J. Appl. Math. & Informatics Vol. 29(2011), No. 5 - 6, pp. 1351 - 1361
Website: http://www.kcam.biz

ON THE CONVERGENCE FOR ND RANDOM VARIABLES WITH APPLICATIONS†

JONG-IL BAEK* AND HYE-YOUNG SEO

Abstract. We in this paper study the complete convergence and almost surely convergence for arrays of rowwise pairwise negatively dependent(ND) random variables ($r.v.'s$) which are dominated randomly by some random variables and obtain a result dealing with complete convergence of linear processes.

AMS Mathematics Subject Classification : 60F05, 62E10, 45E10.
Keywords : Negatively dependent random variables, complete convergence, almost sure convergence, arrays.

1. Introduction

Lehmann(1996) first investigated various concepts of positive and negative dependence in the bivariate case. Esary et al.(1967) introduced a concept of association implying a strong form of positive dependence. Also, Esary and Proschan(1972) introduced strong definitions of bivariate positive and negative dependence. Their concept has been very useful in reliability theory and applications. Harris(1970),Brindly and Thompson(1972) initiated multivariate generalizations for concepts of dependence. Ebrahimi and Ghosh(1981), Block and Ting(1981) developed these concepts. In addition, for other related negative dependence concepts, many authors had been generalized and extended in several directions;(see Jogdeo and Patil(1975), Karlin and Rinott(1980), Joag-Dev and Proschan(1983), Matula(1992), Bozorgnia et al.(1996), Chandra and Ghosal(1996a, 1996b), Amini et al.(2004), Bingham and Nili Sani(2004), and Chen and Zhang(2007)). In particular, Hu et al.(1989) had obtained the following result on complete convergence and they had established Theorem 1.1 for non identically random variables when assumptions of independence between rowwise of the array is made and Thrum(1974) had obtained the following almost surely convergence in conditions of Theorem 1.2.

Received April 18, 2011. Revised, June 6, 2011. Accepted June 10, 2011. *Corresponding author †This paper was supported by Wonkwang University Research Grant in 2011.

Theorem 1.1. *Let $\{X_{nk}|1 \leq k \leq n, n \geq 1\}$ be an array of rowwise independent random variables with $EX_{nk} = 0$. Suppose that $\{X_{nk}|1 \leq k \leq n, n \geq 1\}$ are uniformly bounded by some random variable X. If $E|X|^{2p} < \infty$ for some $1 \leq p < 2$, then*

$$n^{-1/p} \sum_{k=1}^{n} X_{nk} \to 0 \quad \text{completely as } n \to \infty$$

if and only if $E|X_{11}|^{2p} < \infty$.

Theorem 1.2. *Let $\{X_i|i \geq 1\}$ be a sequence of independent identically distributed (i.i.d.) $r.v.'s$ with $EX_1 = 0$ and let $E|X_1|^{2/\alpha} < \infty$ for some $0 < \alpha \leq 1$, $\sum_{i=1}^{n} a_{ni}^2 = 1$ and $|a_{ni}| \leq n^{-\alpha/2}$.*
Then,

$$\sum_{i=1}^{n} a_{nk} X_i \to 0 \ a.s \quad \text{as } n \to \infty.$$

The main goal of our paper is to establish the above results for arrays of rowwise pairwise ND $r.v.'s$. As an application, the corresponding results of Zhang(1996) on *i.i.d.* random variables are extended to the ND setting. This paper is organized as follows. In Section 2, we provide some definitions and lemmas used in the proof of the main theorems. In Section 3, we derive a general result for the complete convergence and strong convergence of weighted sums of arrays of rowwise pairwise ND $r.v.'s$ which are dominated randomly by some random variables. In Section 4, we obtain a result dealing with complete convergence of linear processes. Finally, we recall that a sequence $\{X_n|n \geq 1\}$ of random variables is called stochastically bounded by a random variable X if there exists a positive constant C such that $P(|X_n| > t) \leq CP(|X| > t)$ for all $n \geq 1, t > 0$. In this case we write $\{X_n\} < X$. Throughout this paper, C represents positive constants whose values may change from one place to another.

