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BOOLEAN RANK INEQUALITIES AND THEIR EXTREME PRESERVERS

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Abstract. The $m \times n$ Boolean matrix A is said to be of Boolean rank r if there exist $m \times r$ Boolean matrix B and $r \times n$ Boolean matrix C such that $A = BC$ and r is the smallest positive integer that such a factorization exists. We consider the the sets of matrix ordered pairs which satisfy extremal properties with respect to Boolean rank inequalities of matrices over nonbinary Boolean algebra. We characterize linear operators that preserve these sets of matrix ordered pairs as the form of $T(X) = PXP^T$ with some permutation matrix P.

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1. Introduction

One of the most active and fertile subjects in matrix theory during the past one hundred years is the linear preserver problem, which concerns the characterization of linear operators on matrix spaces that leave certain functions, subsets, relations, etc., invariant. In 1896, Frobenius characterized the linear operators that preserve determinant of matrices over the real field, which was the first results on linear preserver problems. After his result, many researchers have studied linear operators that preserve some matrix functions, say, rank and the permanent of matrices([9]). Beasley and Guterman([1]) investigated rank inequalities of matrices over semirings. And they characterized the equality cases for some inequalities in [2]. The investigation of the corresponding problems over semirings for the column rank function was done in [3]. The complete classification of linear operators that preserve equality cases in matrix inequalities over fields was obtained in [5]. For details on linear operators preserving matrix invariants one can see [8] and [9]. Almost all research on linear preserver problems over semirings have dealt with those semirings without zero-divisors

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to avoid the difficulties of multiplication arithmetic for the elements in those semirings([2]-[6]). But nonbinary Boolean algebra is not the case. That is, all elements except 0 and 1 in the nonbinary Boolean algebra are zero-divisors. So there are few results on the linear preserver problems for the matrices over nonbinary Boolean algebra([7], [10]). Kirkland and Pullman characterized the linear operators that preserve rank of matrices over nonbinary Boolean algebra in [7].

In this paper, we characterize the linear operators that preserve the sets of matrix ordered pairs which satisfy extremal properties with respect to Boolean rank inequalities of matrices over nonbinary Boolean algebra.

2. Preliminaries and basic results

A semiring S consists of a set S with two binary operations, addition and multiplication, such that:

- · S is an Abelian monoid under addition (the identity is denoted by 0);
- · S is a monoid under multiplication (the identity is denoted by 1, $1 \neq 0$);
- · multiplication is distributive over addition on both sides;
- · $s0 = 0s = 0$ for all $s \in S$.

A semiring $\mathcal{S}$ is called *antinegative* if the zero element is the only element with an additive inverse.

A semiring $\mathcal{S}$ is called a *Boolean algebra* if $\mathcal{S}$ is equivalent to a set of subsets of a given set M, the sum of two subsets is their union, and the product is their intersection. The zero element is the empty set and the identity element is the whole set M. Let $S_k = \{a_1, a_2, \cdots, a_k\}$ be a set of k-elements, $\mathcal{P}(S_k)$ be the set of all subsets of S_k and $\mathbb{B}_k$ be a Boolean algebra of subsets of $S_k = \{a_1, a_2, \cdots, a_k\}$, which is a subset of $\mathcal{P}(S_k)$. It is straightforward to see that a Boolean algebra $\mathbb{B}_k$ is a commutative and antinegative semiring. If $\mathbb{B}_k$ consists of only the empty subset and M then it is called a *binary Boolean algebra*. If $\mathbb{B}_k$ is not binary Boolean algebra then it is called a *nonbinary Boolean algebra*. Let $\mathbb{M}_{m,n}(\mathbb{B}_k)$ denote the set of $m \times n$ matrices with entries from the Boolean algebra $\mathbb{B}_k$. If $m = n$, we use the notation $\mathbb{M}_n(\mathbb{B}_k)$ instead of $\mathbb{M}_{n,n}(\mathbb{B}_k)$.

