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MODIFIED HALPERN ITERATIVE ALGORITHMS FOR NONEXPANSIVE MAPPINGS

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Abstract. Halpern iterative algorithm is one of the most cited in the literature of approximation of fixed points of nonexpansive mappings. Different authors modified this iterative algorithm in Banach spaces to approximate fixed points of nonexpansive mappings. One of which is Hu [8] and Yao et al [21] modification of Halpern iterative algorithm for nonexpansive mappings in Banach spaces. It is the purpose of this paper to thoroughly analyze this modification and its convergence conditions. Unfortunately, Hu [8] and Yao et al [21] control conditions imposed on the modified Halpern iterative algorithm to have strong convergence are found to be not sufficient. In this paper, counterexamples are constructed to prove that the strong convergence conditions of Hu [8] and Yao et al [21] are not sufficient. It is also proved that with some additional conditions on the control parameters, strong convergence of the defined iterative algorithm is obtained in different Banach space settings.

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1. Halpern Iterative Algorithm in Hilbert Spaces

Let K be a nonempty closed convex subset of a real Hilbert space H with inner product denoted by $\langle \, , \, \rangle$ and induced norm by $\| \, . \, \|$. Recall that a mapping $T : K \to K$ is said to be *nonexpansive* if and only if for each $x, y \in K$

$$\|Tx - Ty\| \leq \|x - y\| \, .$$

Let $Fix(T)$ denote the set of fixed points of T; that is, $Fix(T) = \{x \in K : Tx = x\}$.

Assume that $u \in K$ is a fixed anchor. Then for any initial point $x_0 \in K$, the explicit iterative algorithm $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \; n \geq 0, \tag{1.1}$$

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where $\{\alpha_n\}$ is a sequence in $(0,1)$, is referred as Halpern iterative algorithm (or simply Halpern iteration). Convergence of the Halpern iteration to a fixed point depends on the variety of Banach spaces and also on control conditions imposed on the parameter. In 1967, Halpern [7] proved the following theorem.

Theorem 1.1 (Halpern [7]). *Let K be a nonempty bounded closed convex subset of a real Hilbert space H and suppose $T : K \to K$ is a nonexpansive mapping. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ that satisfies the following control conditions:*

$$(C1)\ \lim_{n\to\infty} \alpha_n = 0;$$

$$(C2)\ \sum_{n=0}^{\infty} \alpha_n = +\infty \ \textit{or, equivalently,}\ \prod_{n=0}^{\infty}(1-\alpha_n) = 0.$$

Suppose that there is strictly increasing sequence of nonnegative integers $\{n_j\}$ such that

$$(i)\ \lim_{j\to\infty} \frac{\alpha_{j+n_j}}{\alpha_j} = 1;$$

$$(ii)\ \lim_{j\to\infty} n_j\alpha_j = +\infty.$$

Then the sequence $\{x_n\}$ defined in (1.1) converges strongly to a fixed point of T.

Halpern [7] also investigated that control conditions $(C1)$ and $(C2)$ are necessary, in the sense that, if a sequence defined in iterative algorithm (1.1) converges strongly for every nonexpansive mapping $T : K \to K$ such that $Fix(T) \neq \emptyset$ then $(C1)$ and $(C2)$ must be satisfied.

In 1977, Lions [9] improved the control conditions of Halpern, and proved the following theorem.

Theorem 1.2 (Lions [9]). *Let K be a nonempty bounded closed convex subset of a real Hilbert space H and suppose $T : K \to K$ is a nonexpansive mapping. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ satisfying the control conditions $(C1)$, $(C2)$ and*

$$(C3)\ \lim_{n\to\infty} \frac{\alpha_{n+1}-\alpha_n}{\alpha_{n+1}^2} = 0.$$

Then the sequence $\{x_n\}$ defined in (1.1) converges strongly to a fixed point of T.

In 1992, Wittmann [17] further improved control conditions to obtain the following result.

Theorem 1.3 (Wittmann [17]). *Let K be a nonempty closed convex subset of a real Hilbert space H and suppose $T : K \to K$ is a nonexpansive mapping with nonempty fixed point set $Fix(T)$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ satisfying the control conditions $(C1)$, $(C2)$ and*

$$(C4)\ \sum_{n=0}^{\infty} |\alpha_{n+1}-\alpha_n| < +\infty.$$

Then the iterative sequence $\{x_n\}$ defined in (1.1) converges strongly to a fixed point of T.

The following examples reveal that the control conditions ($C3$) and ($C4$) are independent, in the sense that, one does not imply the other.

Example 1.1. The canonical choice

$$\alpha_n = \frac{1}{n+1},$$

satisfies ($C4$) but fails ($C3$).