2. Preliminaries

For the proof of the theorems, we need to restate some definitions and lemmas for easy reference.

Definition 2.1 (Ebrahimi et al. [11])**.** Random variables X and Y are ND if

$$P[X \leq x,\ Y \leq y] \leq P[X \leq x]\ P[Y \leq y] \tag{1}$$

for all $x, y \in R$. A collection of random variables is said to be pairwise ND if every pair of random variables in the collection satisfies (1).
It is important to note that Definition 2.1 implies

$$P[X > x,\ Y > y] \leq P[X > x]\ P[Y > y] \tag{2}$$

for the $x, y \in R$. Moreover, it follows that (2) implies (1); hence, they are equivalent for pairwise ND $r.v's$. Ebrahimi et al.(1981) showed that(1) and (2) are not equivalent for random vectors of dimension greater than 2. Consequently, we need the following definition to define sequences of negatively dependent random variables.

Definition 2.2. The random variables $X_1, X_2, \cdots$ are said to be
(a) lower negatively dependent (LND) if for each n

$$P[X_1 \leq x_1, \ldots, X_n \leq x_n] \leq \prod_{i=1}^{n} P[X_i \leq x_i] \tag{3}$$

for all $x_1, \ldots, x_n \in R$,
(b) upper negatively dependent (UND) if for each n

$$P[X_1 > x_1, \ldots, X_n > x_n] \leq \prod_{i=1}^{n} P[X_i > x_i] \tag{4}$$

for all $x_1, \ldots, x_n \in R$,
(c) negatively dependent (ND) if both (3) and (4) hold.

The following properties are listed for reference in obtaining the main result in the next section and detailed proofs can be found in their paper.

Lemma 2.1 (Ebrahimi et al.[11]). *If $\{X_n | n \geq 1\}$ is a sequence of ND r.v.'s and $\{f_n | n \geq 1\}$ is a sequence of monotone increasing with Borel functions, then $\{f_n(X_n) | n \geq 1\}$ is a sequence of ND r.v.'s.*

Lemma 2.2 (Taylor et al.[21]). *(a) If $X_1, \ldots, X_n$ are ND r.v.'s and for any real numbers $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ such that $a_i < b_i$, $1 \leq i \leq n$, then $\{Y_i,\ 1 \leq i \leq n\}$ are ND r.v.'s, where $Y_i = b_i I(X_i > b_i) + X_i I(a_i \leq X_i \leq b_i) + a_i I(X_i < a_i)$. (b) If $X_1, \ldots, X_n$ are pairwise ND r.v.'s, then $EX_iX_j \leq EX_iEX_j, i \neq j$.*

Lemma 2.3 (Burton and Dehling[7]). *Let $\sum_{i=-\infty}^{\infty} a_i$ be an absolutely convergent series of real numbers with $a = \sum_{i=-\infty}^{\infty} a_i, b = \sum_{i=-\infty}^{\infty} |a_i|$. Suppose $\Phi : [-b, b] \to R$ is a function satisfying the following conditions:*
(i) Φ is bounded and continuous at a.
(ii) There exist $\delta > 0$ and $C > 0$ such that for all $|x| \leq \delta, |\Phi(x)| \leq C|x|$.
Then $\lim_{n\to\infty} \frac{1}{n} \sum_{i=-\infty}^{\infty} \Phi(\sum_{j=i+1}^{i+n} a_j) = \Phi(a)$.

