J. Appl. Math. & Informatics Vol. **29***(2011), No. 5 - 6, pp. 1311 - 1325*
Website: http://www.kcam.biz

STABILITY OF THE MILSTEIN METHOD FOR STOCHASTIC DIFFERENTIAL EQUATIONS WITH JUMPS†

LIN HU* AND SIQING GAN

Abstract. In this paper the Milstein method is proposed to approximate the solution of a linear stochastic differential equation with Poisson-driven jumps. The strong Milstein method and the weak Milstein method are shown to capture the mean square stability of the system. Furthermore using some technique, our result shows that these two kinds of Milstein methods can well reproduce the stochastically asymptotical stability of the system for all sufficiently small time-steps. Some numerical experiments are given to demonstrate the conclusions.

AMS Mathematics Subject Classification : 60H10, 65C20, 65L20.
Key words and phrases : Milstein method, mean square stability, asymptotical stability, jump-diffusion, stochastic differential equation.

1. Introduction

Stochastic differential equations (SDEs) with jumps have many important applications in the areas such as biology, finance, economics and so on. In general, SDEs with jumps have no explicit solutions. Thus, it is necessary to develop numerical methods and to study the properties of these numerical schemes. There is a vast literature on the stability of numerical methods for SDEs, see, for example, Higham [5], Milstein [12], Higham et al.[7] and for stochastic differential delay equations (SDDEs), for example, see [17, 9]. There are some results concerned with the weak approximate schemes for SDEs, see, for example, Saito and Mistui [14], Burrage et al. [2], Cao and Liu [4]. Recently, there has been some work done about the numerical schemes for SDEs with jumps [6, 3] and SDDEs with jumps [10, 15].

Received July 19, 2010. Revised November 29, 2010. Accepted December 4, 2010.
*Corresponding author. †This work was supported by: 1. The National Natural Science Foundation of China (Grant No.10871207); 2. The Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry.

However, the numerical methods in [5, 7, 6, 3, 10, 15] converge with strong order of one half. In this paper, we will focus on the Milstein method which has strong convergence rate of one, see, for example, [11]. There is an extensive literature on the numerical analysis of the Milstein method (e.g., [16, 13]). Our aim in this work is to investigate the mean square stability and the asymptotical stability of the Milstein method for SDEs with jumps.

In this paper, we consider the linear, scalar SDEs with jumps:

$$dx(t) = ax(t^-)dt + bx(t^-)dW(t) + cx(t^-)dN(t) \tag{1}$$

with $x(0) = x_0$. Here $t > 0$, $x_0 \neq 0$ with probability one, $x(t^-)$ denotes $\lim_{s\to t^-} x(s)$. Here, $W(t)$ is a scalar Brownian motion and $N(t)$ is a scalar Poisson process with intensity $\lambda(\lambda > 0)$, both defined on an appropriate complete probability space $(\Omega,\ \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$, with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions(i.e. it is increasing and right-continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). $W(t)$ is independent of $N(t)$. The coefficients $a,\ b,\ c \in \mathbb{R}$ and $c \neq 0$ (if $c = 0$, the model reduces to a SDEs without jumps).

The main results is organized as follows. Sections 2 and 3 deal with the mean square stabilities of the strong Milstein method and the weak Milstein method respectively. In Sections 4 and 5, we apply our asymptotic stability analysis to the strong Milstein method and the weak Milstein method. In addition, some numerical results are presented to confirm the theoretical results in Section 6.

2. mean square stability of the strong Milstein method

Given a stepsize $h > 0$, the strong Milstein scheme [11] is proposed for (1) as follows.

$$\begin{aligned} Y_{n+1} = [&1 + (a - \tfrac{1}{2}b^2)h + b\Delta W_n + \tfrac{1}{2}b^2(\Delta W_n)^2 + \tfrac{1}{2}(2c - c^2)\Delta N_n \\ &+ bc\Delta W_n \Delta N_n + \tfrac{1}{2}c^2(\Delta N_n)^2]Y_n \end{aligned} \tag{2}$$

with $Y(0) = x_0$. Here Y_n is an approximation to $x(t_n)$ with $t_n = nh,\ \Delta W_n = W(t_{n+1}) - W(t_n),\ \Delta N_n = N(t_{n+1}) - N(t_n)$ and ΔW_n is independent of ΔN_n.

To begin with, we cite here a linear mean square stability result of underlying the system (1)

Lemma 2.1 ([6]). *The analytical solution of the system* (1) *is mean square stable, that is to say,* $\lim_{t\to\infty} \mathbb{E}(x(t))^2 = 0$, *if and only if*

$$2a + b^2 + \lambda c(2 + c) < 0. \tag{3}$$

Given parameters $a,\ b,\ c,\ \lambda$ and stepsize h, we say the Milstein method is mean-square stable if $\lim_{n\to\infty} \mathbb{E}(Y_n)^2 = 0$ for any Y_0. We are now in a position to state our result on the mean-square stability of the strong Milstein method.

Theorem 2.1. *Assume the condition* (3) *is satisfied. Then the strong Milstein method* (2) *is mean square stable.*

Proof. It follows from (2) that

$$\begin{aligned}
&\mathbb{E}Y_{n+1}^2 \\
&= \{1 + (a - \tfrac{1}{2}b^2)^2h^2 + b^2\mathbb{E}(\Delta W_n)^2 + \tfrac{1}{4}b^4\mathbb{E}(\Delta W_n)^4 + \tfrac{1}{4}(2c - c^2)^2\mathbb{E}(\Delta N_n)^2 \\
&+b^2c^2\mathbb{E}[(\Delta W_n)^2(\Delta N_n)^2] + \tfrac{1}{4}c^4\mathbb{E}(\Delta N_n)^4 + 2(a - \tfrac{1}{2}b^2)h + 2b\mathbb{E}\Delta W_n \\
&+b^2\mathbb{E}(\Delta W_n)^2 + (2c - c^2)\mathbb{E}\Delta N_n + 2bc\mathbb{E}[\Delta W_n\Delta N_n] + c^2\mathbb{E}(\Delta N_n)^2 \\
&+2(a - \tfrac{1}{2}b^2)hb\mathbb{E}\Delta W_n + (a - \tfrac{1}{2}b^2)hb^2\mathbb{E}(\Delta W_n)^2 + (a - \tfrac{1}{2}b^2)(2c - c^2)h\mathbb{E}\Delta N_n \\
&+2(a - \tfrac{1}{2}b^2)bch\mathbb{E}[\Delta W_n\Delta N_n] + (a - \tfrac{1}{2}b^2)c^2h\mathbb{E}(\Delta N_n)^2 + b^3\mathbb{E}(\Delta W_n)^3 \\
&+(2c - c^2)b\mathbb{E}[\Delta W_n\Delta N_n] + 2b^2c\mathbb{E}[(\Delta W_n)^2\Delta N_n] + c^2b\mathbb{E}[(\Delta N_n)^2\Delta W_n] \\
&+\tfrac{1}{2}b^2(2c - c^2)\mathbb{E}[(\Delta W_n)^2\Delta N_n] + b^3c\mathbb{E}[(\Delta W_n)^3\Delta N_n] + \tfrac{1}{2}b^2c^2\mathbb{E}[(\Delta W_n)^2(\Delta N_n)^2] \\
&+(2c - c^2)bc\mathbb{E}[\Delta W_n(\Delta N_n)^2] + \tfrac{1}{2}c^2(2c - c^2)\mathbb{E}(\Delta N_n)^3 + bc^3\mathbb{E}[\Delta W_n(\Delta N_n)^3]\}\mathbb{E}Y_n^2 \\
&= [1 + R_1(a, b, c, \lambda)h + R_2(a, b, c, \lambda)h^2 + R_3(a, b, c, \lambda)h^3 + R_4(a, b, c, \lambda)h^4]\mathbb{E}Y_n^2,
\end{aligned}$$