Throughout the paper, we assume that $m \leq n$ and $\mathbb{B}_k$ denotes the nonbinary Boolean algebra, which contains at least 3 elements. The matrix I_n is the $n \times n$ identity matrix, $J_{m,n}$ is the $m \times n$ matrix of all ones and $O_{m,n}$ is the $m \times n$ zero matrix. We omit the subscripts when the order is obvious from the context and we write I, J and O, respectively. The matrix $E_{i,j}$, which is called a *cell*, denotes the matrix with exactly one nonzero entry, that being a one in the $(i,j)^{th}$ entry. A *weighted cell* is any nonzero scalar multiple of a cell, that is, $\alpha E_{i,j}$ is a weighted cell for any $0 \neq \alpha \in \mathbb{B}_k$. Let R_i denote the matrix whose i^{th} row is all ones and is zero elsewhere, and C_j denote the matrix whose j^{th} column is all ones and is zero elsewhere. We denote by $\#(A)$ the number of nonzero entries in the matrix A. We denote by $A[i,j|r,s]$ the 2×2 submatrix of A which lies in the intersection of the i^{th} and j^{th} rows with the r^{th} and s^{th} columns.

The matrix $A \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ is said to be of *Boolean rank r* if there exist matrices

$B \in \mathbb{M}_{m,r}(\mathbb{B}_k)$ and $C \in \mathbb{M}_{r,n}(\mathbb{B}_k)$ such that $A = BC$ and r is the smallest positive integer that such a factorization exists. We denote $b(A) = r$. By definition, the unique matrix with Boolean rank equal to 0 is the zero matrix O.

A *line* of a matrix A is a row or a column of the matrix A.

For $X, Y \in \mathbb{M}_{m,n}(\mathcal{S})$, the matrix $X \circ Y$ denotes the *Hadamard or Schur product*, i.e., the $(i,j)^{th}$entry of $X \circ Y$ is $x_{i,j}y_{i,j}$.

We say that the matrix A *dominates* the matrix B if and only if $b_{i,j} \neq 0$ implies that $a_{i,j} \neq 0$, and we write $A \geq B$ or $B \leq A$.

An operator $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ is called *linear* if it satisfies $T(X+Y) = T(X) + T(Y)$ and $T(\alpha X) = \alpha T(X)$ for all $X, Y \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ and $\alpha \in \mathbb{B}_k$.

We say that an operator T *preserves* a set $\mathcal{P}$ if $X \in \mathcal{P}$ implies that $T(X) \in \mathcal{P}$ or if $\mathcal{P}$ is the set of ordered pairs (triples) such that $(X, Y) \in \mathcal{P}$ (respectively, $(X, Y, Z) \in \mathcal{P}$) implies $((T(X), T(Y)) \in \mathcal{P}$ (respectively, $(T(X), T(Y), T(Z)) \in \mathcal{P}$).

An operator T *strongly preserves* a set $\mathcal{P}$ if $X \in \mathcal{P}$ if and only if $T(X) \in \mathcal{P}$ or if $\mathcal{P}$ is the set of ordered pairs (triples) such that $(X, Y) \in \mathcal{P}$ (respectively, $(X, Y, Z) \in \mathcal{P}$) if and only if $(T(X), T(Y)) \in \mathcal{P}$ (respectively, $(T(X), T(Y), T(Z)) \in \mathcal{P}$).

An operator T is called a (P, Q, B)-*operator* if there exist permutation matrices P and Q and a matrix $B \in \mathbb{M}_{m,n}(\mathcal{S})$ with no zero entries such that $T(X) = P(X \circ B)Q$ for all $X \in \mathbb{M}_{m,n}(\mathcal{S})$ or if for $m = n$, $T(X) = P(X \circ B)^T Q$ for all $X \in \mathbb{M}_{m,n}(\mathcal{S})$. A (P, Q, B)-operator is called a (P, Q)-*operator* if $B = J$, the matrix of all ones.