Remark 2.1. Taking $\Phi(x) = |x|^q, q \geq 1$, from Lemma 2.3 we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=-\infty}^{\infty} |\sum_{j=i+1}^{i+n} a_j|^q = |a|^q.$$

3. Main Results

This result extends and generalizes the result of Hu et al.(1989) from the *i.i.d.* case to the ND *r.v.'s*

Theorem 3.1. *Let $\alpha > 1/2$, $0 < p < 2$, $\alpha p \geq 1$ and let $\{X_{ni}|1 \leq i \leq n, n \geq 1\}$ be an array of rowwise pairwise ND r.v.'s with $EX_{ni} = 0$ for some $\alpha \leq 1$. Suppose that $h(x) > 0$ is a slowly varying function as $x \to \infty$ and let $h(x) \geq C > 0$ for $\alpha p = 1$.*

If

$$\{X_n\} < X \text{ and } E|X|^p h(|X|^{1/\alpha}) < \infty,$$

then we have

$$\sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P(|\sum_{i=1}^{n} X_{ni}| \geq \varepsilon n^{\alpha}) < \infty \text{ for all } \varepsilon > 0.$$

Proof. Let $Y_i = n^{\alpha} I(X_{ni} > n^{\alpha}) + X_{ni} I(|X_{ni}| \leq n^{\alpha}) - n^{\alpha} I(X_{ni} < -n^{\alpha})$.
Then for any $\varepsilon > 0$,

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P(|\sum_{i=1}^{n} X_{ni}| \geq \varepsilon n^{\alpha}) \\ &\leq \sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P(|\sum_{i=1}^{n} Y_i| \geq n^{\alpha}\varepsilon/2) \\ &+ \sum_{n=1}^{\infty} n^{\alpha p-2} h(n) \sum_{i=1}^{n} P(|X_{ni}| \geq n^{\alpha}) \\ &= A_n + B_n \text{ (say)} . \end{aligned}$$

It is omitted, since we can easily prove that $B_n < \infty$.
Next, in order to prove that $A_n < \infty$, we first show that

$$n^{-\alpha} |\sum_{i=1}^{n} EY_i| \to 0, \ n \to \infty.$$

$$\begin{aligned} & n^{-\alpha} |\sum_{i=1}^{n} EY_i| \\ \leq \quad & n^{-\alpha} \sum_{i=1}^{n} E|X_{ni}| I(|X_{ni}| \leq n^{\alpha}) + nP(|X_{ni}| > n^{\alpha}) \\ = \quad & I_1(n) + I_2(n) \text{ (say)} . \end{aligned}$$

It is omitted, since we also can easily get that $I_1(n) \to 0$, $n \to \infty$ and $I_2(n) \to 0$, $n \to \infty$.
Thus it suffices to show that

$$C_n = \sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P(|\sum_{i=1}^{n} (Y_i - EY_i)| \geq \varepsilon n^{\alpha}) < \infty \text{ for all } \varepsilon > 0.$$

Thus, $\{(Y_i - EY_i)|i \geq 1\}$ is still a sequence of ND $r.v.'s$ by Lemma 2.1. From Lemma 2.2, we get that for any $\varepsilon > 0$ and $0 < p < 2$,

$$\begin{aligned}
C_n &= \sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P(|\sum_{i=1}^{n}(Y_i - EY_i)| \geq \varepsilon n^{\alpha}) \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha(p-2)-2} h(n) \sum_{i=1}^{n} E|Y_i|^2 \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha(p-2)-2} h(n) \sum_{i=1}^{n} [E|X_{ni}|^2 I(|X_{ni}| \leq n^{\alpha}) + n^{2\alpha} P(|X_{ni}| > n^{\alpha})] \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha(p-2)-1} h(n) [E|X|^2 I(|X| \leq n^{\alpha}) + n^{2\alpha} P(|X| > n^{\alpha})] \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha(p-2)-1} h(n) [\sum_{k=1}^{n} k^{2\alpha} P(k^{\alpha} \leq |X| < (k+1)^{\alpha}) \\
&\quad + n^{2\alpha} \sum_{k=1}^{\infty} P(k^{\alpha} \leq |X| < (k+1)^{\alpha})] \\
&\leq C\sum_{k=1}^{\infty} k^{2\alpha} P(k^{\alpha} \leq |X| < (k+1)^{\alpha}) \sum_{n=k}^{\infty} n^{\alpha(p-2)-1} h(n) \\
&\quad + \sum_{k=1}^{\infty} P(k^{\alpha} \leq |X| < (k+1)^{\alpha}) \sum_{n=1}^{k} n^{\alpha p-1} h(n) \\
&\leq C\sum_{k=1}^{\infty} k^{\alpha p} h(k) P(k^{\alpha} \leq |X| < (k+1)^{\alpha}) \\
&\leq CE|X|^p h(|X|^{1/\alpha}) < \infty,
\end{aligned}$$

so, we have

$$C_n = \sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P(|\sum_{i=1}^{n}(Y_i - EY_i)| \geq \varepsilon n^{\alpha}) < \infty \text{ for all } \varepsilon > 0$$