where $R_1(a, b, c, \lambda) = 2a + b^2 + \lambda c(c + 2)$, $R_3(a, b, c, \lambda) = b^2c^2\lambda^2 + c^4\lambda^3 + ac^2\lambda^2 + c^3\lambda^3$, $R_2(a, b, c, \lambda) = \frac{1}{2}b^4 + a^2 + b^2c^2\lambda + \frac{1}{2}\lambda^2c^4 + 2\lambda^2c^2 + 2b^2c\lambda + 2ac\lambda + 2c^3\lambda^2$, $R_4(a, b, c, \lambda) = \frac{1}{4}c^4\lambda^4$. Here the fact has been used that $\mathbb{E}(\Delta W_n) = \mathbb{E}(\Delta W_n)^3 = 0$, $\mathbb{E}(\Delta W_n)^2 = h$, $\mathbb{E}(\Delta W_n)^4 = 3h^2$, and the Poisson increments satisfy $\mathbb{E}(\Delta N_n) = \lambda h$, $\mathbb{E}(\Delta N_n)^2 = \lambda h(1 + \lambda h)$, $\mathbb{E}(\Delta N_n)^3 = \lambda h + 3\lambda^2h^2 + \lambda^3h^3$, $\mathbb{E}(\Delta N_n)^4 = \lambda h + 7\lambda^2h^2 + 6\lambda^3h^3 + \lambda^4h^4$. Thus the numerical solution is mean square stable if and only if

$$R_1(a, b, c, \lambda)h + R_2(a, b, c, \lambda)h^2 + R_3(a, b, c, \lambda)h^3 + R_4(a, b, c, \lambda)h^4 < 0.$$

That is to say

$$R_1(a, b, c, \lambda) + R_2(a, b, c, \lambda)h + R_3(a, b, c, \lambda)h^2 + R_4(a, b, c, \lambda)h^3 < 0.$$

Letting $\psi(h) = R_1(a, b, c, \lambda) + R_2(a, b, c, \lambda)h + R_3(a, b, c, \lambda)h^2 + R_4(a, b, c, \lambda)h^3$, by (3), we easily obtain $\psi(0) < 0$. Because of the continuity of ψ with respect to h, there must exist a $h_0(a, b, c, \lambda) \in (0, \mu)$, μ is small enough, such that for any $h \in (0, h_0(a, b, c, \lambda))$

$$\psi(h) < 0.$$

The proof is completed. □

3. Mean square stability of the weak Milstein method

Applying the weak Milstein method, which is equipped with two-point random variables for the driving process, to system (1) leads to

$$\begin{aligned}
Y_{n+1} = [&1 + (a - \tfrac{1}{2}b^2)h + b\widehat{\Delta W_n} + \tfrac{1}{2}b^2(\widehat{\Delta W_n})^2 + \tfrac{1}{2}(2c - c^2)\widehat{\Delta N_n} \\
&+bc\widehat{\Delta W_n}\,\widehat{\Delta N_n} + \tfrac{1}{2}c^2(\widehat{\Delta N_n})^2]Y_n
\end{aligned} \tag{4}$$

with $Y(0) = x_0$. Here $\mathbb{P}(\widehat{\Delta W_n} = \sqrt{h}) = \mathbb{P}(\widehat{\Delta W_n} = -\sqrt{h}) = 1/2$ and $\mathbb{P}(\widehat{\Delta N_n} = 0) = 1 - \lambda h$, $\mathbb{P}(\widehat{\Delta N_n} = 1) = \lambda h$.

Now we give the main theorem in this section.

Theorem 3.1. *Assume the condition* (3) *is satisfied. Then the weak Milstein method* (4) *is mean square stable.*

Proof. It follows from (4) that

$$\begin{aligned} &\mathbb{E}Y_{n+1}^2 \\ &= [1+\widehat{R}_1(a,\ b,\ c,\ \lambda)h+\widehat{R}_2(a,\ b,\ c,\ \lambda)h^2]\mathbb{E}Y_n^2, \end{aligned} \tag{5}$$

where $\widehat{R}_1(a,b,c,\lambda)=2(a-\frac{1}{2}b^2)+\lambda c(2+c),\ \widehat{R}_2(a,b,c,\lambda)=a^2+b^2c^2\lambda+2ac\lambda+2b^2c\lambda$. Here the fact has been used that $\mathbb{E}(\widehat{\Delta W_n})=\mathbb{E}(\widehat{\Delta W_n})^3=0,\ \mathbb{E}(\widehat{\Delta W_n})^2=h,\ \mathbb{E}(\widehat{\Delta W_n})^4=h^2,\ \mathbb{E}(\widehat{\Delta N_n})^i=\lambda h(i=1,\cdots,4)$. Hence the weak Milstein method is mean square stable if and only if

$$\widehat{R}_1(a,b,c,\lambda)+\widehat{R}_2(a,b,c,\lambda)h<0. \tag{6}$$

If $\widehat{R}_2(a,b,c,\lambda)\le 0$, by (3), then (6) holds for all $h\in(0,1/\lambda)$. If $\widehat{R}_2(a,b,c,\lambda)>0$, by (3), then (6) holds for all $h\in(0,-\frac{\widehat{R}_1(a,b,c,\lambda)}{\widehat{R}_2(a,b,c,\lambda)})$. Thus, there exists $\widehat{h}_0(a,b,c,\lambda)=\min\{\frac{1}{\lambda},\ -\frac{\widehat{R}_1(a,b,c,\lambda)}{\widehat{R}_2(a,b,c,\lambda)}\}$, for any $h\in(0,\ \widehat{h}_0(a,b,c,\lambda))$, the inequality (6) holds. This completes the proof. □

Theorem 2.1 and *Theorem* 3.1 show that the strong Milstein method (2) and the weak Milstein method (4) can preserve the mean square stability of the system (1).

4. Asymptotic stability of the strong Milstein method

In Sections 2 and 3, we consider the mean square stabilities of the two kinds of Milstein methods. This section is concerned with the asymptotic stability. We begin by giving the necessary and sufficient condition for the stochastically asymptotical stability in the large (hereafter, asymptotical stability) of the system (1).

Lemma 4.1 ([3])**.** *The system* (1) *is stochastically asymptotically stable in the large (hereafter, asymptotically stable) if and only if*

$$a-\frac{1}{2}b^2+\lambda\ln|1+c|<0, \tag{7}$$

where $\ln|1+c|=-\infty$*, as* $c=-1$*. Thus when* $c=-1$*, under the condition* (7)*, the system* (1) *is asymptotically stable for any* $a,\ b\in\mathbb{R}$*.*

We say the Milstein method (2) are asymptotically stable for a particular choice of $a,\ b,\ c,\ \lambda$ and h if $\lim_{n\to\infty}|Y_n|=0$, with probability one, for any Y_0.

We now establish *Lemma* 4.2, which will play a key role in the proof of Theorem 4.1. In the similar way as the *Theorem* 3.2 in [1] we can obtain *Lemma* 4.2.