If $\mathcal{S}$ is a field, then there is the usual rank function $\rho(A)$ for any matrix $A \in \mathbb{M}_{m,n}(\mathcal{S})$. It is well-known that the behavior of the function ρ with respect to matrix addition and multiplication is given by the following inequalities([4]):

- the rank-sum inequalities:
$$|\rho(A) - \rho(B)| \leq \rho(A + B) \leq \rho(A) + \rho(B),$$
- Sylvester's laws:
$$\rho(A) + \rho(B) - n \leq \rho(AB) \leq \min\{\rho(A), \rho(B)\},$$
- and the Frobenius inequality:
$$\rho(AB) + \rho(BC) \leq \rho(ABC) + \rho(B),$$

where A, B are conformal matrices with entries from a field.

The arithmetic properties of Boolean rank inequalities are restricted by the following list of inequalities([1]), since the Boolean algebra is antinegative:

(1) $\mathrm{b}(A + B) \leq \min\{\ \mathrm{b}(A) + \mathrm{b}(B), \mathrm{m}, \mathrm{n}\ \}$;

(2) $\mathrm{b}(AB) \leq \min\{\mathrm{b}(A), \mathrm{b}(B)\}$.

(3) $\mathrm{b}(A + B) \geq \begin{cases} \mathrm{b}(A) & \text{if } B = O, \\ \mathrm{b}(B) & \text{if } A = O, \\ 1 & \text{if } A \neq O \text{ and } B \neq O; \end{cases}$

(4) $\mathrm{b}(AB) \geq \begin{cases} 0 & \text{if } \mathrm{b}(A) + \mathrm{b}(B) \leq n, \\ 1 & \text{if } \mathrm{b}(A) + \mathrm{b}(B) > n. \end{cases}$

Below, we use the following notation in order to denote sets of matrix pairs that arise as extremal cases in the inequalities listed above:
$\mathcal{R}_{SD}(\mathbb{B}_k) = \{(X,Y) \in \mathbb{M}_{m,n}(\mathbb{B}_k)^2 |\ \mathrm{b}(X+Y) = |\mathrm{b}(X) - \mathrm{b}(Y)|\}$,
$\mathcal{R}_{MA}(\mathbb{B}_k) = \{(X,Y) \in \mathbb{M}_n(\mathbb{B}_k)^2 |\ \mathrm{b}(XY) = \mathrm{b}(X) + \mathrm{b}(Y) - n\}$,
$\mathcal{R}_{M3}(\mathbb{B}_k) = \{(X,Y,Z) \in \mathbb{M}_n(\mathbb{B}_k)^3 |\ \mathrm{b}(XYZ) + \mathrm{b}(Y) = \mathrm{b}(XY) + \mathrm{b}(YZ)\}$.

In this paper, we characterize the linear operators that preserve $\mathcal{R}_{SD}(\mathbb{B}_k)$, $\mathcal{R}_{MA}(\mathbb{B}_k)$ and $\mathcal{R}_{M3}(\mathbb{B}_k)$.

Theorem 2.1 ([10], Theorem 2.15)**.** *Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a linear operator. Then the following conditions are equivalent:*
(a) T is bijective;
(b) T is surjective;
(c) T is injective;
(d) there exists a permutation σ on $\{(i,j)|i = 1,2,\ldots,m; j = 1,2,\ldots,n\}$ such that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all $1 \le i \le m$ and $1 \le j \le n$.

Remark 2.2. One can easily verify that if $m = 1$ or $n = 1$, then all operators under consideration are (P,Q,B)-operators and if $m = n = 1$, then all operators under consideration are (P,P^T,B)-operators.

Henceforth we will always assume that $m, n \ge 2$.