The proof is complete. □

Taking $X_{ni} = X_i$ for $1 \leq i \leq n$ and $h(x) = log^{-2}n$ in Theorem 3.1, we can immediately obtain the following corollary.

Corollary 3.1. *Let $\alpha > 1/2$, $0 \leq p \leq 2$, $\alpha p \geq 1$ and let $\{X_i|i \geq 1\}$ be an identically distributed ND $r.v.'s$ with $EX_1 = 0$ for some $\alpha \leq 1$. If $E|X_1|^p < \infty$, then we have*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} log^{-2} n P(\sum_{i=1}^{n} |X_i| \geq \varepsilon n^{\alpha}) < \infty \text{ for all } \varepsilon > 0.$$

Remark 3.1. The condition of identical distribution can be weakened slightly to be uniformly bounded in probability. When $\{X_i|i \geq 1\}$ is a sequence of $i.i.d$ $r.v's$, if we take $p = 1/\alpha$ for some $0 < p < 1$ and $log^{-2}n = 1$, then Corollary 3.1 becomes the result of Bai and Su(1985).

Theorem 3.2. *Let $\{X_{ni}|1 \leq i \leq n, n \geq 1\}$ be an array of rowwise pairwise ND $r.v.'s$ with $EX_{ni} = 0$ and let $E|X|^{2/\alpha} < \infty$ for some $0 < \alpha \leq 1$, $|a_{ni}| \leq n^{-\alpha-\delta}$ for some $0 < \delta < \alpha/2$. If*

$$\{X_n\} < X,$$

then we have

$$\sum_{n=1}^{\infty} P(|\sum_{i=1}^{n} a_{ni}X_{ni}| > \varepsilon) < \infty \text{ for all } \varepsilon < 0.$$

Proof. Let $T_i = n^{-\delta}I(a_{ni}X_{ni} > n^{-\delta}) + a_{ni}X_{ni}I(|a_{ni}X_{ni}| \leq n^{-\delta}) - n^{-\delta}I(a_{ni}X_{ni} < -n^{-\delta})$. Then for any $\varepsilon > 0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} P(|\sum_{i=1}^{n} a_{ni}X_{ni}| > \varepsilon) \\
&\leq \sum_{n=1}^{\infty} P(\bigcup_{i=1}^{n} |a_{ni}X_{ni}| > n^{-\delta}) + \sum_{n=1}^{\infty} P(|\sum_{i=1}^{n} a_{ni}X_{ni}| > \varepsilon) \\
&\leq \sum_{n=1}^{\infty} P(\bigcup_{i=1}^{n} |a_{ni}X_{ni}| > n^{-\delta}) \\
&\quad + \sum_{n=1}^{\infty} P(|\sum_{i=1}^{n} (T_i - ET_i)| > \varepsilon/2) \\
&= D_n + E_n \text{ (say)}
\end{aligned}$$

It is omitted, since we can easily get that $D_n < \infty$.
Next, to prove $E_n < \infty$, we first need to show that

$$|\sum_{i=1}^{n} ET_i| \to 0, n \to \infty.$$

$$\begin{aligned}
|\sum_{i=1}^{n} ET_i| &\leq \sum_{i=1}^{n} E|a_{ni}X_{ni}|I(|a_{ni}X_{ni}| \leq n^{-\delta}) + n^{1-\delta}P(|a_{ni}X| > n^{-\delta}) \\
&= I_3(n) + I_4(n) \text{ (say)} .
\end{aligned}$$