Lemma 4.2. *Let* ξ *be a standard Normal random variable. Then, for the integrable function* $\varphi:\ \mathbb{R}\to\mathbb{R}$*, we have*

$$\begin{aligned} &\mathbb{E}[\varphi(1+(a-\tfrac{1}{2}b^2)h+b\sqrt{h}\xi+\tfrac{1}{2}b^2h\xi^2)] \\ &=\varphi(1)+\varphi'(1)((a-\tfrac{1}{2}b^2)h+\tfrac{1}{2}b^2h)+\tfrac{\varphi''(1)}{2}b^2h+o(h), \end{aligned} \tag{8}$$

where $a, b \in \mathbb{R},\ h \to 0.$

Proof. Consider the function φ, such that there exists $\sigma > 0$ and the integral function $\widetilde{\varphi}:\ \mathbb{R} \to \mathbb{R}$ satisfying

(1) $\widetilde{\varphi} \equiv \varphi$ on $A = [1-\sigma,\ 1+\sigma]$,

(2) $\widetilde{\varphi} \in C^3(\mathbb{R})$ and $|\widetilde{\varphi}^{'''}(x)| \le H$ for some H and all $x \in \mathbb{R}$,

(3) $\int_{\mathbb{R}} |\varphi - \widetilde{\varphi}| dx \le K < +\infty$ for some K.

The following proof is divided into two parts:

Part 1: We proves formula (8) for $\widetilde{\varphi}(1 + (a - \frac{1}{2}b^2)h + b\sqrt{h}\xi + \frac{1}{2}b^2h\xi^2)$. Using Taylor expansion gives

$$\widetilde{\varphi}(1+x) = \widetilde{\varphi}(1) + \widetilde{\varphi}^{'}(1)x + \frac{\widetilde{\varphi}^{''}(1)}{2}x^2 + \frac{\widetilde{\varphi}^{'''}(\eta)}{6}x^3,$$

where η lying between 1 and $1+x$.

Substituting $x = (a - \frac{1}{2}b^2)h + b\sqrt{h}\xi + \frac{1}{2}b^2h\xi^2$ and taking expectation lead to

$$\begin{aligned}&\mathbb{E}[\widetilde{\varphi}(1+x)]\\ &= \widetilde{\varphi}(1) + \widetilde{\varphi}^{'}(1)((a - \tfrac{1}{2}b^2)h + \tfrac{1}{2}b^2h) + \tfrac{\widetilde{\varphi}^{''}(1)}{2}(b^2h + o(h)) + \tfrac{1}{6}\mathbb{E}(\widetilde{\varphi}^{'''}(\eta)x^3).\end{aligned}$$

Here the fact has been used that $\mathbb{E}\xi = 0$, $\mathbb{E}|\xi| = 2/\sqrt{2\pi}$, $\mathbb{E}|\xi|^2 = 1$. Noticing that $\widetilde{\varphi} \in C^3(\mathbb{R})$ and $|\widetilde{\varphi}^{'''}(x)| \le H$ for all $x \in \mathbb{R}$, we have

$$|\frac{1}{6}\mathbb{E}(\widetilde{\varphi}^{'''}(\eta)x^3)| \le \frac{H}{6}\mathbb{E}|x^3| = o(h),$$

which implies

$$\mathbb{E}[\widetilde{\varphi}(1+x)] = \widetilde{\varphi}(1) + \widetilde{\varphi}^{'}(1)((a - \tfrac{1}{2}b^2)h + \tfrac{1}{2}b^2h) + \tfrac{\widetilde{\varphi}^{''}(1)}{2}b^2h + o(h).$$

This proves formula (8) for $\widetilde{\varphi}(1 + (a - \frac{1}{2}b^2)h + b\sqrt{h}\xi + \frac{1}{2}b^2h\xi^2)$.

Part 2: Letting $u_1 = \frac{1}{2} + (a - \frac{1}{2}b^2)h,\ u_2 = b\sqrt{h}$ and, now we estimate the error term $e = \mathbb{E}[\varphi(u_1 + \frac{1}{2}(u_2\xi + 1)^2) - \widetilde{\varphi}(u_1 + \frac{1}{2}(u_2\xi + 1)^2)]$ as following

$$|e| = |\int_{-\infty}^{+\infty}[\varphi(u_1 + \tfrac{1}{2}(u_2z + 1)^2) - \widetilde{\varphi}(u_1 + \tfrac{1}{2}(u_2z + 1)^2)]p(z)dz|,$$

where $p(z)$ is the density of ξ.

In the following, we give the two cases to discuss:

Case 1: $u_2z + 1 \ne 0$. In this case, we suppose $\sigma \in (0,\ 1/2)$. We set $v = u_1 + \frac{1}{2}(u_2z + 1)^2$, since $u_2z + 1 \ne 0$, which means $v - u_1 \ne 0$. Here without loss of generality, we can assume $b \ne 0$ which implies $u_2 = b\sqrt{h} \ne 0$. Noting that A

is the integration range on which $\varphi(v)-\widetilde{\varphi}(v)=0$, we compute

$$\begin{aligned}
|e| &\leq \textstyle\int_{\mathbb{R}\setminus A} |\varphi(v)-\widetilde{\varphi}(v)| p(\frac{-1+\sqrt{2(v-u_1)}}{u_2}) \frac{dv}{\sqrt{2(v-u_1)}|u_2|} \\
&+ \textstyle\int_{\mathbb{R}\setminus A} |\varphi(v)-\widetilde{\varphi}(v)| p(\frac{-1-\sqrt{2(v-u_1)}}{u_2}) \frac{dv}{\sqrt{2(v-u_1)}|u_2|} \\
&\leq \sup_{v\notin A}\{p(\frac{-1+\sqrt{2(v-u_1)}}{u_2}) \frac{1}{\sqrt{2(v-u_1)}|u_2|}\} \textstyle\int_{\mathbb{R}} |\varphi(v)-\widetilde{\varphi}(v)| dv \\
&+ \sup_{v\notin A}\{p(\frac{-1-\sqrt{2(v-u_1)}}{u_2}) \frac{1}{\sqrt{2(v-u_1)}|u_2|}\} \textstyle\int_{\mathbb{R}} |\varphi(v)-\widetilde{\varphi}(v)| dv \\
&\leq K|u_2|^2 \sup_{v\notin A}\{p(\frac{-1+\sqrt{2(v-u_1)}}{u_2}) \frac{1}{\sqrt{2(v-u_1)}|u_2|^3}\} \\
&+ K|u_2|^2 \sup_{v\notin A}\{p(\frac{-1-\sqrt{2(v-u_1)}}{u_2}) \frac{1}{\sqrt{2(v-u_1)}|u_2|^3}\} \\
&= Kb^2h \sup_{v\notin A}\{p(y_1)|y_1|^3 \frac{1}{|-1+\sqrt{2(v-u_1)}|^3} \cdot \frac{1}{\sqrt{2(v-u_1)}}\} \\
&+ Kb^2h \sup_{v\notin A}\{p(y_2)|y_2|^3 \frac{1}{|-1-\sqrt{2(v-u_1)}|^3} \cdot \frac{1}{\sqrt{2(v-u_1)}}\},
\end{aligned} \tag{9}$$

where $y_1 = \frac{-1+\sqrt{2(v-u_1)}}{u_2}$, $y_2 = \frac{-1-\sqrt{2(v-u_1)}}{u_2}$.
Noticing that when $v \notin A$, that is to say, $v > 1+\sigma$, we have

$$\begin{aligned}
&-1+\sqrt{2(v-u_1)} \\
&= -1+\sqrt{2(v-\tfrac{1}{2}-(a-\tfrac{1}{2}b^2)h)} \\
&> -1+\sqrt{1+2\sigma} = \tfrac{2\sigma}{1+\sqrt{1+2\sigma}},\ h\to 0,
\end{aligned} \tag{10}$$

or $v < 1-\sigma$, thus

$$-1+\sqrt{2(v-u_1)} < -1+\sqrt{1-2\sigma} = -\tfrac{2\sigma}{1+\sqrt{1-2\sigma}},\ h\to 0. \tag{11}$$

