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EXISTENCE OF SOLUTIONS FOR GENERALIZED NONLINEAR VARIATIONAL-LIKE INEQUALITY PROBLEMS IN BANACH SPACES†

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Abstract. In this paper, we study a new class of generalized nonlinear variational-like inequalities in reflexive Banach spaces. By using the KKM technique and the concept of the Hausdorff metric, we obtain some existence results for generalized nonlinear variational-like inequalities with generalized monotone multi-valued mappings in Banach spaces. These results improve and generalize many known results in recent literature.

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1. Introduction

Variational inequality theory provides techniques to solve a variety of applied problems in fluid flow through porous media, elasticity, optimization, nonlinear programming, economics, transportation and engineering (see [4,7,8]).

It is well known that the KKM technique has played a very important role in the study of many fields such as optimization, mathematical programming problems, equilibrium problems, game theory, variational inequality theory and so on (see [6,9,10]).

In 1997, by using the KKM technique , Konnov and Yao proved in Ref.[10] some results about the existence of solutions for vector variational inequalities with C_x-pseudomonotone multi-valued mappings. In 1999, Chen [1] obtained the existence of solutions for a class of variational inequalities with semi-monotone single-valued mappings in nonreflexive Banach spaces. In 2003, Fang and Huang [6] considered two classes of variational-like inequalities with generalized monotone and semi-monotone mappings. Utilizing the KKM technique they proved

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the existence of solutions for these variational-like inequalities with relaxed η-α-monotone mappings in reflexive Banach spaces.

In this paper, we introduce and study a new class of generalized variational-like inequality problem with generalized monotone multi-valued mappings. By applying the KKM technique and the concept of the Hausdorff metric, we establish some existence results for generalized variational-like inequalities with generalized monotone multi-valued mappings in reflexive Banach spaces.

2. Preliminaries

Let $\mathbb{R} = (-\infty, +\infty)$, let E be a Banach space with norm $\|\cdot\|$, let E^* be the topological dual space of E and let (u, v) be the pairing between $u \in E^*$ and $v \in E$. Let K be a nonempty closed convex subset of E. Let the functional $b : K \times K \to \mathbb{R}$ satisfy the following conditions:

(2a) for each $u \in K$, $b(u, \cdot)$ is a convex functional,
(2b) $b(u, v)$ is bounded, that is, there exists a constant $\gamma > 0$ such that

$$b(u, v) \leq \gamma \|u\| \|v\|, \quad \forall u, v \in K,$$

(2c) for all $u, v, w \in D$

$$b(u, v) - b(u, w) \leq b(u, v - w).$$

Remark 2.1. In view of (2b) and (2c), we know that

$$\begin{aligned} b(u, v) - b(u, w) &\leq b(u, v - w) \\ &\leq \gamma \|u\| \|v - w\|, \end{aligned}$$

$$\begin{aligned} b(u, w) - b(u, v) &\leq b(u, w - v) \\ &\leq \gamma \|u\| \|w - v\| \end{aligned}$$

for all $u, v, w \in K$. That is,

$$|b(u, v) - b(u, w)| \leq \gamma \|u\| \|v - w\|, \quad \forall u, v, w \in K. \tag{2.1}$$

Let $\Psi : K \times K \times E^* \to \mathbb{R}$, $\alpha : E \times E \to \mathbb{R}$ be functionals and let $T : K \to 2^{E^*}$, $A : E^* \to E^*$ be mappings. Now we consider the following generalized nonlinear variational-like inequality problems with generalized monotone multi-valued mappings: Find $x \in K$ such that for each $y \in K$ there exists $s \in Tx$ satisfying

$$\Psi(y, x; As) + b(x, y) - b(x, x) \geq 0, \tag{2.2}$$

where b satisfies (2a)-(2c).