As to $I_3(n)$, according to $EX_{ni} = 0$, we obtain

$$\begin{aligned}
I_3(n) &= \sum_{i=1}^{n} E|a_{ni}X_{ni}|I(|a_{ni}X_{ni}| \leq n^{-\delta}) \\
&\leq C\sum_{i=1}^{n} E|a_{ni}X|I(|X| > n^{\alpha})
\end{aligned}$$

$$\begin{aligned} &\leq C\sum_{i=1}^{n} n^{(2\delta/\alpha)-\delta} E|a_{ni}X|^{2/\alpha} \\ &\leq Cn^{-1-\delta} \to 0, n \to \infty, \end{aligned}$$

since $|a_{ni}X| = |a_{ni}X|^{2/\alpha}|a_{ni}X|^{1-2/\alpha}$, $|a_{ni}X|^{1-2/\alpha} = |a_{ni}|^{1-2/\alpha}|X|^{1-2/\alpha} \leq n^{(2\delta/\alpha)-\delta}$. As to $I_4(n)$,

$$\begin{aligned} I_4(n) &= n^{1-\delta} P(|a_{ni}X| > n^{-\delta}) \\ &\leq n^{1-\delta} P(|X| > n^{\alpha}) \\ &\leq n^{1-\delta} \sum_{k=n}^{\infty} P(k^{\alpha} \leq |X| < (k+1)^{\alpha}) \\ &\leq C \sum_{k=n}^{\infty} kP(k^{\alpha} \leq |X| < (k+1)^{\alpha}) \to 0, \end{aligned}$$

so for n large enough we have

$$|\sum_{i=1}^{n} ET_i| \to 0, n \to \infty. \tag{5}$$

Thus it suffices to show that for any $\varepsilon > 0$,

$$E_n{}^* = \sum_{n=1}^{\infty} P(|\sum_{i=1}^{n}(T_i - ET_i)| > \varepsilon/2) < \infty.$$

Note that $\{(T_i - ET_i)|1 \leq i \leq n, n \geq 1\}$ is still a sequence of ND $r.v.'s$ by Lemma 2.1. From Lemma 2.2, we get that for any $\varepsilon > 0$,

$$\begin{aligned} E_n{}^* &= \sum_{n=1}^{\infty} P(|\sum_{i=1}^{n}(T_i - ET_i)| > \varepsilon/2) \\ &\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n} E|T_i|^2 \\ &\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n}[E|a_{ni}X_{ni}|^2 I(|a_{ni}X_{ni}| \leq n^{-\delta} \\ &\quad + n^{-2\delta}P(|a_{ni}X_{ni}| > n^{-\delta})] \\ &\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n}[E|a_{ni}X|^2 I(|a_{ni}X| \leq n^{-\delta} \\ &\quad + n^{-2\delta}P(|a_{ni}X| > n^{-\delta})] \\ &\leq C\sum_{n=1}^{\infty} n^{-1-2\delta} E|X|^{2/\alpha} \\ &\leq C\sum_{n=1}^{\infty} n^{-1-2\delta} < \infty, \end{aligned} \tag{6}$$

since $|a_{ni}X|^2 = |a_{ni}X|^{2/\alpha}|a_{ni}X|^{2-2/\alpha} = |a_{ni}|^{2/\alpha}.|X|^{2/\alpha}|a_{ni}X|^{2-2/\alpha}$, $|a_{ni}|^{2/\alpha}.$
$|a_{ni}X|^{2-2/\alpha} \le n^{(-\alpha-\delta)2/\alpha} n^{(-\alpha-\delta)(2-2/\alpha)} n^{\alpha(2-2/\alpha)} = n^{(-2-2\delta)}$
Hence, by (5) and (6), the proof is complete. □

As to strong convergence, by weakening the degree of condition in Theorem 3.2 to its half, we can get the following result for ND $r.v.'s$.

