{g} = (g_1,\ g_2,\ g_3)$. Here, δ, the dimensionless thickness of the layer and * is used to denote the complex conjugate. It can be easily shown that

$$< \overrightarrow{f},\ \overrightarrow{g}^* >=< \overrightarrow{g},\ \overrightarrow{f}^* > \qquad \text{and} \qquad < \overrightarrow{g},\ \overrightarrow{f} >=< \overrightarrow{f},\ \overrightarrow{g} >^* . \tag{27}$$

To obtain the adjoint system, we multiply the equations (19) , (20), (21) and (22) by $u_{P_a},\ \theta_a,\ \phi_a$and $h_{P_a},$ respectively, and add them and then integrate with respect to z from $z = 0$ to $z = \delta$, i.e.,

$$\begin{aligned}
&< L_0 q_0,\ q_a^* > = \\
&\int_0^\delta \left\{\left((k_b + Q)\, D^2 + k_b' D - \alpha^2 k_b\right) u_{P_0} - \mathcal{R}\theta_0 + Q\left(D^2 - \sigma D\right) h_{P_0}\right\} u_{P_a} dz \\
&+ \int_0^\delta \left\{-\alpha^2 \theta_b' u_{P_0} + \left(D^2 + D - \alpha^2 - \sigma\right)\theta_0 - \mathcal{S}\,(D - \sigma)\,\phi_0\right\} \theta_a dz \\
&+ \int_0^\delta \left\{\alpha^2 \theta_b' u_{P_0} + [\phi_b' - (1 - \phi_b)\,(D - \sigma)]\,\theta_0\right. \\
&\qquad \left. + [\theta_b' + (\theta_b - \mathcal{C})\,(D - \sigma)]\,\phi_0\right\} \phi_a dz \\
&+ \int_0^\delta \left\{\tau D u_{P_0} + \left(D^2 + \tau D - \alpha^2 - \tau\sigma\right) h_{P_0}\right\} h_{P_a} dz.
\end{aligned} \tag{28}$$

Boundary conditions for the adjoint system are

$$\begin{aligned}
&u_{P_a} = \theta_a = \phi_a = h_{P_a} = 0 \qquad \text{at} \quad z = 0 \\
&u_{P_a} = \theta_a = h_{P_a} = 0 \qquad \text{at} \quad z = \delta.
\end{aligned}$$

Using integration by parts on the right hand side of (28), and the boundary conditions, we get

$$\begin{aligned}
&< L_0 q_0,\ q_a^* > = \\
&\int_0^\delta \left[\left\{(k_b + Q)\, D^2 + k_b' D - \alpha^2 k_b\right\} u_{P_a} - \alpha^2 \theta_b' \theta_a + \alpha^2 \theta_b' \phi_a - \tau D h_{P_a}\right] u_{P_0} dz \\
&+ \int_0^\delta \left[-\mathcal{R} u_{P_a} + \left\{D^2 + D - \alpha^2 - \sigma\right\}\theta_a + (1 - \phi_b)\,(D + \sigma)\,\phi_a\right] \theta_0 dz \\
&+ \int_0^\delta \left[\mathcal{S}\,(D + \sigma)\,\theta_a + (\mathcal{C} - \theta_b)\,(D + \sigma)\,\phi_a\right] \phi_0 dz \\
&+ \int_0^\delta \left[Q\left(D^2 + \sigma D\right) u_{P_a} + \left(D^2 - \tau D - \alpha^2 - \tau\sigma\right) h_{P_a}\right] h_{P_0} dz = 0
\end{aligned}$$

Now using the relation (27), we have

$$< L_0 q_0,\ q_a^* >=< q_a,\ (L_0 q_0)^* >=< q_a,\ L_0 q_0^* > \qquad \text{as } L_0^* = L_0$$

and using the definition of L_a, we obtain

$$< L_0 q_0,\ q_a^* > = \ 0 = < q_a,\ L_0 q_0^* > = < L_a q_a,\ q_0^* >$$

yielding $L_a q_a = 0$. Hence, we can write the adjoint system as

$$\left\{ (k_b + Q) D^2 + k_b' D - \alpha^2 k_b \right\} u_{P_a} - \alpha^2 \theta_b' \theta_a + \alpha^2 \theta_b' \phi_a - \tau \left(D h_{P_a} \right) = 0 \tag{29}$$

$$-\mathcal{R} u_{P_a} + \left\{ D^2 - D - \alpha^2 - \sigma \right\} \theta_a + (1 - \phi_b)(D + \sigma) \phi_a = 0 \tag{30}$$

$$\mathcal{S} (D + \sigma) \theta_a - (\mathcal{C} - \theta_b)(D + \sigma) \phi_a = 0 \tag{31}$$

$$Q \left(D^2 + \sigma D \right) u_{P_a} + \left(D^2 - \tau D - \alpha^2 - \tau\sigma \right) h_{P_a} = 0 \tag{32}$$

and, in turn, we can write our adjoint operator L_a as

$$L_a = \begin{bmatrix} (k_b + Q) D^2 + k_b' D - \alpha^2 k_b & -\alpha^2 \theta_b' & \alpha^2 \theta_b' & -\tau D \\ -\mathcal{R} & D^2 - D - \alpha^2 - \sigma & (1 - \phi_b)(D + \sigma) & 0 \\ 0 & \mathcal{S}(D + \sigma) & (\mathcal{C} - \theta_b)(D + \sigma) & 0 \\ Q \left(D^2 + \sigma D \right) & 0 & 0 & D^2 - \tau D - \alpha^2 - \tau\sigma \end{bmatrix} \tag{33}$$