Noting that $y_1 = \frac{-1+\sqrt{2(v-\frac{1}{2}-(a-\frac{1}{2}b^2)h)}}{b\sqrt{h}}$, thus using (10) and (11) gives $y_1\to\infty$ uniformly on $v\notin A$, as $h\to 0$. By (10) and (11), we find that

$$\begin{aligned}
&(-1+\sqrt{2(v-\tfrac{1}{2})})^3\cdot\sqrt{2(v-u_1)} \to (-1+\sqrt{2(v-\tfrac{1}{2})})^3\cdot\sqrt{2(v-\tfrac{1}{2})} \\
&> \tfrac{8\sigma^3}{(1+\sqrt{1+2\sigma})^3}\sqrt{1+2\sigma},\ h\to 0,
\end{aligned}$$

or

$$(-1+\sqrt{2(v-\tfrac{1}{2})})^3\cdot\sqrt{2(v-u_1)} < \tfrac{-8\sigma^3}{(1+\sqrt{1-2\sigma})^3}\sqrt{1-2\sigma},\ h\to 0,$$

which implies $|-1+\sqrt{2(v-\frac{1}{2})}|^3\cdot\sqrt{2(v-\frac{1}{2})}$ is bounded away from zero. With the properties of $p(z)$, we can prove $p(y_1)|y_1|^3\to 0$, as $y_1\to\infty$, thus

$$\sup_{v\notin A}\{p(y_1)|y_1|^3\tfrac{1}{|-1+\sqrt{2(v-u_1)}|^3}\cdot\tfrac{1}{\sqrt{2(v-u_1)}}\} = o(1),\ h\to 0. \tag{12}$$

Similarly

$$\sup_{v\notin A}\{p(y_2)|y_2|^3\tfrac{1}{|-1-\sqrt{2(v-u_1)}|^3}\cdot\tfrac{1}{\sqrt{2(v-u_1)}}\} = o(1),\ h\to 0. \tag{13}$$

Inserting (12) and (13) into (9) yields

$$|e| = o(h).$$

Case 2: $u_2 z + 1 = 0$. In this case, we suppose $\sigma \in (1/2,\ 1)$. Noticing that $u_1 = \frac{1}{2} + (a - \frac{1}{2}b^2)h \in A$, when $h \to 0$, we have

$$\begin{aligned} |e| &= |\int_{-\infty}^{+\infty}[\varphi(u_1 + \tfrac{1}{2}(u_2 z + 1)^2) - \widetilde{\varphi}(u_1 + \tfrac{1}{2}(u_2 z + 1)^2)]p(z)dz| \\ &= |\int_{-\infty}^{+\infty}[\varphi(u_1) - \widetilde{\varphi}(u_1)]p(z)dz| = 0. \end{aligned}$$

Combining *Part* 1 and *Part* 2 leads the assertion (8). □

Based on *Lemma* 4.1, we can now begin to establish the main theorem in this section.

Theorem 4.1. *Given a, b, c, λ, the system* (1) *is asymptotically stable if and only if there exists a $h_1^* > 0$, the strong Milstein method* (2) *is asymptotically stable for all $0 < h < h_1^*$.*

Proof. It follows from (2) that

$$\begin{aligned} Y_{n+1} = [&1 + (a - \tfrac{1}{2}b^2)h + b\sqrt{h}\xi + \tfrac{1}{2}b^2 h\xi^2 + \tfrac{1}{2}(2c - c^2)\Delta N_n \\ &+bc\sqrt{h}\xi\Delta N_n + \tfrac{1}{2}c^2(\Delta N_n)^2]Y_n, \end{aligned} \tag{14}$$

where ξ is standard Normal random variable. By the Lemma 5.1 of [5], we deduce immediately that the asymptotic stability of the strong Milstein method (14) is equivalent to

$$\begin{aligned} &\mathbb{E}[\ln|1 + (a - \tfrac{1}{2}b^2)h + b\sqrt{h}\xi + \tfrac{1}{2}b^2 h\xi^2 + \tfrac{1}{2}(2c - c^2)\Delta N_n \\ &+bc\sqrt{h}\xi\Delta N_n + \tfrac{1}{2}c^2(\Delta N_n)^2|] < 0. \end{aligned} \tag{15}$$

Multiplying the expected value in (15) by $e^{\lambda h}$ yields

$$\begin{aligned} &e^{\lambda h}\mathbb{E}[\ln|1 + (a - \tfrac{1}{2}b^2)h + b\sqrt{h}\xi + \tfrac{1}{2}b^2 h\xi^2 + \tfrac{1}{2}(2c - c^2)\Delta N_n \\ &+bc\sqrt{h}\xi\Delta N_n + \tfrac{1}{2}c^2(\Delta N_n)^2|] \\ &= \mathbb{E}[\ln|1 + (a - \tfrac{1}{2}b^2)h + b\sqrt{h}\xi + \tfrac{1}{2}b^2 h\xi^2|] + \lambda h\mathbb{E}[\ln|1 + (a - \tfrac{1}{2}b^2)h \\ &+b\sqrt{h}\xi + \tfrac{1}{2}b^2 h\xi^2 + \tfrac{1}{2}(2c - c^2) + bc\sqrt{h}\xi + \tfrac{1}{2}c^2|] \\ &+\sum_{k=2}^{\infty}\tfrac{(\lambda h)^k}{k!}\mathbb{E}[\ln|1 + (a - \tfrac{1}{2}b^2)h + b\sqrt{h}\xi + \tfrac{1}{2}b^2 h\xi^2 + \tfrac{1}{2}(2c - c^2)k \\ &+bc\sqrt{h}\xi k + \tfrac{1}{2}c^2 k^2|]. \end{aligned} \tag{16}$$

Using *Lemma* 4.2 with $\varphi = \ln|\cdot|$ yields

$$\begin{aligned} &\mathbb{E}[\ln|1 + (a - \tfrac{1}{2}b^2)h + b\sqrt{h}\xi + \tfrac{1}{2}b^2 h\xi^2|] \\ &= (a - \tfrac{1}{2}b^2)h + \tfrac{1}{2}b^2 h - \tfrac{1}{2}b^2 h + o(h) = (a - \tfrac{1}{2}b^2)h + o(h). \end{aligned} \tag{17}$$

To discuss the second term at the right side of (16), we consider the following two different cases:

Case 1: $c \neq -1$ then

$$\begin{aligned} &\lambda h\mathbb{E}[\ln|1 + (a - \tfrac{1}{2}b^2)h + b\sqrt{h}\xi + \tfrac{1}{2}b^2 h\xi^2 + \tfrac{1}{2}(2c - c^2) + bc\sqrt{h}\xi + \tfrac{1}{2}c^2|] \\ &= \lambda h(\ln|1 + c| + \mathbb{E}[\ln|1 + \tfrac{a - \frac{1}{2}b^2}{1+c}h + b\sqrt{h}\xi + \tfrac{b^2 h\xi^2}{2(1+c)}|]). \end{aligned}$$