Example 1.2 (Bauschke [1])**.** If $\{\alpha_n\}$ is given by

$$\alpha_n = \begin{cases} (k+1)^{\frac{-1}{4}} \text{ if } n = 2k, \\ (k+1)^{\frac{-1}{4}} + \frac{1}{k+1} \text{ if } n = 2k+1 \end{cases}$$

then it satifies ($C3$) but fails ($C4$) since

$$\sum_{k=1}^{\infty} |\alpha_{2k+1} - \alpha_{2k}| = +\infty.$$

In 2003, Xu [20] replaced conditions ($C3$) and ($C4$) by the following new control condition:

$$(C5)\ \lim_{n\to\infty} \frac{\alpha_{n+1} - \alpha_n}{\alpha_{n+1}} = 0; \text{ or equivalently, } \lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1.$$

We observe that the canonical choice $\alpha_n = \frac{1}{n+1},\ n \geq 1$ satisfies ($C5$), and since control condition ($C5$) removes the square in the denominator of condition ($C3$), it is thus more general than ($C3$). However, the following examples reveal that coupled with conditions ($C1$) and ($C2$), conditions ($C4$) and ($C5$) are not comparable in general.

Example 1.3 (Xu [18])**.** If the sequence $\{\alpha_n\}$ is given by

$$\alpha_n = e^{-n^2},\ n \geq 1,$$

then $\{\alpha_n\}$ satisfies ($C1$), ($C2$) and ($C4$) but fails to satisfy ($C5$).

Example 1.4 (Xu [18])**.** If the sequence $\{\alpha_n\}$ is given by

$$\alpha_n = \begin{cases} \frac{1}{\sqrt{n}}, & \text{if } n \text{ is odd;} \\ \frac{1}{\sqrt{n}-1}, & \text{if } n \text{ is even,} \end{cases}$$

then $\{\alpha_n\}$ satisfies ($C1$), ($C2$) and ($C5$) but fails to satisfy ($C4$).

Generalization of Halpern iterative algorithm to finite family of nonexpansive mappings in Hilbert spaces was studied by Wittmann [17], Bauschke [1], and Xu [20].

In the last half century, one of the key question regarding Halpern iterative algorithm was the following:

> *Are the conditions* $(C1)$ *and* $(C2)$ *sufficient for the strong convergence of the sequence* $\{x_n\}$ *generated by (1.1) to a fixed point of a nonexpansive mapping* $T : K \to K$*?*

However, in 2009, Suzuki [15] constructed the following counterexample which shows that $(C1)$ and $(C2)$ are not sufficient for the strong convergence.

Example 1.5 (Suzuki [15])**.** Let $X = \Re$ denote the set of reals, $K = [-1, 1]$ and $u = 1$. Define a nonexpansive mapping $T : K \to K$ by

$$Tx = -x, \; x \in K.$$

Then $p = 0$ is the unique fixed point of T in K. Now fix $\delta > 1$ and define a sequence $\{\alpha_n\}$ in $(0, 1)$ by

$$\alpha_n = \begin{cases} \frac{1}{n\delta}, & \text{if } n \text{ is odd,} \\ \frac{1}{n}, & \text{if } n \text{ is even.} \end{cases}$$

Then it is obvious that the sequence $\{\alpha_n\}$ satisfies the control conditions $(C1)$ and $(C2)$.

Take the initial point $x_1 = \frac{\delta-1}{\delta+1}$. Then it is easily shown by induction that

$$x_{2n+1} = \frac{\delta - 1}{\delta + 1} \text{ for each } n \geq 1.$$

Thus, the sequence $\{x_n\}$ does not converge to 0 which is the only fixed point of the mapping T.

Suzuki [15] also investigated some of strong necessary conditions for convergence of Halpern iterative algorithm to fixed points of nonexpansive mappings.

Remark 1.1. The question "What are necessary and sufficient conditions to have strong convergence of Halpern iterative algorithm?" is still open in general Banach space setting.

2. Halpern Iterative Algorithm in Banach Spaces

The Banach space version of convergence of Halpern iterative algorithm was first investigated by Reich [11] in 1980, who proved the following theorem.

Theorem 2.1 (Reich [11])**.** *Let* K *be a nonempty closed convex subset of a uniformly smooth Banach space* X*, and let* $T : K \to K$ *be a nonexpansive mapping with a nonempty fixed point set* $Fix(T)$*. Let* $\{\alpha_n\}$ *be a sequence in* $(0, 1)$ *defined by*

$$\alpha_n = (n + 2)^{-a},$$

where $0 < a < 1$*. Then for a fixed anchor* $u \in K$ *and any initial point* $x_0 \in K$*, iterative sequence* $\{x_n\}$ *defined by (1.1) converges strongly to a fixed point of the mapping* T*.*

In 1983, Reich [12] posed the following question:

> **Problem**. Let X be a Banach space. Is there a sequence $\{\alpha_n\}$ such that whenever a weakly compact, convex subset K of X possesses the fixed point property for nonexpansive mappings, then the sequence $\{x_n\}$ defined by (1.1) converges to a fixed point of T for all $x \in K$ and all nonexpansive $T : K \to K$?