Thus, the adjoint system becomes

$$L_a q_a \quad = \quad 0 \tag{34}$$

It can be shown that

$$\left(L_a \right)_a \quad = \quad L_0$$

as

$$< L_0 q_0,\ q_a > = < q_0,\ L_a q_a > = < \left(L_a \right)_a q_0,\ q_a > .$$

5. Numerical Results

Here we present computational results for two cases: (i) passive layer (layer with constant permeability) and (ii) active layer (layer with variable permeability). Following Worster, we use the expression for K as $\frac{1}{(1-\phi)^n}$ with $n = 3$ for variable permeability and $n = 0$ for constant permeability where ϕ represents the solid volume fraction. We chose the following parameters for further calculations: 3.2 for Stefan number, 9.0 for concentration ratio, 0.1 for far-field temperature, these are based on previous experimental and theoretical studies ([1], [2], [3], [4], [7], [8]). We chose the value of the Chandrasekhar number, Q

as 1.0 for our numerical results. First we obtain the basic state solutions by performing computation on the equations (7) and (7) using Muller's method. We use JMSL library to compute the zeros of (7). The thickness of the layer is computed from (7) by using the boundary condition $\theta_b = 0$ at $z = \delta$ for further use. After computing the basic state solutions, we compute the critical Rayleigh number and the critical wavenumber by solving the linear problem numerically using the fourth-order Runge-Kutta method in combination of shooting method ([18]) at the onset of the motion. In order to solve the linear system (23), we first convert this system into a system of seven first-order linear ordinary differential equations. We obtain 2.034 and 26.6032 as critical wavenumber and Rayleigh number, respectively, for constant permeability case and 2.009 and 29.0555 as critical wavenumber and Rayleigh number, respectively, for variable permeability case. Linear marginal stability curves are shown in figure 2.

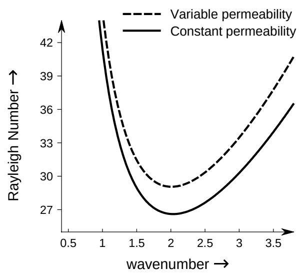


FIGURE 2. Comparison of the marginal stability curves

Then, we compute the solutions of the adjoint system given by (34) by converting it into a system of seven first-order linear ordinary differential equations by using the fourth-order Runge-Kutta method. Some of the solutions of the adjoint system are presented in figure 3 through 5.

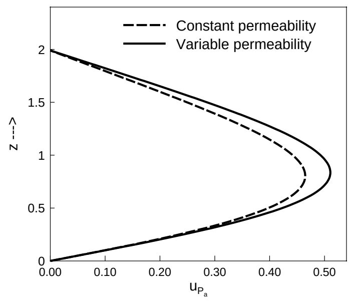


FIGURE 3. Adjoint solutions related to the vertical velocity component

These results indicate that linear theory proposes the critical wavenumber and critical Rayleigh number are higher for the variable permeability layer than the constant permeability layer implying more efficient and stable active mush, which is observed in experiments. Figure 3 presents the results of the adjoint solution related to the vertical velocity component and figure 4 displays the adjoint solution for the solid volume fraction for both active and passive mushy layer cases.

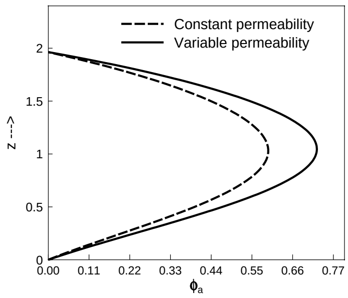


FIGURE 4. Adjoint solutions for the solid volume fraction

Figure 5 shows the adjoint solutions for the magnetic component for both the cases.

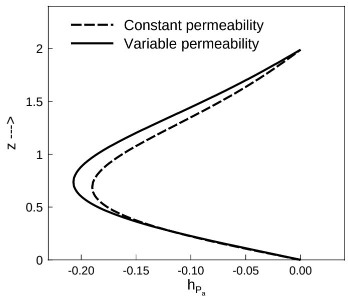


FIGURE 5. Adjoint solutions for the magnetic component

6. Conclusion

Here we have considered magneto-convection in a mushy layer which is formed during solidification of binary alloys. We have derived the adjoint system analytically for the mushy layer with an impermeable mush-liquid interface. The mushy layer has been treated as an active porous media with variable permeability. We have derived the linear system using perturbation technique and basic state solutions. Adjoint system is derived analytically using linear system by introducing an inner product. Numerical results for the adjoint system are presented for passive and active mushy layers at the onset of the motion using a set of parameters experimentalists use.

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