Recalling the fundamental inequality

$$\ln|u| \le |u-1|,\ \ u \ge 1. \tag{18}$$

Thus

$$\begin{aligned}&|\mathbb{E}[\ln|1+\tfrac{a-\frac{1}{2}b^2}{1+c}h+b\sqrt{h}\xi+\tfrac{b^2h\xi^2}{2(1+c)}|]|\\ \le\ &\mathbb{E}[\ln(1+|\tfrac{a-\frac{1}{2}b^2}{1+c}h+b\sqrt{h}\xi+\tfrac{b^2h\xi^2}{2(1+c)}|)]\\ \le\ &|\tfrac{a-\frac{1}{2}b^2}{1+c}|h+|b|\sqrt{h}\mathbb{E}|\xi|+\tfrac{b^2h}{2|1+c|}\mathbb{E}\xi^2=O(\sqrt{h}),\end{aligned}$$

which implies

$$\begin{aligned}&\lambda h\mathbb{E}[\ln|1+(a-\tfrac{1}{2}b^2)h+b\sqrt{h}\xi+\tfrac{1}{2}b^2h\xi^2\\ &+\tfrac{1}{2}(2c-c^2)+bc\sqrt{h}\xi+\tfrac{1}{2}c^2|]\\ &=\lambda h\ln|1+c|+o(h).\end{aligned} \tag{19}$$

Without loss of generality, we suppose $0<h\le h_1<1$. By using the inequality (18), the third term in the expansion of (16) becomes

$$\begin{aligned}&|\textstyle\sum_{k=2}^{\infty}\frac{(\lambda h)^k}{k!}\mathbb{E}[\ln|1+(a-\frac{1}{2}b^2)h+b\sqrt{h}\xi+\frac{1}{2}b^2h\xi^2+\frac{1}{2}(2c-c^2)k\\ &+bc\sqrt{h}\xi k+\tfrac{1}{2}c^2k^2|]|\\ &<(\lambda h)^2[(|a-\tfrac{1}{2}b^2|h+\tfrac{2|b|}{\sqrt{2\pi}}\sqrt{h}+\tfrac{1}{2}b^2h)\textstyle\sum_{k=2}^{\infty}\frac{\lambda^{k-2}}{k!}\\ &+(\tfrac{1}{2}|2c-c^2|+\tfrac{2}{\sqrt{2\pi}}|bc|\sqrt{h})\textstyle\sum_{k=2}^{\infty}\frac{\lambda^{k-2}}{(k-1)!}+\frac{1}{2}c^2\sum_{k=2}^{\infty}\frac{\lambda^{k-2}k}{(k-1)!}]\\ &<(\lambda h)^2[(|a-\tfrac{1}{2}b^2|h+\tfrac{2|b|}{\sqrt{2\pi}}\sqrt{h}+\tfrac{1}{2}b^2h)M_1+(\tfrac{1}{2}|2c-c^2|\\ &+\tfrac{2}{\sqrt{2\pi}}|bc|\sqrt{h})M_2+\tfrac{1}{2}c^2M_3]=o(h).\end{aligned} \tag{20}$$

where $\sum_{k=2}^{\infty}\frac{\lambda^{k-2}}{k!}\le M_1, \sum_{k=2}^{\infty}\frac{\lambda^{k-2}}{(k-1)!}\le M_2, \sum_{k=2}^{\infty}\frac{\lambda^{k-2}k}{(k-1)!}\le M_3$. Here the fact has been used that $\mathbb{E}|\xi|=2/\sqrt{2\pi},\ \mathbb{E}|\xi|^2=1$. Combining (17), (19), (20) and (16) yields

$$\begin{aligned}&e^{\lambda h}\mathbb{E}[\ln|1+(a-\tfrac{1}{2}b^2)h+b\sqrt{h}\xi+\tfrac{1}{2}b^2h\xi^2+\tfrac{1}{2}(2c-c^2)\Delta N_n\\ &+bc\sqrt{h}\xi\Delta N_n+\tfrac{1}{2}c^2(\Delta N_n)^2|]\\ &=(a-\tfrac{1}{2}b^2+\lambda\ln|1+c|)h+o(h).\end{aligned} \tag{21}$$

From (21), the sign of $\mathbb{E}[\ln|1+(a-\frac{1}{2}b^2)h+b\sqrt{h}\xi+\frac{1}{2}b^2h\xi^2+\frac{1}{2}(2c-c^2)\Delta N_n+bc\sqrt{h}\xi\Delta N_n+\frac{1}{2}c^2(\Delta N_n)^2|]$ is same to the sign of $a-\frac{1}{2}b^2+\lambda\ln|1+c|$ when h is sufficiently small, and hence the assertion follows.

Case 2: $c=-1$. In this case, we will show that the Milstein method (2) is also asymptotically stable with sufficiently small stepsize h for all values of a, b and $\lambda>0$. It follows from (16) that

$$\begin{aligned}&e^{\lambda h}\mathbb{E}[\ln|1+(a-\tfrac{1}{2}b^2)h+b\sqrt{h}\xi+\tfrac{1}{2}b^2h\xi^2+\tfrac{1}{2}(2c-c^2)\Delta N_n\\ &+bc\sqrt{h}\xi\Delta N_n+\tfrac{1}{2}c^2(\Delta N_n)^2|]\\ &=\mathbb{E}[\ln|1+(a-\tfrac{1}{2}b^2)h+b\sqrt{h}\xi+\tfrac{1}{2}b^2h\xi^2|]+\lambda h\mathbb{E}[\ln|(a-\tfrac{1}{2}b^2)h\\ &+\tfrac{1}{2}b^2h\xi^2|]+\textstyle\sum_{k=2}^{\infty}\frac{(\lambda h)^k}{k!}\mathbb{E}[\ln|1+(a-\frac{1}{2}b^2)h+b\sqrt{h}\xi+\frac{1}{2}b^2h\xi^2\\ &-\tfrac{3k}{2}-b\sqrt{h}\xi k+\tfrac{1}{2}k^2|].\end{aligned} \tag{22}$$

By (18)

$$\begin{array}{l}\lambda h\mathbb{E}[\ln|(a-\frac{1}{2}b^2)h+\frac{1}{2}b^2h\xi^2|]\\ =\lambda h\mathbb{E}[\ln h+\ln|(a-\frac{1}{2}b^2)+\frac{1}{2}b^2\xi^2|]\\ \le\lambda h[\ln h+\mathbb{E}(|a-\frac{1}{2}b^2|+\frac{1}{2}b^2\xi^2)]\\ =\lambda h[\ln h+|a-\frac{1}{2}b^2|+\frac{1}{2}b^2].\end{array}\tag{23}$$

In a similar way as in deriving (20), we have

$$\begin{array}{l}|\sum_{k=2}^{\infty}\frac{(\lambda h)^k}{k!}\mathbb{E}[\ln|1+(a-\frac{1}{2}b^2)h+b\sqrt{h}\xi+\frac{1}{2}b^2h\xi^2\\ -\frac{3k}{2}-b\sqrt{h}\xi k+\frac{1}{2}k^2|]|=o(h).\end{array}\tag{24}$$

Combining (17), (23), (24) with (22) gives

$$\begin{array}{l}\mathbb{E}[\ln|1+(a-\frac{1}{2}b^2)h+b\sqrt{h}\xi+\frac{1}{2}b^2h\xi^2+\frac{1}{2}(2c-c^2)\Delta N_n\\ +bc\sqrt{h}\xi\Delta N_n+\frac{1}{2}c^2(\Delta N_n)^2|]\\ \le e^{-\lambda h}\mathbb{E}[\ln|1+(a-\frac{1}{2}b^2)h+b\sqrt{h}\xi+\frac{1}{2}b^2h\xi^2|]+\lambda h(\ln h+|a-\frac{1}{2}b^2|+\frac{1}{2}b^2)\\ +\sum_{k=2}^{\infty}\frac{(\lambda h)^k}{k!}\mathbb{E}[\ln|1+(a-\frac{1}{2}b^2)h+b\sqrt{h}\xi+\frac{1}{2}b^2h\xi^2-\frac{3k}{2}-b\sqrt{h}\xi k+\frac{1}{2}k^2|]\\ =e^{-\lambda h}[(a-\frac{1}{2}b^2)h+\lambda h(\ln h+|a-\frac{1}{2}b^2|+\frac{1}{2}b^2)+o(h)].\end{array}$$