Those theorems in Section 1 gave us partial answer on a Hilbert space setting.

Reich [13] also extended Wittmann's [17] result to the class of uniformly smooth Banach spaces having weakly sequentially continuous duality mapping by further assuming that the control sequence $\{\alpha_n\}$ satisfies $(C1)$, $(C2)$ and to be decreasing (and hence $(C4)$ is satisfied).

In 1997, Shioji and Takahashi [14] extended Wittmann's [17] results to the framework of a Banach space whose norm is uniformly Gâteaux differentiable.

In 2002, Xu [18] proved the following theorem in the framework of uniformly smooth Banach spaces.

Theorem 2.2 (Xu [18])). *Let K be a nonempty closed convex subset of a uniformly smooth Banach space X, and let $T : K \to K$ be a nonexpansive mapping with a nonempty fixed point set. Let $u,\ x_0 \in K$ be given. Assume $\{\alpha_n\} \subseteq (0,1)$ satisfies the control conditions (Cl), $(C2)$ and $(C5)$. Then the iteration process $\{x_n\}$ defined by (1.1) converges strongly to a fixed point of T.*

We now come up with more recent developments due to Hu [8] and Yao et al [21].

Hu [8] and Yao et al [21] modified Halpern's iterative algorithm to the following general iterative algorithm defined explicitly as:

$$\begin{cases} x_0 \in K \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T x_n,\ n \geq 0 \end{cases} \tag{2.1}$$

where $u \in K$ is a fixed anchor and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $(0,1)$ such that

$$\alpha_n + \beta_n + \gamma_n = 1.$$

Their motivation behind modifying Halpern iteration was to obtain strong convergence only by imposing control conditions $(C1)$ and $(C2)$ on a sequence $\{\alpha_n\}$. The following theorems were stated and proved.

Theorem 2.3 (Hu [8]). *Let K be a nonempty closed convex subset of a real Banach space X whose norm is uniformly Gâteaux differentiable. Let $T : K \to K$ be a nonexpansive mapping with nonempty fixed points set $Fix(T)$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0,1)$ satisfying the following control conditions:*

$$(i)\ \alpha_n + \beta_n + \gamma_n = 1,\ for\ all\ n \geq 0;$$

$$(ii)\ \lim_{n\to\infty} \alpha_n = 0,\ and\ \sum_{n=0}^{\infty} \alpha_n = +\infty;$$

$$(iii)\ \lim_{n\to\infty} \gamma_n = 0.$$

Given a fixed anchor $u \in K$, assume that

$$\lim_{t \to 0} z_t = p$$

for some $p \in Fix(T)$, where z_t is the unique element of K which satisfies

$$z_t = tu + Tz_t, \ t \in (0, 1).$$

Then for any initial point $x_0 \in K$, the sequence $\{x_n\}$ defined in (2.1) converges strongly to a fixed point of T.

Theorem 2.4 (Yao et al [21]). *Let K be a nonempty closed convex subset of a real uniformly smooth Banach space X and let $T : K \to K$ be a nonexpansive mapping with a nonempty fixed points set $Fix(T)$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0, 1)$ satisfying the control conditions:*

$$(i)\ \alpha_n + \beta_n + \gamma_n = 1, \ for\ all\ n \geq 0;$$

$$(ii)\ \lim_{n \to \infty} \alpha_n = 0,$$

$$(iii)\ \sum_{n=0}^{\infty} \alpha_n = +\infty; \text{ and}$$

$$(iv)\ \lim_{n \to \infty} \gamma_n = 0.$$

Then for a fixed anchor $u \in K$ and for any initial point $x_0 \in K$, $\{x_n\}$ defined in (2.1) converges strongly to a fixed point of T.

Unfortunately, in Section 3, counterexamples are constructed to show that in Theorem 2.3 and Theorem 2.4 the given assumptions of control conditions on parameters are not sufficient to obtain strong convergence of the explicit iterative sequence defined in (2.1).

3. Counterexamples

In this section, counterexamples are given to show that the assumed control conditions on parameters of Theorem 2.3 and Theorem 2.4 are not sufficient for the strong convergence of modified Halpern iterative algorithm. Throughout this section $\Re$ denotes the set of all real numbers.