Special cases

(I) If $\Psi(x, y, z^*) = \langle z^*, \eta(x, y) \rangle$, T is single-valued and $Ax = N(Sx, Bx) - w^*$ for a given $w^* \in E^*$, where $N : E^* \times E^* \to E^*$, $S, B : K \to E^*$, $\eta : K \times K \to E$

are four mappings, then problem (2.2) reduces the following nonlinear mixed variational-like inequality problem: find $x \in K$ such that

$$\langle N(Sx, Bx) - w^*, \eta(y,x)\rangle + b(x,y) - b(x,x) \geq 0, \quad \forall y \in K. \tag{2.3}$$

The problem (2.3) is introduced and studied by Ding [3].

(II) If $N(Sx, Bx) = Sx - Bx$ for all $x \in K$, then problem (2.3) reduces to the following variational-like inequality problem: find $x \in K$ such that

$$\langle Sx - Bx - w^*, \eta(y,x)\rangle + b(x,y) - b(x,x) \geq 0, \quad \forall y \in K. \tag{2.4}$$

The problem (2.4) with $w^* = 0$ is introduced and studied by Ding [2] in reflexive Banach spaces.

Definition 2.1. Let K be a nonempty subset of a Banach space E with the dual space E^*. Let $\Psi : K \times K \times E^* \to \mathbb{R}$, $\alpha : E \times E \to \mathbb{R}$ be functionals and let $A : E^* \to E^*$, $T : K \to 2^{E^*}$ be mappings. Then

(1) T is called generalized α-monotone with respect to Ψ and A if for any $x, y \in K$ we have

$$\Psi(y, x; At) - \Psi(y, x; As) \geq \alpha(x, y)$$

for each $s \in Tx$ and $t \in Ty$, where $\lim_{t\to 0^+} \frac{\alpha(x, x+t(y-x))}{t} = 0$.

(2) Ψ is b-coercive with respect to T and A if there exists $y_0 \in K$ such that

$$\lim_{\|x\|\to\infty} \inf_{s\in Tx} \frac{\Psi(x, y_0; As) - \Psi(x, y_0; At_0) + b(x,x) - b(x, y_0)}{|\Psi(x, y_0; At_0)|} = +\infty$$

for some $t_0 \in Ty_0$.

Remark 2.2. If $\Psi(x, y; z^*) = \langle z^*, \eta(x,y)\rangle$ for each $(x, y, z^*) \in K \times K \times E^*$, $A = I$ is the identity mapping of E^*, T is single-valued and $\alpha(x,y) = \beta(y-x)$, where $\beta : K \to \mathbb{R}$ with $\beta(\lambda z) = \lambda^p \beta(z)$ for $\lambda > 0$, $p > 1$, then the generalized α-monotonicity of mapping T reduces to relaxed η-α monotonicity of mapping T (see [6]).

Example 2.1. *Let* $K = (-\infty, +\infty)$, $Tx = \{-x, x\}$, $Ax = x$, $\Psi(y, x; At) = \langle -At, \eta(x,y)\rangle$ *and*

$$\eta(x,y) = \begin{cases} -c(x-y) & \text{if } x \geq y, \\ c(x-y) & \text{if } x < y \end{cases}$$

for every $x, y, t \in K$, *where* $c > 0$ *is a constant. It is easy to check that* T *is generalized* α*-monotone with respect to* Ψ *and* A *with* $\alpha(x,y) = -c\|x-y\|^2$ *for all* $x, y \in K$.

Lemma 2.1([11]). *Let* $(E, \|\cdot\|)$ *be a normed vector space and* H *be a Hausdorff metric on* $CB(E)$*, the family of all closed and bounded subsets of* E*. If* A *and* B *are two members in* $CB(E)$*, then for each* $\varepsilon > 0$ *and each* $x \in A$*, there exists*

$y \in B$ such that

$$\|x - y\| \le (1 + \varepsilon) H(A, B).$$

In particular, if A and B are any two compact subsets in E, then for each $x \in A$, there exists $y \in B$ such that

$$\|x - y\| \le H(A, B).$$

Lemma 2.2([5]). *Let K be a nonempty subset of a Hausdorff topological vector space E and let $F : K \to 2^E$ be a KKM mapping. If $F(x)$ is closed in E for every x in K and is compact for some $x \in K$, then $\cap_{x \in K} F(x) \neq \phi$.*

3. Main results

In this section, we suppose always that E is a real reflexive Banach space with the dual space E^* and K is a nonempty closed convex subset of E.