For all sufficiently small h, we immediately obtain

$$\begin{array}{l}\mathbb{E}[\ln|1+(a-\frac{1}{2}b^2)h+b\sqrt{h}\xi+\frac{1}{2}b^2h\xi^2+\frac{1}{2}(2c-c^2)\Delta N_n\\ +bc\sqrt{h}\xi\Delta N_n+\frac{1}{2}c^2(\Delta N_n)^2|]<0.\end{array}$$

Combining these different cases together leads the desired results. The proof is completed. □

5. Asymptotical stability of the weak Milstein method

In section 3, we have discussed the mean square stability of the weak Milstein method (4). In this section, we will investigate whether the weak Milstein method (4) can share the asymptotical stability of the system (1).

Theorem 5.1. *Given* a,b,c,λ, *the system* (1) *is asymptotically stable if and only if there exists a* $\widehat{h}_1^*>0$, *the weak Milstein method* (4) *is asymptotically stable for all* $0<h<\widehat{h}_1^*$.

Proof. In view of *Lemma* 5.1 of [5], the weak Milstein method (4) is asymptotically stable if and only if

$$\begin{array}{l}\mathbb{E}[\ln|1+(a-\frac{1}{2}b^2)h+b\sqrt{h}\widehat{\xi}+\frac{1}{2}b^2h\widehat{\xi}^2+\frac{1}{2}(2c-c^2)\widehat{\Delta N_n}\\ +bc\sqrt{h}\widehat{\xi}\widehat{\Delta N_n}+\frac{1}{2}c^2(\widehat{\Delta N_n})^2|]<0,\end{array}\tag{25}$$

where $\mathbb{P}(\widehat{\xi}=-1)=\mathbb{P}(\widehat{\xi}=1)=1/2$. Noticing $\widehat{\Delta N_n}$ comes from a two point distribution:$\mathbb{P}(\widehat{\Delta N_n}=0)=1-\lambda h,\ \mathbb{P}(\widehat{\Delta N_n}=1)=\lambda h$, we have

$$\begin{array}{l}\mathbb{E}[\ln|1+(a-\frac{1}{2}b^2)h+b\sqrt{h}\widehat{\xi}+\frac{1}{2}b^2h\widehat{\xi}^2+\frac{1}{2}(2c-c^2)\widehat{\Delta N_n}\\ +bc\sqrt{h}\widehat{\xi}\widehat{\Delta N_n}+\frac{1}{2}c^2(\widehat{\Delta N_n})^2|]\\ =(1-\lambda h)\mathbb{E}[\ln|1+(a-\frac{1}{2}b^2)h+b\sqrt{h}\widehat{\xi}+\frac{1}{2}b^2h\widehat{\xi}^2|]+(\lambda h)\mathbb{E}[\ln|1+\\ (a-\frac{1}{2}b^2)h+b\sqrt{h}\widehat{\xi}+\frac{1}{2}b^2h\widehat{\xi}^2+\frac{1}{2}(2c-c^2)+bc\sqrt{h}\widehat{\xi}+\frac{1}{2}c^2|]\end{array}\tag{26}$$

Note that for any $h \in (0,\ \widehat{h}_1)$ with $\widehat{h}_1 = \min\{1/(|a - \frac{1}{2}b^2| + |b| + \frac{1}{2}b^2)^2,\ 1\}$,

$$|(a - \frac{1}{2}b^2)h + b\sqrt{h}\widehat{\xi} + \frac{1}{2}b^2h\widehat{\xi}^2| < (|a - \frac{1}{2}b^2| + |b| + \frac{1}{2}b^2)\sqrt{h} < 1.$$

Here the fact has been used that $\mathbb{E}\widehat{\xi} = 0$, $\mathbb{E}|\widehat{\xi}| = 1$, $\mathbb{E}|\widehat{\xi}|^2 = 1$. Thus using Taylor expansion gives

$$\begin{aligned}
&\mathbb{E}[\ln|1 + (a - \tfrac{1}{2}b^2)h + b\sqrt{h}\widehat{\xi} + \tfrac{1}{2}b^2h\widehat{\xi}^2|]\\
&= \mathbb{E}((a - \tfrac{1}{2}b^2)h + b\sqrt{h}\widehat{\xi} + \tfrac{1}{2}b^2h\widehat{\xi}^2) - \tfrac{1}{2}\mathbb{E}((a - \tfrac{1}{2}b^2)h\\
&+b\sqrt{h}\widehat{\xi} + \tfrac{1}{2}b^2h\widehat{\xi}^2)^2 + \tfrac{1}{3}\mathbb{E}[((a - \tfrac{1}{2}b^2)h + b\sqrt{h}\widehat{\xi} + \tfrac{1}{2}b^2h\widehat{\xi}^2)\varsigma]^3\\
&= (a - \tfrac{1}{2}b^2)h + o(h),
\end{aligned} \tag{27}$$

where $0 < \varsigma < 1$.
To discuss the second term in the expansion of (26), let us discuss the following two possible cases:
Case 1: $c \neq -1$. In this case, similarly to (19), we have

$$\begin{aligned}
&(\lambda h)\mathbb{E}[\ln|1 + (a - \tfrac{1}{2}b^2)h + b\sqrt{h}\widehat{\xi} + \tfrac{1}{2}b^2h\widehat{\xi}^2 + \tfrac{1}{2}(2c - c^2)\\
&+bc\sqrt{h}\widehat{\xi} + \tfrac{1}{2}c^2|]\\
&= \lambda h\ln|1 + c| + \lambda h\mathbb{E}[\ln|1 + \tfrac{(a-\frac{1}{2}b^2)h}{1+c} + b\sqrt{h}\widehat{\xi} + \tfrac{b^2h\widehat{\xi}^2}{2(1+c)}|]\\
&= \lambda h\ln|1 + c| + o(h).
\end{aligned} \tag{28}$$

Combing (27), (28) and (26) yields

$$\begin{aligned}
&\mathbb{E}[\ln|1 + (a - \tfrac{1}{2}b^2)h + b\sqrt{h}\widehat{\xi} + \tfrac{1}{2}b^2h\widehat{\xi}^2 + \tfrac{1}{2}(2c - c^2)\widehat{\Delta N_n}\\
&+bc\sqrt{h}\widehat{\xi}\widehat{\Delta N_n} + \tfrac{1}{2}c^2(\widehat{\Delta N_n})^2|]\\
&= [a - \tfrac{1}{2}b^2 + \lambda\ln|1 + c|]h + o(h).
\end{aligned}$$

With sufficiently small stepsize h, we find that (25) holds if and only if

$$a - \tfrac{1}{2}b^2 + \lambda\ln|1 + c| < 0.$$

Case 2: c= -1. In this case, in a similar way as in deriving (23), we have

$$\begin{aligned}
&\lambda h\mathbb{E}[\ln|1 + (a - \tfrac{1}{2}b^2)h + b\sqrt{h}\widehat{\xi} + \tfrac{1}{2}b^2h\widehat{\xi}^2 + \tfrac{1}{2}(2c - c^2)\\
&+bc\sqrt{h}\widehat{\xi} + \tfrac{1}{2}c^2|] \leq \lambda h[\ln h + |a - \tfrac{1}{2}b^2| + \tfrac{1}{2}b^2].
\end{aligned} \tag{29}$$