Theorem 3.3. *Let* $\{X_{ni}|1 \le i \le n, n \ge 1\}$ *be an array of rowwise pairwise* ND $r.v.'s$ *with* $EX_{ni} = 0$ *and let* $E|X|^{2/\alpha} < \infty$ *for some* $0 < \alpha \le 1$, $|a_{ni}| \le n^{-\alpha/2-\delta}$ *for some* $0 < \delta < \alpha/2$. *If*

$$\{X_n\} < X,$$

then we have

$$\sum_{i=1}^{n} a_{ni}X_{ni} \to 0 \quad a.s. \ as \quad n \to \infty.$$

Proof. Let $S_i = \Gamma_n I(X_{ni} > \Gamma_n) + X_{ni}I(|X_{ni}| \le \Gamma_n) - \Gamma_n I(X_{ni} < -\Gamma_n)$, where $\Gamma_n = n^{-\delta/2}/a_{ni}$. Then

$$\begin{aligned}&\sum_{i=1}^{n} a_{ni}X_{ni}\\&= \sum_{i=1}^{n} a_{ni}(X_{ni} - S_i) + \sum_{i=1}^{n} a_{ni}(S_i - ES_i) + E\sum_{i=1}^{n} a_{ni}S_i\\&= F_n + G_n + H_n \text{ (say)}\end{aligned}$$

For any $\varepsilon > 0$,

$$\begin{aligned}&P(\sum_{i=1}^{\infty} a_{ni}(X_{ni} - S_i) > \varepsilon/3)\\&\le P(\bigcup_{i=1}^{\infty} X_{ni} \ne S_i)\\&\le \sum_{i=1}^{\infty} P(|a_{ni}X_{ni}| > n^{-\delta/2})\\&\le C\sum_{i=1}^{\infty} P(|a_{ni}X| > n^{-\delta/2})\\&\le C\sum_{i=1}^{\infty} n^{-1-\delta/\alpha}E|X|^{2/\alpha}\\&\le C\sum_{i=1}^{\infty} n^{-1-\delta/\alpha} < \infty,\end{aligned}$$

by Borel-Cantelli Lemma, we conclude that

$$F_n \to 0 \ a.s \ \text{ as } \ n \to \infty. \tag{7}$$

Note that $\{a_{ni}(S_i - ES_i)|1 \le i \le n, n \ge 1\}$ is still a sequence of ND $r.v.'s$ by Lemma 2.1. Using Lemma 2.2, we get that for any $\varepsilon > 0$,

$$\begin{aligned}
&P(\sum_{i=1}^{n} a_{ni}(S_i - ES_i) > \varepsilon/3) \\
&\le C \sum_{i=1}^{n} E|a_{ni}S_i|^2 \\
&\le C \sum_{i=1}^{n} [E|a_{ni}X_{ni}|^2 I(|a_{ni}X_{ni}| \le n^{-\delta/2}) + n^{-\delta} P(|a_{ni}X_{ni}| > n^{-\delta/2})] \\
&\le C \sum_{i=1}^{n} [E|a_{ni}X|^2 I(|a_{ni}X| \le n^{-\delta/2}) + n^{-\delta} P(|a_{ni}X| > n^{-\delta/2})] \\
&\le C \sum_{i=1}^{n} n^{-1-\delta-\delta/\alpha} E|X|^{2/\alpha} \\
&\le C n^{-\delta-\delta/\alpha} \to 0 \ \text{ as } \ n \to \infty,
\end{aligned}$$

since $|a_{ni}X|^2 = |a_{ni}X|^{2/\alpha}|a_{ni}X|^{2-2/\alpha} = |a_{ni}|^{2/\alpha}|X|^{2/\alpha}|a_{ni}X|^{2-2/\alpha}$, $|a_{ni}|^{2/\alpha}\cdot |a_{ni}X|^{2-2/\alpha} \le n^{-1-\delta-\delta/\alpha}$.
Hence, we obtain that