Theorem 3.1. *Let $T : K \to 2^{E^*}$ be a nonempty compact-valued mapping such that for any $x, y \in K$*

$$H(T(x + \lambda(y - x)), Tx) \to 0 \quad as \quad \lambda \to 0^+,$$

where H is the Hausdorff metric defined on $CB(E^*)$. Assume that:
(i) $A : E^* \to E^*$ is a continuous mapping;
(ii) $b : K \times K \to (-\infty, +\infty)$ satisfies conditions (2a), (2b) and (2c);
(iii) $\Psi(x, \cdot; \cdot) : K \times E^* \to (-\infty, +\infty)$ is continuous for each fixed $x \in K$;
(iv) $\Psi(x, y; z^*) + \Psi(y, x; z^*) = 0$ for each $(x, y, z^*) \in K \times K \times E^*$;
(v) $\Psi(\cdot, y; At)$ is a convex functional on K for each $y \in K$ and $t \in Ty$;
(vi) T is generalized α-monotone with respect to Ψ and A.
Then the following problems (3.1) and (3.2) are equivalent:
(1) Find $x \in K$ such that for each $y \in K$, there exists $s \in Tx$ satisfying

$$\Psi(y, x; As) + b(x, y) - b(x, x) \ge 0. \tag{3.1}$$

(2) Find $x \in K$ such that

$$\Psi(y, x; At) + b(x, y) - b(x, x) \ge \alpha(x, y), \quad \forall y \in K, \quad t \in Tx. \tag{3.2}$$

Proof. Let $x_0 \in K$ be a solution of problem (3.1), i.e., for any $y \in K$ there is $s_0 \in Tx_0$ satisfying

$$\Psi(y, x_0; As_0) + b(x_0, y) - b(x_0, x_0) \ge 0.$$

Since T is generalized α-monotone with respect to Ψ and A, we have

$$\begin{aligned} &\Psi(y, x_0; At) + b(x_0, y) - b(x_0, x_0) \\ &\quad \le \Psi(y, x_0; As_0) + \alpha(x_0, y) + b(x_0, y) - b(x_0, x_0) \\ &\quad \le \alpha(x_0, y). \end{aligned}$$

Thus $x_0 \in K$ is a solution of problem (3.2).

Conversely, let $x_0 \in K$ be a solution of problem (3.2), i.e.,

$$\Psi(y, x_0; At) + b(x_0, y) - b(x_0, x_0) \geq \alpha(x_0, y), \quad \forall y \in K, t \in Ty.$$

Let $y_\lambda = (1-\lambda)x_0 + \lambda y, t \in (0,1)$. Then $y_\lambda \in K$. Since $x_0 \in K$ is a solution of problem (3.2), it follows that for all $t_\lambda \in Ty_\lambda$

$$\Psi(y_\lambda, x_0; At_\lambda) + b(x_0, y_\lambda) - b(x_0, x_0) \geq \alpha(x_0, y_\lambda). \tag{3.3}$$

Conditions (ii) and (v) imply that

$$\begin{aligned}
0 &= \Psi(y_\lambda, y_\lambda; At_\lambda) + b(x_0, y_\lambda) - b(x_0, y_\lambda) \\
&= \Psi((1-\lambda)x_0 + \lambda y, y_\lambda; At_\lambda) + b(x_0, (1-\lambda)x_0 + \lambda y) - b(x_0, y_\lambda) \\
&\leq (1-\lambda)\Psi(x_0, y_\lambda; At_\lambda) + \lambda\Psi(y, y_\lambda; At_\lambda) + (1-\lambda)b(x_0, x_0) \\
&\quad + \lambda b(x_0, y) - b(x_0, y_\lambda) \\
&= \lambda\Psi(y, y_\lambda; At_\lambda) + \lambda b(x_0, y) - \lambda b(x_0, y_\lambda) + (1-\lambda)\Psi(x_0, y_\lambda; At_\lambda) \\
&\quad + (1-\lambda)b(x_0, x_0) - (1-\lambda)b(x_0, y_\lambda).
\end{aligned}$$