Example 3.1. Put $X = \Re$, $K = [-1, 1]$ and u=1.
Suppose $T : [-1, 1] \to [-1, 1]$ is defined by

$$Tx = -x, \ x \in K.$$

Then it is easily shown that T is a nonexpansive mapping with the only fixed point 0. Let $\{\alpha_n\}$ and $\{\gamma_n\}$ be real sequences in the interval $(0, \frac{1}{3})$ such that:

$$(1)\ \alpha_n = \gamma_n, \ \forall n \geq 0;$$

$$(2)\ \lim_{n \to \infty} \alpha_n = 0, \ and$$

$$(3)\ \sum_{n=0}^{\infty} \alpha_n = +\infty.$$

Let $\{\beta_n\}$ be a sequence in $(\frac{2}{3}, 1)$ such that

$$\alpha_n + \beta_n + \gamma_n = 1,$$

for each $n \geq 0$. Then $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the control conditions:

$$(i)\ \alpha_n + \beta_n + \gamma_n = 1,\ \textit{for all } n \geq 0;$$
$$(ii)\ \lim_{n\to\infty} \alpha_n = 0,$$
$$(iii)\ \sum_{n=0}^{\infty} \alpha_n = +\infty;\ \textit{and}$$
$$(iv)\ \lim_{n\to\infty} \gamma_n = 0.$$

Define a sequence $\{x_n\}$ in K by

$$\begin{cases} x_0 = \frac{1}{3} \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T x_n,\ n \geq 0. \end{cases}$$

Consider that

$$\begin{aligned} x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n T x_n = \alpha_n + \beta_n x_n - \gamma_n x_n \\ &= \alpha_n + (\beta_n - \gamma_n) x_n = \alpha_n + (1 - \alpha_n - 2\gamma_n) x_n \\ &= \alpha_n + (1 - 3\alpha_n) x_n. \end{aligned}$$

for each $n \geq 0$. By induction, it is easily shown that

$$x_n = \frac{1}{3},\ \forall n \geq 1.$$

Therefore, $\{x_n\}$ does not converge to 0.

Example 3.2. Let X, K, T and u be as in Example 3.1. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0, 1)$ such that

$$\alpha_n = \frac{1}{n+5},\ \beta_n = 1 - \frac{3}{n+5},\ \text{and } \gamma_n = \frac{2}{n+5},$$

for each positive integer n. Then $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the control conditions:

$$(i)\ \alpha_n + \beta_n + \gamma_n = 1,\ \textit{for all } n \geq 0;$$
$$(ii)\ \lim_{n\to\infty} \alpha_n = 0,\ \textit{and } \sum_{n=0}^{\infty} \alpha_n = +\infty;$$
$$(iii)\ \lim_{n\to\infty} \gamma_n = 0.$$

Define a sequence $\{x_n\}$ in K by

$$\begin{cases} x_0 = \frac{1}{5} \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T x_n,\ n \geq 0. \end{cases}$$

Consider that

$$\begin{aligned} x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n T x_n = \alpha_n + \beta_n x_n - \gamma_n x_n \\ &= \alpha_n + (\beta_n - \gamma_n) x_n \\ &= \frac{1}{n+5} + (1 - \frac{5}{n+5}) x_n \end{aligned}$$

for each $n \geq 0$. By induction, it follows that

$$x_n = \frac{1}{5}, \ \forall n \geq 1.$$

Therefore, $\{x_n\}$ does not converge to 0 which is the only fixed point of the mapping T.

Example 3.3 (Wang [16]). Let X be a real Banach space whose norm is uniformly Gâteaux differentiable. Let K be a nonempty closed and convex subset of X defined by

$$K = \{\lambda y : \lambda \in [0, 3]\},$$

where $y \neq 0$ with $\|y\| = 1$ is a fixed element of X. Let $T : K \to K$ be a mapping defined by $Tx = 0$ for all $x \in K$. It is obvious that T is a nonexpansive mapping and $Fix(T) = \{0\}$. Take $\alpha_n = \frac{1}{n+2}$, $\beta_n = 1 - \frac{2}{n+2}$, and $\gamma_n = \frac{1}{n+2}$, for all $n \geq 0$ and $x_0 = y, \ u = 3y$. It is also shown easily that

$$\lim_{t \to 0} z_t = \lim_{t \to 0} 3ty = 0.$$

Observe that all conditions of Theorem 2.3 are satisfied. However, the iterative sequence $\{x_n\}$ does not converge strongly to the fixed point $z = 0$ of the mapping T.

The above examples reveal that additional control conditions on the parameters must be imposed for the validity of Theorem 2.3 and Theorem 2.4.