$$G_n \to 0 \ a.s \ \text{ as } \ n \to \infty. \tag{8}$$

Finally, by $EX_{ni} = 0$, we obtain

$$\begin{aligned}
H_n &= \sum_{i=1}^{n} E|a_{ni}S_i| \\
&\le \sum_{i=1}^{n} E|a_{ni}X_{ni}| I(|a_{ni}X_{ni}| \le n^{-\delta/2}) + \sum_{i=1}^{n} n^{-\delta/2} P(|a_{ni}X_{ni}| > n^{-\delta/2}) \\
&\le \sum_{i=1}^{n} E|a_{ni}X| I(|a_{ni}X| > n^{-\delta/2}) + \sum_{i=1}^{n} n^{-\delta/2} P(|a_{ni}X| > n^{-\delta/2}) \\
&\le C n^{-\delta/2-\delta/\alpha} E|X|^{2/\alpha} \\
&\le C n^{-\delta/2-\delta/\alpha} \to 0 \ \text{ as } \ n \to \infty, \qquad (9)
\end{aligned}$$

since $|a_{ni}X| = |a_{ni}X|^{2/\alpha}|a_{ni}X|^{1-2/\alpha}$, $|a_{ni}|^{2/\alpha}|a_{ni}X|^{1-2/\alpha} \le n^{-1-\delta/2-\delta/\alpha}$.
Hence, by (7), (8) and (9), the proof is complete. □

4. Complete convergence of linear processes

In this section, we present one result about the complete convergence of linear processes which follows from Theorem 3.1. We gave a general version of Zhang[22] from the identically distributed ϕ-mixing assumptions case to the ND $r.v.'s$.

Let $\{X_i, i \in Z\}$, where $Z_+ = \{1, 2, 3, \cdots\}$ denote a sequence of random variables and $\{a_i | i \in Z_+\}$ a sequence of real numbers with $\sum_{j=-\infty}^{\infty} |a_j| < \infty$. Define a linear process of the form

$$Y_i = \sum_{i=-\infty}^{\infty} a_{i+k} X_i,\ k \in Z_+, \quad \text{where } Z_+ = \{1, 2, 3, \ldots\}. \tag{10}$$

Theorem 4.1. *Assume that* $\{X_i| -\infty < i < \infty\}$ *is a sequence of rowwise pairwise ND r.v.'s with* $EX_i = 0$. *Let* $h(x) > 0$ *be a slowly varying function as* $x \to \infty$ *and* $r \geq 1, 1 \leq t < 2, h(x)$ *is nondecreasing function for* $r = 1$ *and* $\{Y_i | i \geq 1\}$ *be satisfied as in (10) of this section . If*

$$\{X_n\} < X \text{ and } E|X|^{rt} h(|X|^t) < \infty,$$

then we have

$$\sum_{n=1}^{\infty} n^{r-2} h(n) P(|\sum_{i=1}^{n} Y_i| \geq \varepsilon n^{1/t}) < \infty \text{ for all } \varepsilon > 0.$$

Proof. Let $X_{ni} = X_i$ and $Y_i = n^{1/t} I(a_{ni} X_i > n^{1/t}) + a_{ni} X_i I(|a_{ni} X_i| \leq n^{1/t}) - n^{1/t} a_{ni} I(a_{ni} X_i < -n^{1/t})$ and note that

$$(1/n^{1/t}) \sum_{i=1}^{n} Y_i = \sum_{i=-\infty}^{\infty} a_{ni} X_{ni} = \sum_{i=-\infty}^{\infty} a_{ni} X_i$$

By taking Lemma 2.3, Remark 2.1, and $p = rt, t = 1/\alpha$ and $\alpha p = r$ in Theorem 3.1, similarly to proof of Theorem 3.1, we can obtain the result of Theorem 4.1. The proof is complete. □