It follows from condition (iv) and (3.3) that

$$\begin{aligned}
&\Psi(y, y_\lambda; At_\lambda) + b(x_0, y) - b(x_0, y_\lambda) \\
&\quad \leq \frac{1-\lambda}{\lambda}[-\Psi(x_0, y_\lambda; At_\lambda) + b(x_0, y_\lambda) - b(x_0, x_0)] \\
&\quad = \frac{1-\lambda}{\lambda}[\Psi(y_\lambda, x_0; At_\lambda) + b(x_0, y_\lambda) - b(x_0, x_0)] \\
&\quad \geq \frac{1-\lambda}{\lambda}\alpha(x_0, y_\lambda).
\end{aligned} \tag{3.4}$$

By Lemma 2.1, for each $t_\lambda \in Ty_\lambda$ we can find an $s_\lambda \in Tx_0$ such that

$$\|t_\lambda - s_\lambda\| \leq H(Ty_\lambda, Tx_0).$$

Since Tx_0 is compact, without loss of generality, we may assume that

$$s_\lambda \to s_0 \in Tx_0 \quad \text{as} \quad \lambda \to 0^+.$$

Since $H(Ty_\lambda, Tx_0) \to 0$ as $\lambda \to 0^+$, we obtain

$$\begin{aligned}
\|t_\lambda - s_0\| &\leq \|t_\lambda - s_\lambda\| + \|s_\lambda - s_0\| \\
&\leq H(Ty_\lambda, Tx_0) + \|s_\lambda - s_0\| \\
&\to 0 \quad \text{as} \quad \lambda \to 0^+.
\end{aligned}$$

So, $t_\lambda \to s_0$. Since $A : E^* \to E^*$ is continuous and $b : K \times K \to \mathbb{R}$ is continuous in the second argument, $At_\lambda \to As_0$ and $b(x_0, y_\lambda) \to b(x_0, x_0)$ as $\lambda \to 0^+$. It follows from Definition 2.1 and (3.4) that

$$\begin{aligned}&\Psi(y,x_0;As_0)+b(x_0,y)-b(x_0,x_0)\\&=\lim_{\lambda\to 0^+}[\Psi(y,y_\lambda;At_\lambda)+b(x_0,y)-b(x_0,y_\lambda)]\\&\geq \lim_{\lambda\to 0^+}\frac{\alpha(x_0,y_\lambda)}{\lambda}(1-\lambda)\\&=0.\end{aligned}$$

Therefore $x_0\in K$ is also a solution of problem (3.1). □

Corollary 3.1. *Let $T: K\to 2^{E^*}$ be a nonempty compact-valued mapping such that for any $x,y\in K$*

$$H(T(x+\lambda(y-x)),Tx)\to 0 \quad \text{as} \quad \lambda\to 0^+,$$

where H is the Hausdorff metric defined on $CB(E^*)$. Assume that:

(i) $A: E^*\to E^*$ is a continuous mapping;

(ii) $b: K\times K\to(-\infty,\infty)$ satisfies conditions (2a), (2b) and (2c);

(iii) $\eta(x,\cdot): K\to E$ is continuous for each fixed $x\in K$;

(iv) $\eta(x,y)+\eta(y,x)=0$ for each $(x,y)\in K\times K$;

(v) $\langle At,\eta(\cdot,y)\rangle: K\to\mathbb{R}$ is a convex functional on K for each $y\in K$ and $t\in Ty$;

(vi) T is generalized η-α-monotone with respect to A.