4. Convergence Theorems

Let K be a nonempty closed convex subset of a Banach space X and $T : K \to K$ a nonexpansive mapping. Let $Fix(T)$ to denote the set of fixed points of T; that is,

$$Fix(T) = \{x \in K : T(x) = x\}.$$

Given a point $u \in K$ and $t \in (0, 1)$, the mapping $T_t : K \to K$ defined by

$$T_t(x) = tu + (1 - t)Tx, \ x \in K,$$

is a contraction. By Banach contraction mapping principle T_t has a unique fixed point, say $y_t \in K$; that is,

$$y_t = tu + (1 - t)Ty_t.$$

Note that if $Fix(T) \neq \emptyset$, then $\{y_t : t \in (0, 1)\}$ is bounded.

The study of convergence of $\{y_t\}$ as $t \to 0$ plays a key role in the study of convergence of iterative algorithms for nonexpansive mappings in Banach spaces.

There is a vast literature on this subject and a complete summary of the topic is not possible in this paper work only.

In the next paragraphs, we mention some of research results which are closely linked to our main results of the paper.

Browder [2] and Halpern [7], simultaneously, studied the convergence of $\{y_t\}$ as $t \to 0$ in Hilbert space setting and proved the following result.

Theorem 4.1 (Browder [2], Halpern [7]). *Let K be a nonempty closed convex subset of a real Hilbert space H and let $T : K \to K$ be a nonexpansive mapping with nonempty fixed points set $Fix(T)$. Then for some $p \in Fix(T)$*

$$\lim_{t\to 0} y_t = p$$

and satisfies the variational inequality

$$\langle p - u, p - z\rangle \leq 0,$$

for each $z \in Fix(T)$.

The Banach space version of Theorem 4.1 was obtained by Reich [11] in which he proved the following result.

Theorem 4.2 (Reich [11]). *Let K be a nonempty closed convex subset of a real uniformly smooth Banach space X and suppose $T : K \to K$ is a nonexpansive mapping with a nonempty fixed points set $Fix(T)$. Then for some $p \in Fix(T)$*

$$\lim_{t\to 0} y_t = p,$$

and satisfies the variational inequality

$$\langle p - u, J(p - z)\rangle \leq 0, \; z \in Fix(T),$$

where J is the normalized duality mapping of X.

We recall that the normalized duality mapping $J : X \to 2^{X^*}$ is given by

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}.$$

One of the important properties of normalized duality mapping is stated in the following lemma.

Lemma 4.3 (Chang [5]). *Let X be a real Banach space with the normalized duality mapping J. Then, for each pair $x, y \in X$ and for each $j(x+y) \in J(x+y)$, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle.$$

The proof of the following technical lemma can be found in Xu ([18], [19], [20]).

Lemma 4.4. *Assume $\{\lambda_n\}$ is a sequence of nonnegative real numbers such that*

$$\lambda_{n+1} \leq (1 - \alpha_n)\lambda_n + \beta_n, \; n \geq 0,$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *and* $\{\beta_n\}$ *is a sequence of real numbers such that*

$$(i)\ \sum_{n=1}^{\infty}\alpha_n = +\infty;$$

$$(ii)\ \limsup_{n\to\infty}\frac{\beta_n}{\alpha_n} \le 0\ or\ \sum_{n=1}^{\infty}\beta_n < +\infty.$$

Then $\lim_{n\to\infty}\lambda_n = 0$.

Theorem 4.5. *Let* K *be a nonempty closed convex subset of a real uniformly smooth Banach space* X, *and suppose that* $T : K \to K$ *is a nonexpansive mapping with nonempty fixed points set* $Fix(T)$. *Let* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *be sequences in* $(0,1)$ *which satisfy the following control conditions:*

$$\alpha_n + \beta_n + \gamma_n = 1,\ for\ all\ n \ge 0; \tag{4.1}$$

$$\lim_{n\to\infty}\alpha_n = 0,\ and\ \sum_{n=0}^{\infty}\alpha_n = +\infty; \tag{4.2}$$

$$\lim_{n\to\infty}\beta_n = 0; \tag{4.3}$$

$$\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n| < +\infty; \tag{4.4}$$

$$\sum_{n=0}^{\infty}|\beta_{n+1}-\beta_n| < +\infty. \tag{4.5}$$

Then for any given point $u \in K$, *the sequence* $\{x_n\}$ *defined in (2.1) converges strongly to a fixed point of* T.

Proof. First, we prove that $\{x_n\}$ is bounded. Let $z \in Fix(T)$ be given. Then for each positive integer n, we get

$$\begin{aligned}\|x_{n+1}-z\| &= \|\alpha_n(u-z)+\beta_n(x_n-z)+\gamma_n(Tx_n-z)\|\\ &\le \alpha_n\|u-z\|+\beta_n\|x_n-z\|+\gamma_n\|Tx_n-z\|\\ &\le \alpha_n\|u-z\|+(\beta_n+\gamma_n)\|x_n-z\|\\ &= \alpha_n\|u-z\|+(1-\alpha_n)\|x_n-z\|\\ &\le \max\{\|u-z\|,\|x_n-z\|\}.\end{aligned}$$

By induction we conclude that

$$\|x_n-z\| \le \max\{\|u-z\|,\|x_0-z\|\},$$

for each nonnegative integer n.