Substituting (27), (29) into (26), for sufficiently small stepsize h, we have

$$\begin{aligned}
&\mathbb{E}[\ln|1 + (a - \tfrac{1}{2}b^2)h + b\sqrt{h}\widehat{\xi} + \tfrac{1}{2}b^2h\widehat{\xi}^2 + \tfrac{1}{2}(2c - c^2)\widehat{\Delta N_n}\\
&+bc\sqrt{h}\widehat{\xi}\widehat{\Delta N_n} + \tfrac{1}{2}c^2(\widehat{\Delta N_n})^2|]\\
&\leq (1 - \lambda h)\mathbb{E}[\ln|1 + (a - \tfrac{1}{2}b^2)h + b\sqrt{h}\widehat{\xi} + \tfrac{1}{2}b^2h\widehat{\xi}^2|]\\
&+\lambda h[\ln h + |a - \tfrac{1}{2}b^2| + \tfrac{1}{2}b^2]\\
&= (a - \tfrac{1}{2}b^2)h + \lambda h[\ln h + |a - \tfrac{1}{2}b^2| + \tfrac{1}{2}b^2] + o(h) < 0.
\end{aligned}$$

Thus, combining these two different cases completes the proof. □

Theorem 4.1 and *Theorem* 5.1 show that the strong Milstein method (2) and the weak Milstein method (4) can well reproduce the asymptotical stability of the system (1).

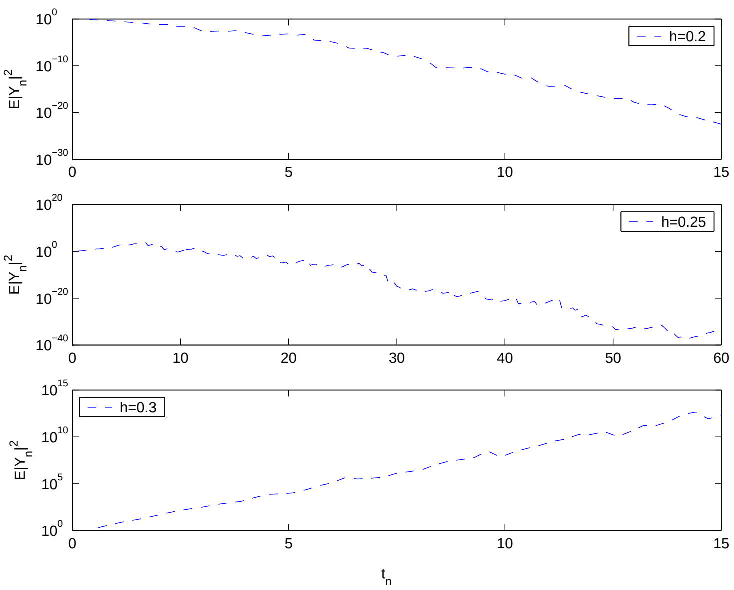

FIGURE 1. The mean square stability of the strong Milstein method.

6. Numerical experiments

Consider the linear equation with real and scalar coefficients

$$dx(t) = ax(t^-)dt + bx(t^-)dW(t) + cx(t^-)dN(t),\ t > 0 \tag{30}$$

with $x(0) = 1$.

It is know that (30) has the solution[8]

$$x(t) = (1 + c)^{N(t)} \exp[(a - \frac{1}{2}b^2)t + bW(t)]. \tag{31}$$

Now we test the mean square stability behavior of the Milstein method. In the experiments examining the mean square stability, the expectations are estimated by averaging 2000 different discretized Brownian paths. In the figures, the blue broken lines represent the numerical solutions produced by the Milstein method and we plot the graphs with the vertical axis scaled logarithmically.

In Figures 1, 2, we consider the equation (30) with the parameters $a = -5,\ b = 1,\ c = -1.1,\ \lambda = 2$ which satisfy the condition (3), hence the system is mean square stable.

In Figure 1, we see that the strong Milstein method (2) is mean square stable on $h = 0.2,\ 0.25$. With stepsize increasing to 0.3, the solution produced by the strong Milstein method grows rapidly to the scale of 10^{12} and instability is displayed without doubt. It is shown that the strong Milstein method (2) is unstable if h is large enough and mean square stable if $h \leq 0.25$.

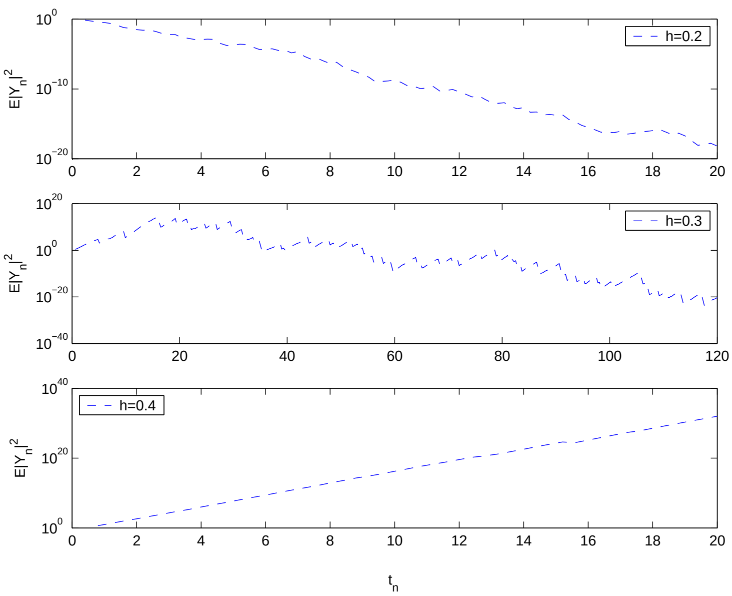

FIGURE 2. The mean square stability of the weak Milstein method.

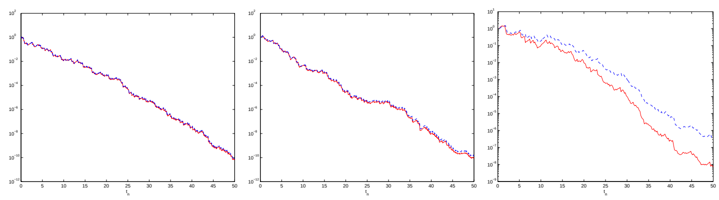

FIGURE 3. The asymptotical stabilities of the stable explicit solution and the strong Milstein method. *left*: $h = 0.01$; *middle* : $h = 0.02$; *right*: $h = 0.2$.

In Figure 2, we draw the numerical solutions produced by the weak Milstein method (4). From *Theorem* 3.1, we compute $\widehat{h}_0(a, b, c, \lambda) = 0.3195$. Figure 2 shows that the weak Milstein method (4) is mean square stable on $h = 0.2,\ 0.3 < 0.3195$ and unstable on $h = 0.4 > 0.3195$. Hence we conclude that the weak Milstein method (4) is mean square stable when $h < 0.3195$.

In order to show the asymptotical stability of the Milstein method, we plot the numerical solutions and the explicit solution (31) by one sample trajectory in all the figures. we use the red real lines and the blue broken lines to represent

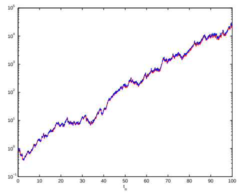

FIGURE 4. The asymptotical stabilities of the unstable explicit solution and the strong Milstein method. *left*: $h = 0.01$; *middle* : $h = 0.02$; *right*: $h = 0.2$.