Then the following problems (3.5) and (3.6) are equivalent:

(1) Find $x\in K$ such that for each $y\in K$ there exists $s\in Tx$ satisfying

$$\langle As,\eta(y,x)\rangle+b(x,y)-b(x,x)\geq 0. \tag{3.5}$$

(2) Find $x\in K$ such that

$$\langle At,\eta(y,x)\rangle+b(x,y)-b(x,x)\geq\alpha(x,y),\quad \forall y\in K,\quad t\in Tx. \tag{3.6}$$

Theorem 3.2. *Let K be a nonempty bounded closed convex subset of a real reflexive Banach space E and let $T: K\to 2^{E^*}$ be a nonempty compact-valued mapping such that for any $x,y\in K$*

$$H(T(x+\lambda(y-x)),Tx)\to 0 \quad \text{as} \quad \lambda\to 0^+,$$

where H is the Hausdorff metric defined on $CB(E^*)$. Assume that

(i) $A: E^*\to E^*$ is a continuous mapping;

(ii) $b: K\times K\to(-\infty,+\infty)$ satisfies conditions (2a), (2b) and (2c);

(iii) $\Psi(x,\cdot,\cdot): K\times E^*\to(-\infty,+\infty)$ is continuous for each fixed $x\in K$;

(iv) $\Psi(x,y,z^*)+\Psi(y,x;z^*)=0$ for each $(x,y,z^*)\in K\times K\times E^*$;

(v) $\Psi(\cdot,y;At)$ is a convex and lower semicontinuous functional on K for each fixed $y\in K$ and $t\in Ty$;

(vi) T is generalized α-monotone with respect to Ψ and A;

(vii) $\alpha(\cdot,y)$ is weakly lower semicontinuous for each fixed $y\in K$.

Then problem (3.1) has a solution.

Proof. Define two set-valued mappings $F, G : K \to 2^K$ as follows:

$$F(y) = \{x \in K : \quad \text{there exists} \quad s \in Tx \quad \text{such that}$$
$$\Psi(y, x; As) + b(x, y) - b(x, x) \geq 0\}, \quad \forall y \in K,$$

$$G(y) = \{x \in K : \Psi(y, x; At) + b(x, y) - b(x, x) \geq \alpha(x, y), \forall t \in Ty\}, \quad \forall y \in K.$$

We claim first that F is a KKM mapping.

If F is not a KKM mapping, then there exist $\{y_1, y_2, \cdots, y_n\} \subset K$ and $\lambda_i > 0$, $i = 1, 2, \cdots, n$ with $\sum_{i=1}^n \lambda_i = 1$ such that $y = \sum_{i=1}^n \lambda_i y_i \notin \cup_{i=1}^n F(y_i)$. By the definition of F, we have

$$\Psi(y_i, y; As) + b(y, y_i) - b(y, y) < 0, \quad \forall s \in Ty \tag{3.7}$$

for each $i = 1, 2, \cdots, n$. It follows from (ii), (v) and (3.7) that

$$\begin{aligned} 0 &= \Psi(y, y; As) + b(y, y) - b(y, y) \\ &= \Psi\left(\sum_{i=1}^n \lambda_i y_i, y; As\right) + b\left(y, \sum_{i=1}^n \lambda_i y_i\right) - b(y, y) \\ &\leq \sum_{i=1}^n \lambda_i \Psi(y_i, y; As) + \sum_{i=1}^n \lambda_i b(y, y_i) - b(y, y) \\ &= \sum_{i=1}^n \lambda_i [\Psi(y_i, y; As) + b(y, y_i) - b(y, y)] \\ &< 0, \end{aligned}$$

which is contradiction. This implies that F is a KKM mapping.

Now we prove that $F(y) \subset G(y)$ for all y in K.

For any given y in K, letting $x \in F(y)$, then there exists $s \in Tx$ such that

$$\Psi(y, x; As) + b(x, y) - b(x, x) \geq 0.$$

Since T is generalized α-monotone with respect to Ψ and A, we have

$$\begin{aligned} &\Psi(y, x; At) + b(x, y) - b(x, x) \\ &\quad \geq \Psi(y, x; As) + \alpha(x, y) + b(x, y) - b(x, x) \\ &\quad \geq \alpha(x, y). \end{aligned}$$

It follows that $x \in G(y)$ and so $F(y) \subset G(y)$ for all $y \in K$. This implies that G is also a KKM mapping. From the assumptions we know that $G(y)$ is weakly closed for all y in K.