Hence $\{x_n\}$ is bounded and thus $\{Tx_n\}$ too. Now we claim that

$$\lim_{n\to\infty}\|x_n-Tx_n\| = 0. \tag{4.6}$$

Since $\{x_n\}$ and $\{Tx_n\}$ are bounded, it follows from control conditions (4.1), (4.2), and (4.3) that

$$\lim_{n\to\infty} \|x_{n+1} - Tx_n\| = 0. \tag{4.7}$$

Note that for each positive integer n, we have

$$\begin{aligned}|\gamma_n - \gamma_{n-1}| &= |(1-\alpha_n - \beta_n) - (1 - \alpha_{n-1} - \beta_{n-1})| \\ &= |(\alpha_{n-1} - \alpha_n) + (\beta_{n-1} - \beta_n)| \\ &\leq |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|.\end{aligned}$$

Therefore, for each positive integer n,

$$|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| \leq 2[|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|].$$

Noting the above fact, for each positive integer n we get

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq |\alpha_n - \alpha_{n-1}| \, \|u\| \\ &\quad + \|\beta_n x_n + \gamma_n T x_n - \beta_{n-1} x_{n-1} - \gamma_{n-1} T x_{n-1}\| \\ &\leq |\alpha_n - \alpha_{n-1}| \, \|u\| + |\beta_n - \beta_{n-1}| \, \|x_{n-1}\| \\ &\quad + |\gamma_n - \gamma_{n-1}| \, \|Tx_{n-1}\| + \beta_n \|x_n - x_{n-1}\| \\ &\quad + \gamma_n \|Tx_n - Tx_{n-1}\| \, .\end{aligned}$$

Since $\{x_n\}$ *and* $\{Tx_n\}$ are bounded, there is a nonnegative real number M such that

$$M = \max\{\|u\| \, , \, \sup_n \|x_n\| \, , \, \sup_n \|Tx_n\|\}.$$

Therefore,

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq (1 - \alpha_n) \|x_n - x_{n-1}\| \\ &\quad + M[|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}|] \\ &\leq (1 - \alpha_n) \|x_n - x_{n-1}\| \\ &\quad + 2M[|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|].\end{aligned}$$

Therefore, by control conditions (4.2), (4.4), (4.5) and Lemma 4.4 we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{4.8}$$

Since $\|x_n - Tx_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \, , \; n \geq 0$, (4.6) follows trivially from (4.7) and (4.8).

For $t \in (0, 1)$ there is a unique $y_t \in K$ such that

$$y_t = tu + (1 - t)Ty_t.$$

It follows form Theorem 4.2 that

$$\lim_{t\to 0} y_t = p$$

for some $p \in Fix(T)$. Now we claim that

$$\limsup_{n\to\infty} \langle u - p, J(x_n - p)\rangle \leq 0. \tag{4.9}$$

Using Lemma 4.3, for each $n \geq 1$ and $t \in (0,1)$, we get

$$\begin{aligned}
\|y_t - x_n\|^2 &= \|t(u - x_n) + (1-t)(Ty_t - x_n)\|^2 \\
&\leq (1-t)^2 \|Ty_t - x_n\|^2 + 2t\langle u - x_n, J(y_t - x_n)\rangle \\
&\leq (1-t)^2[\|Ty_t - Tx_n\| + \|Tx_n - x_n\|]^2 \\
&\quad + 2t\langle u - x_n, J(y_t - x_n)\rangle \\
&\leq (1-t)^2 \|y_t - x_n\|^2 \\
&\quad + \|Tx_n - x_n\| [2\|y_t - x_n\| + \|Tx_n - x_n\|] \\
&\quad + 2t[\langle y_t - x_n, J(y_t - x_n)\rangle + \langle u - y_t, J(y_t - x_n)\rangle] \\
&= (1-t)^2 \|y_t - x_n\|^2 \\
&\quad + \|Tx_n - x_n\| [2\|y_t - x_n\| + \|Tx_n - x_n\|] \\
&\quad + 2t[\|y_t - x_n\|^2 + \langle u - y_t, J(y_t - x_n)\rangle] \\
&= (1+t^2)\|y_t - x_n\|^2 + 2t\langle u - y_t, J(y_t - x_n)\rangle \\
&\quad + \|Tx_n - x_n\| [2\|y_t - x_n\| + \|Tx_n - x_n\|].
\end{aligned}$$

It follows that

$$\langle y_t - u, J(y_t - x_n)\rangle \leq \frac{t}{2}\|y_t - x_n\|^2 + \frac{1}{2t}\|Tx_n - x_n\| A_n, \tag{4.10}$$

where $A_n = 2\|y_t - x_n\| + \|Tx_n - x_n\|$.