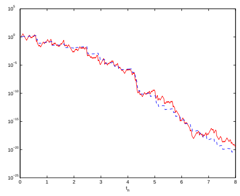

FIGURE 5. The asymptotical stabilities of the stable explicit solution and the weak Milstein method. *left*: $h = 0.02$; *middle* : $h = 0.04$; *right*: $h = 0.1$.

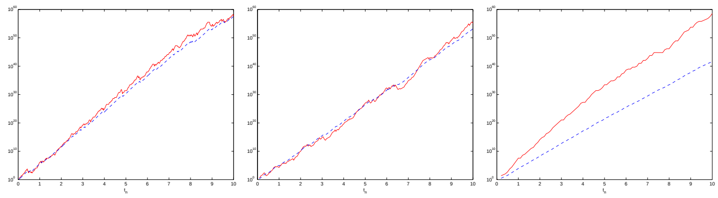

FIGURE 6. The asymptotical stabilities of the unstable explicit solution and the weak Milstein method. *left*: $h = 0.02$; *middle* : $h = 0.04$; *right*: $h = 0.1$.

the explicit solution and the numerical solutions respectively in all the pictures. All the the graphs are drawn with the vertical axis scaled logarithmically.

To verify our result concerning asymptotical stability for the strong Milstein method, we take the coefficients $a = 0.5,\ b = 0.1,\ c = -0.2,\ \lambda = 4$. Note that these parameters satisfy the condition (7), thus the system is asymptotically stable. In Figure 3, we plot the explicit solution (31) and the numerical solutions produced by the strong Milstein method (2) with different stepsize h. We replace the parameter $c = -0.2$ by $c = -0.1$, then the the coefficients $a = 0.5,\ b = 0.1,\ c = -0.1,\ \lambda = 4$ don't satisfy the condition (7), so the system becomes unstable. In this case, we draw the explicit solution (31) and the numerical solutions of the strong Milstein method (2) changing the stepsize h in Figure 4.. Figures 3 and 4 suggest that the two curves match well on $h = 0.01,\ 0.02$, whereas, when for some large stepsize $h = 0.2$, the two curves can't match well. We conclude that the strong Milstein method (2) can well behave the same stability of the explicit solution (31) with sufficiently small stepsize $h \leq 0.02$.

To examine the asymptotical stability of the weak Milstein method (4), we choose $a = 1,\ b = 0.4,\ c = -0.8,\ \lambda = 4$ which satisfy the condition (7), so the system is asymptotically stable. We change the parameter $a = 1$ with $a = 20$ which leads to the unstability of the system. In Figure 6, we simulate the explicit solution (31) and the numerical solutions of the weak Milstein method by the different stepsize h. In Figures 5 and 6, we find that the two curves match well on $h = 0.02,\ 0.04$, whereas, when for some large stepsize $h = 0.1$, the two curves can't match well. This implies that the weak Milstein method (4) can well reproduce the same stability of the explicit solution (31) with sufficiently small stepsize $h \leq 0.04$.

7. Conclusion

In this paper, we have investigated the mean square stability and the asymptotical stability of the strong Milstein method and the weak Milstein method for the stochastic differential equations with jumps. The forgoing stability results show that both kinds of the Milstein methods can reproduce the mean square stability. Moreover, it is shown that the asymptotical stability for stochastic jump-diffusion differential equations is inherited by the two kinds of Milstein methods with sufficiently small stepsizes h.

References

1. J. A. D. Appleby, G. Berkolaiko and A. Rodkina, *Non-exponential stability and decay rates in non-linear stochastic difference equation with unbounded noises*, Stochastics. **81**(2009), 99-127.
2. K. Burrage, P. Burrage and T. Mitsui, *Numerical solutions of stochastic differential equations-implementation and stability issues*, J. Comput. Appl. Math. **125**(2000), 171-182.
3. G. D. Chalmers and D. J. Higham, *Asymptotic stability of a jump-diffusion equation and its numerical approximation*, SIAM J. Sci. Comput. **31**(2008), 1141-1155.
4. W. R. Cao and M. Z. Liu, *T-stability of the Euler-maruyama numerical method for the stochastic differential delay equations*, Journal of Harbin Institute of Technology. **3**(2005), 303-306.(In Chinese)
5. D. J. Higham, *Mean-square and asymptotical stability of the stochastic theta method*, SIAM J. Numer. Anal. **38**(2000), 753-769.
6. D. J. Higham and P. E. Kloeden, *Convergence and stability of implicit methods for jump-diffusion systems*, Int. J. Numer. Anal. Mod. **3**(2006), 125-140.
7. D. J. Higham, X. R. Mao and A. M. Stuart, *Exponential mean-square stability of numerical solutions to stochastic differential equations*, London Mathematical Society J. Comput. Math. **6**(2003), 297-313.

8. D. Lamberton and B. Lapeyre, *Introduction to Stochastic Calculus Applied to Finance*, Chapman and Hall, London, 1996.
9. M. Z. Liu, W. R. Cao and Z. C. Fan, *Convergence and stability of the semi-implicit Euler method for a linear stochastic differential delay equation*, J. Comput. Appl. Math. **170**(2004), 255-268.
10. R. H. Li, H. B. Meng and Y. H. Dai, *Convergence of numerical solutions to stochastic delay differential equations with jumps*, Appl. Math. Comput. **172**(2006), 584-602.
11. Y. Maghsoodi, *Mean square efficient numerical solution of jump-diffusion stochastic differential equations*, Indian J. Statis. **58**(1996), 25-47.
12. G. N. Milstein, *Numerical integration of stochastic differential equations*, Kulwer Acadenic, London, 1998.
13. A. Rathinasamy and K. Balachandran, *Mean-square stability of Milstein method for linear hybrid stochastic delay integro-differential equations*, Nonlinear Anal. Hybrid Syst. **2**(2008), 1256-1263.
14. Y. Saito and T. Mitsui, *T-stability of numerical schemes for stochastic differential equations*, World Sci. Ser. Appl. Anal. **2**(1993), 333-344.
15. L. S. Wang, C. L. Mei and H. Xue, *The semi-implicit Euler method for the stochastic differential delay equations with jumps*, Appl. Math. Comput. **192**(2007), 567-578.
16. Z. Y. Wang and C. J. Zhang, *An analysis of stability of Milstein method for stochastic differential equations with delay*, J. Comput. Math. Appl. **51**,(2006), 1445-1452.
17. H. M. Zhang, S. Q. Gan and L. Hu, *The split-step backward Euler method for linear stochastic delay differential equations*, J. Comput. Appl. Math. **225**(2009), 558-568.

Lin Hu received M.Sc. from Harbin Institute of Technology University. Since 2002 she has been at Northeast Forestry University. Her research interests include numerical numerical analysis on the stochastic differential equations.

1.Department of Mathematical Sciences and Computing Technology, Central South University, Changsha, Hunan 410075, P.R.China; 2.College of Science, Northeast Forestry University, Harbin, Heilongjiang 150040, P.R.China.
e-mail: littleleave05@163.com

Siqing Gan received M.Sc. from Xiangtan University, and Ph.D. from Academy of Mathematics and Systems Science, Chinese Academy of Sciences. He is currently a professor at Central South University since 2003. His research interests include numerical analysis of deterministic and stochastic differential equations.

Department of Mathematical Sciences and Computing Technology, Central South University, Changsha, Hunan 410075, P.R.China.
e-mail: siqinggan@yahoo.com.cn