In fact, since $x \mapsto \Psi(x, y, At)$ is lower semicontinuous for each fixed $y \in K$ and $t \in Ty$ and b is continuous in the second argument, we known that they are both weakly lower semicontinuous. From the definition of G and the weakly lower semicontinuity of α we obtain that for all $y \in K$

$$\begin{aligned} G(y) &= \{x \in K : \Psi(y,x;At) + b(x,y) - b(x,x) \geq \alpha(x,y), \forall t \in Ty\} \\ &= \{x \in K : \Psi(x,y;At) + b(x,x) - b(x,y) + \alpha(x,y) \leq 0, \forall t \in Ty\} \end{aligned}$$

is weakly closed. Since K is a bounded closed and convex subset of E, we know from the reflexivity of E that K is weakly compact in K and so $G(y)$ is weakly compact in K for each $y \in K$. It follows from Lemma 2.2 and Theorem 3.1 that

$$\cap_{y\in K} F(y) = \cap_{y\in K} G(y) \neq \phi.$$

Hence there exists $x \in K$ such that for any $y \in K$ there is $s \in Tx$ satisfying

$$\Psi(y,x;As) + b(x,y) - b(x,x) \geq 0.$$

This completes the proof. □

Corollary 3.2. *Let K be a nonempty bounded closed convex subset of a real reflexive Banach space E and let $T : K \to 2^{E^*}$ be a nonempty compact-valued mapping such that for any $x, y \in K$*

$$H(T(x + \lambda(y - x)), Tx) \to 0 \quad as \quad \lambda \to 0^+,$$

where H is a Hausdorff metric defined on $CB(E^*)$. Assume that
(i) $A : E^* \to E^*$ is a continuous mapping;
(ii) $b : K \times K \to (-\infty, +\infty)$ satisfies conditions (2a), (2b) and (2c);
(iii) $\eta(x, \cdot) : K \to E$ is continuous for each fixed $x \in K$;
(iv) $\eta(x, y) + \eta(y, x) = 0$ for each $(x, y) \in K \times K$;
(v) $\langle At, \eta(\cdot, y)\rangle : K \to \mathbb{R}$ is a convex and lower semicontinuous functional on K for each fixed $y \in K$ and $t \in Ty$;
(vi) T is generalized η-α-monotone with respect to A;
(vii) $\alpha(\cdot, y)$ is weakly semicontinuous for each fixed $y \in K$.
Then problem (3.3) has a solution.

Now we consider the case of unbounded closed convex domains.

Theorem 3.3. *Let K be a nonempty unbounded closed convex subset of a real reflexive Banach space E and let $T : K \to 2^{E^*}$ be a nonempty compact-valued mapping such that for any $x, y \in K$*

$$H(T(x + \lambda(y - x)), Tx) \to 0 \quad as \quad \lambda \to 0^+,$$

where H is a Hausdorff metric defined on $CB(E^*)$. Assume that:
(i) $A : E^* \to E^*$ is a continuous mapping;
(ii) $b : K \times K \to (-\infty, +\infty)$ satisfies conditions (2a), (2b) and (2c);
(iii) $\Psi(x, \cdot, \cdot) : K \times E^* \to (-\infty, +\infty)$ is continuous for each fixed $x \in K$;
(iv) $\Psi(x, y; z^*) + \Psi(y, x; z^*) = 0$ for each $(x, y, z^*) \in K \times K \times E^*$;
(v) $\Psi(\cdot, y; At)$ is a convex and lower semicontinuous functional on K for each fixed $y \in K$ and $t \in Ty$;
(vi) Ψ is b-coercive with respect to T and A;
(vii) T is generalized α-monotone with respect to Ψ and A;

(viii) $\alpha(\cdot, y)$ *is weakly lower semicontinuous for each fixed* $y \in K$.
Then problem (3.1) has a solution.