Since $\{y_t : t \in (0,1)\}$ and $\{x_n\}$ are bounded, there exists a positive real number Γ such that

$$A_n \leq \Gamma,$$

for each positive integer n.

Hence it follows from (4.10) that

$$\langle y_t - u, J(y_t - x_n)\rangle \leq \frac{t}{2}\Gamma^2 + \frac{1}{2t}\|Tx_n - x_n\|\Gamma. \tag{4.11}$$

For fixed $t \in (0,1)$, letting $n \to \infty$ in (4.11) and noting that (4.6), we have

$$\limsup_{n\to\infty}\langle y_t - u, J(y_t - x_n)\rangle \leq \frac{t}{2}\Gamma^2. \tag{4.12}$$

Thus it follows from (4.12) that

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle y_t - u, J(y_t - x_n)\rangle \leq 0. \tag{4.13}$$

Since the normalized duality mapping J is norm-to-norm uniformly continuous on bounded subsets of a uniformly smooth Banach spaces, the two limits in (4.13) can be interchanged and (4.9) follows as a result.

Finally we show that $\{x_n\}$ converges strongly to p. Applying Lemma 4.3 we get

$$\|x_{n+1} - p\|^2 \leq (1-\alpha_n)\|x_n - p\|^2 + 2\alpha_n\langle u - p, J(x_{n+1} - p)\rangle. \tag{4.14}$$

Noting (4.2), (4.9) and applying Lemma 4.4 to (4.14), we obtain the required conclusion

$$\lim_{n\to\infty} \|x_n - p\| = 0.$$

This completes the proof.

□

By a **gauge** we mean a continuous strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = +\infty$. Let X be a Banach space. The duality mapping (generally multivalued) $J_\phi : X \to X^*$ associated with a gauge ϕ is defined by

$$J_\phi(x) = \{u^* \in X^* : \langle x, u^* \rangle = \|x\| \, \|u^*\| \,, \ \|u^*\| = \phi(\|x\|)\},$$

for each $x \in X$, where $\langle \ , \ \rangle$ denotes the generalized duality pairing. In the particular case $\phi(t) = t$, the duality mapping is called the normalized duality mapping.

A Banach space X is said to have a weakly continuous duality mapping if and only if there exists a gauge ϕ such that the duality mapping J_ϕ is single-valued and (sequentially) continuous from X with the weak topology to X^* with the weak* topology.

Definition 4.6. Let K be a nonempty closed convex subset of a Banach space X and $Q : X \to K$ a surjective mapping. Then Q is said to be sunny if and only if

$$Q(Qx + t(x - Qx)) = Qx \ \text{ for all } x \in X, \ t \geq 0.$$

A mapping $Q : X \to X$ is said to be a retraction if and only if $Q^2 = Q$; that is, $Qz = z$ for every $z \in R(Q)$, range of Q. A subset K of X is said to be a sunny nonexpansive retract of X if and only if there exists a sunny nonexpansive retraction of X onto K; and it is said to be a nonexpansive retract of X if and only if there exists a nonexpansive retraction of X onto K.

Zegeye and Shahzad [22] proved the following theorem whose application is our second main result of the paper.

Theorem 4.7 (Zegeye and Shahzad [22])**.** *Let K be a nonempty closed convex subset of a real reflexive Banach space X having weakly continuous duality mapping J_ϕ for some gauge ϕ. Let $T : K \to K$ be a nonexpansive mapping and $f : K \to K$ be a contraction with constant β. Then for each $t \in (0,1)$, there exists $y_t \in K$ satisfying the following condition:*

$$y_t = tf(y_t) + (1-t)Ty_t.$$

Suppose further that $\{y_t\}$ is bounded (or $Fix(T) \neq \emptyset$). Then $\lim_{t\to 0} y_t$ exists and is a fixed point of T. Suppose π_K denotes the set of all contraction self-mappings of K. Hence if we define $Q : \pi_K \to Fix(T)$ by

$$Q(f) = \lim_{t\to 0} y_t, \ \ f \in \pi_K,$$

then $Q(f)$ solves the variational inequality

$$\langle (I-f)Q(f), J_\phi(Q(f)-y)\rangle,\ y\in Fix(T).$$

In particular, if $f=u\in K$ is a constant, then $Q:K\to Fix(T)$ is reduced to the sunny nonexpansive retraction of K onto $Fix(T)$,

$$\langle Q(u)-u, J_\phi(Q(u)-y)\rangle,\ y\in Fix(T).$$

In 1967, Opial [10] introduced the following concept by exploiting some geometric properties of Hilbert spaces.