References

1. Amini, M., Bozorgnia, A., Zareei, H. (2004). *Strong convergence for weighted sums of negatively dependent random variables.* Wseas trans. Math. **3**, no. 3, 637-640.
2. Bai, J., Su, C.(1985). *The complete convergence for partial sums of IID random variables.* Scientia.Sinica **28**, 1261-1277.
3. Bingham, N. H., Nili Sani, H. R. (2004). *Summability methods and negatively associated random variables.* J. Appl. Probab. **41A**, 231-238.
4. Block, H. W., Ting, M. L. (1981). *Some concepts of multivariate dependence.* Comm. Stat. Ther. Math. **A10:8**, 749-762.
5. Bozorgnia, A., Patterson, R. F., Taylor, R. L. (1996). *Limit theorems for dependent random variables.* World Congress Nonlinear Analysis. **92**, 1639-1650.
6. Brindly, E., Thompson, W. (1972). *Dependent and aging aspects of multivariate survival.* J. Amer. Stat. Assoc. **67**, 822-830.
7. Burton, R. M., Dehling, H. (1990). *Large deviations for some weakly dependent random processes.* Statist. Probab. Lett. **9**, 397-401.
8. Chandra, T. K., Ghosal, S. (1996 a). *Extensions of the strong law of large numbers of Marcinkiewicz-Zygmund for dependent variables.* Acta Math.Hungar. **71**, 327-336.
9. Chandra, T. K., Ghosal, S. (1996 b). *The strong law of large numbers for weighted average under dependent assumptions.* J. Theoret. Prob **9**, 797-809.
10. Chen, Y., Zhang. W. (2007). *Large deviations for random sums of negatively dependent random variables with consitently varying tails.* Statist.Probab.Lett. **77**, no.5, 530-538.

11. Ebrahimi, N., Ghosh, M. (1981). *Multivariate Negative Dependence.* Commun. Stat. Theory and Methods. **89**, 625-641.
12. Esary, J. D., Proschan, F. (1972). *Relationships among some concepts of bivariate dependence.* Ann. Math. Stat. **43**, 651-655.
13. Esary, J. D., Proschan, F., Walkup, D. W. (1967). *Association of random variables with applications.* Ann. Math. Stat. **38**, 1466- 1477.
14. Harris, R. (1970). *A multivariate definition for increasing hazard rate distributions.* Ann. Math. Stat. **41**, 713-717.
15. Hu,T., Moricz, F., Taylor, R.L.(1989). *Strong laws of large numbers for arrays of rowwise independent random variables.* Acta Math. Hungar. **54**, no. 1-2, 153-162.
16. Joag-Dev, K., Proschan, F. (1983). *Negative association of random variables with applications.* Ann. Statist. **11**, 286-295.
17. Jogdeo, K., Patil, G. P. (1975). *Probability inequalities for certain multivariate discrete distributions.* Sankhya B. **37**, 158-164.
18. Karlin, S., Rinott, Y. (1980). *Classes of ordering of measures and related correlation inequalities.* J. Multi. Anal. **10**, 499-516.
19. Lehmann, E. L.(1966). *Some concepts of dependence.* Ann. Math. Statist. 43, 1137-1153.
20. Matula, P.(1992). *A note on the almost sure convergence of sums of negatively dependent random variables.* Probab. Lett. **15,** 209-213.
21. Taylor,R.L., Hu.T.C.(1987). *Sub-Gaussian techniques in proving strong laws of large numbers.* Amer. Math. Monthly **94,** no. 3, 295-299.
22. Thrum R. (1987). *A remark on almost sure convergence of weighted sums.* J. Probab. The.Rela.fields **79:** 425-430.
23. Zhang, L. X. (1996). *Complete convergence of moving average processes under dependence assumptions.* Statist Probab. Lett. **30**, 165-170.

Baek Jong Il is working at School of Mathematrics and Informational statistics and Institute of Basic Natural Science in Wonkwang University. His research interests focus on the probability theory, time series and reliability theory.

School of mathematics and Informational statistics and Institute of Basic Natural Science, Wonkwang University, Iksan 570-749, Korea.
`e-mail: jibaek@wku.ac.kr`

Seo Hye Young is a part time professor at School of Mathematics and Informational statistics and she is interest in probability theory.

School of mathematics and Informational Statistics, Wonkwang University, Iksan 570-749, Korea.
`e-mail: seofaith@hanmail.net`