Proof. Let $B_r = \{y \in E : \|y\| \geq r\}$. Consider the following problem: Find $x_r \in K \cap B_r$ such that for any $y \in K \cap B_r$ there is $s_r \in Tx_r$ satisfying

$$\Psi(y, x_r; As_r) + b(x_r, y) - b(x_r, x_r) \geq 0. \tag{3.6}$$

By Theorem 3.2, we know that problem (3.6) has a solution $x_r \in K \cap B_r$. Choose $r > \|y_0\|$, where y_0 is given by the b-coercivity condition of Ψ with respect to T and A. Then we have from (3.6) that

$$\Psi(y_0, x_r; As_r) + b(x_r, y_0) - b(x_r, x_r) \geq 0 \tag{3.7}$$

for some $s_r \in Tx_r$. Moreover, by condition (iv), we have

$$\begin{aligned}
&\Psi(y_0, x_r; As_r) + b(x_r, y_0) - b(x_r, x_r) \\
&= -\Psi(x_r, y_0; As_r) + b(x_r, y_0) - b(x_r, x_r) \\
&= -[\Psi(x_r, y_0; As_r) - \Psi(x_r, y_0; At_0) + b(x_r, x_r) - b(x_r, y_0)] \\
&\quad + \Psi(y_0, x_r; At_0) \\
&\leq -[\Psi(x_r, y_0; As_r) - \Psi(x_r, y_0; At_0) + b(x_r, x_r) - b(x_r, y_0)] \\
&\quad + |\Psi(x_r, y_0; At_0)| \\
&= -|\Psi(x_r, y_0; At_0)| \left[\frac{\Psi(x_r, y_0; As_r) - \Psi(x_r, y_0; At_0) + b(x_r, x_r) - b(x_r, y_0)}{|\Psi(x_r, y_0; At_0)|} - 1 \right] \\
&\leq -|\Psi(x_r, y_0; At_0)| \left[\inf_{s \in Tx_r} \frac{\Psi(x_r, y_0; As) - \Psi(x_r, y_0; At_0) + b(x_r, x_r) - b(x_r, y_0)}{|\Psi(x_r, y_0; At_0)|} - 1 \right]
\end{aligned} \tag{3.8}$$

for some $t_0 \in Ty_0$. Now, if $\|x_r\| = r$ for all r, we may choose r large enough so that (3.8) and the b-coercivity of Ψ with respect to T and A imply that

$$\Psi(y_0, x_r; As_r) + b(x_r, y_0) - b(x_r, x_r) < 0,$$

which contracts (3.7). Hence there exists r such that $\|x_r\| < r$. For any $y \in K$ we can choose $\varepsilon \in (0, 1)$ small enough such that

$$x_r + \varepsilon(y - x_r) \in K \cap B_r.$$

It follows from (3.8), (ii) and (v) that

$$\begin{aligned}
0 &\leq \Psi(x_r + \varepsilon(y - x_r), x_r; As_r) + b(x_r, x_r + \varepsilon(y - x_r)) - b(x_r, x_r) \\
&\leq \varepsilon\Psi(y, x_r, As_r) + (1 - \varepsilon)\Psi(x_r, x_r; As_r) + \varepsilon b(x_r, y) \\
&\quad + (1 - \varepsilon)b(x_r, x_r) - b(x_r, x_r) \\
&= \varepsilon[\Psi(y, x_r, As_r) + b(x_r, y) - b(x_r, x_r)].
\end{aligned}$$

This implies that

$$\Psi(y, x_r; As_r) + b(x_r, y) - b(x_r, x_r) \geq 0$$

for all $y \in K$. So, $x_r \in K$ is a solution of problem (3.1). □

Remark 3.1. (i) It is not necessary that b is linear in the first argument in Theorem 3.1, Theorem 3.2 and Theorem 3.3.

(ii) Theorem 3.2 improves and generalizes Theorem 3.1 of Ding [2] and the corresponding results of [3,12-14].

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