Definition 4.8. A Banach space X is said to satisfy Opial's condition if for any sequence $\{x_n\}$ in X, the condition that $\{x_n\}$ convreges weakly to $x\in X$ implies that

$$\limsup_{n\to\infty}\|x_n-x\| < \limsup_{n\to\infty}\|x_n-y\|,\ for\ all\ y\in X,\ y\neq x.$$

Opial [10] proved that sequential spaces $\ell_p,\ 1<p<+\infty$, satisfy Opial's condition; however, the analogous functional spaces $L_p[0,2\pi],\ p\neq 2$, do not have the property.

In 1967, Browder [3] proved the following fact.

Lemma 4.9. *A Banach space with a weakly continuous duality mapping satisfies Opial's condition.*

Definition 4.10. Let K be a nonempty subset of a Banach space X. A mapping $T:K\to X$ is said to be **demiclosed** (at y) if for any sequence $\{x_n\}$ in K, the conditions $x_n\to x$ weakly and $Tx_n\to y$ strongly imply $x\in K$ and $Tx=y$.

One of the fundamental and celebrated results in the study of nonexpansive mappings is Browder's Demiclosedness principle which states that

Theorem 4.11 (Browder [4])**.** *If K is a nonempty closed convex subset of a uniformly convex Banach space X and if $T:K\to X$ is a nonexpansive mapping, then $I-T$ is demiclosed, where I is the identity operator of X.*

Furthermore, the following theorem was proved in Goebel and Kirk [6].

Theorem 4.12 (Goebel and Kirk [6])**.** *A Banach space X with Opial's property satisfies Browder's Demiclosedness principle; that is, if K is a nonempty closed convex subset of a Banach space X satsifying Opial's condition and if $T:K\to X$ is a nonexpansive mapping, then $I-T$ is demiclosed.*

As an application of Theorem 4.7, we prove the following convergence theorem of modified Halpern iterative algorithm defined in equation (2.1).

Theorem 4.13. *Let K be a nonempty closed convex subset of a real reflexive Banach space X having weakly continuous normalized duality mapping J. Let $T:K\to K$ be a nonexpansive mapping with $Fix(T)\neq\emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0,1)$ satisfying the following control conditions:*

$$\alpha_n+\beta_n+\gamma_n=1,\ for\ all\ n\geq 0; \tag{4.15}$$

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=0}^{\infty} \alpha_n = +\infty; \ \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < +\infty; \tag{4.16}$$

$$\lim_{n\to\infty} \beta_n = 0 \ \sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < +\infty. \tag{4.17}$$

Then for any given point $u \in K$, the sequence $\{x_n\}$ defined in (2.1) converges strongly to a fixed point of T.

Proof. It is already proved in Theorem 4.5 that the sequence $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{4.18}$$

By Theorem 4.7 there exists $p \in Fix(T)$ such that

$$\lim_{t\to 0} y_t = p, \tag{4.19}$$

where $y_t = tu + Ty_t$ for each $t \in (0,1)$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle p - u, J(p - x_n)\rangle = \lim_{j\to\infty} \langle p - u, J(p - x_{n_j})\rangle. \tag{4.20}$$

Since X is a reflexive Banach space, without loss of generality, we assume that

$$w - \lim_{j\to\infty} x_{n_j} = z$$

for some $z \in K$. It follows from Theorem 4.12 and noting that (4.18), $z = Tz$. Again using Theorem 4.7 and noting (4.19), we have

$$\langle p - u, J(p - z)\rangle \le 0. \tag{4.21}$$

Since J is weakly continuous, we have

$$\lim_{j\to\infty} \langle p - u, J(p - x_{n_j})\rangle = \langle p - u, J(p - z)\rangle. \tag{4.22}$$

Combining (4.20), (4.21), and (4.22) yields

$$\limsup_{n\to\infty} \langle p - u, J(p - x_n)\rangle \le 0. \tag{4.23}$$

Now we claim that $\{x_n\}$ converges strongly to p. Applying Lemma 4.3 we get

$$\|x_{n+1} - p\|^2 \le (1 - \alpha_n) \|x_n - p\|^2 + 2\alpha_n \langle u - p, J(x_{n+1} - p)\rangle. \tag{4.24}$$

Noting that (4.15), (4.23) and applying Lemma 4.4 to (4.24), we obtain the required conclusion

$$\lim_{n\to\infty} \|x_n - p\| = 0.$$

This completes the proof. □

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