Lemma 2.3 ([10], Theorem 2.17)**.** *Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a linear operator which maps a line to a line and T be defined by the rule $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$, where σ is a permutation on the set $\{(i,j)|i = 1,2,\ldots,m; j = 1,2,\ldots,n\}$ and $b_{i,j} \in \mathbb{B}_k$ are nonzero elements for $i = 1,2,\ldots,m; j = 1,2,\ldots,n$. Then T is a (P,Q,B)-operator.*

3. Linear preservers of $\mathcal{R}_{SD}(\mathbb{B}_k)$

Recall that

$$\mathcal{R}_{SD}(\mathbb{B}_k) = \{(X,Y) \in \mathbb{M}_{m,n}(\mathbb{B}_k)^2 |\ \mathrm{b}(X+Y) = |\mathrm{b}(X) - \mathrm{b}(Y)|\}.$$

Lemma 3.1. *Let σ be a permutation of the set $\{(i,j)|i = 1,2,\ldots, m; j = 1,2,\ldots,n\}$, and $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be defined by $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ with nonzero $b_{i,j} \in \mathbb{B}_k$ for all $1 \le i \le m$ and $1 \le j \le n$, and min$\{m,n\} \ge 3$. If T preserves $\mathcal{R}_{SD}(\mathbb{B}_k)$, then T maps a line to a line.*

Proof. Since the sum of three weighted distinct cells has Boolean rank at most 3, it follows that $\mathrm{b}(T(E_{1,1} + E_{1,2} + E_{2,1})) \le 3$. Now, for $X = E_{1,1} + E_{1,2} + E_{2,1}$ and $Y = E_{2,2}$, we have that $(X,Y) \in \mathcal{R}_{SD}(\mathbb{B}_k)$, and the image of Y under T is a single weighted cell, and hence $\mathrm{b}(T(Y)) = 1$. Now, if $\mathrm{b}(T(X)) = 3$, then $T(X)$ is the sum of three weighted cells that lie in distinct lines. Thus $T(X+Y)$ must have Boolean rank 3 or 4, and hence $(T(X),T(Y)) \notin \mathcal{R}_{SD}(\mathbb{B}_k)$, a contradiction. If $\mathrm{b}(T(X)) = 1$, then $T(X+Y) \ne O$ and $(T(X),T(Y)) \notin \mathcal{R}_{SD}(\mathbb{B}_k)$, a contradiction. Consequently we have that $\mathrm{b}(T(X)) = 2$, and hence $\mathrm{b}(T(X+Y)) = 1$ from $(T(X),T(Y)) \in \mathcal{R}_{SD}(\mathbb{B}_k)$. However it is obvious that if a sum of four cells has the Boolean rank 1, then they lie either in a line or in the intersection of two rows and two columns. The matrix $T(X+Y)$ is a sum of four cells. These cells do not lie in a line since $\mathrm{b}(T(X)) = 2$. Thus $T(X+Y)$ must be the sum of four cells which lie in the intersection of two rows and two columns. Similarly, for any i, j, h and l, $T(E_{i,j} + E_{i,h} + E_{l,j} + E_{l,h})$ must lie in the intersection of two rows and two columns. It follows that any two rows must be mapped into two lines. By the bijectivity of T, if some pair of two rows is mapped into two rows (resp. columns), any pair of two rows is mapped into two rows(resp. columns). Similarly, if some pair of two columns is mapped into two rows(resp. columns), any pair of two columns is mapped into two rows (resp. columns). Now, the image of three rows is contained in three lines, two of which are the image of two rows, thus every row is mapped into a line. Similarly for columns. Thus T maps a line to a line. □

Theorem 3.2. *Let $m, n \geq 2$ and T be a surjective linear operator on $\mathbb{M}_{m,n}(\mathbb{B}_k)$. Then T preserves $\mathcal{R}_{SD}(\mathbb{B}_k)$ if and only if T is a (P, Q)-operator.*

Proof. Assume that T is surjective and a (P, Q)-operator. For any $(X, Y) \in \mathcal{R}_{SD}(\mathbb{B}_k)$, we have $\mathrm{b}(X+Y)= |\mathrm{b}(X)-\mathrm{b}(Y)|$. Thus $\mathrm{b}(T(X)+T(Y)) = \mathrm{b}(T(X+Y)) = \mathrm{b}(P(X+Y)Q) = \mathrm{b}(X+Y) = |\mathrm{b}(X)-\mathrm{b}(Y)| = |\ \mathrm{b}(PXQ) - \mathrm{b}(PYQ)| = |\mathrm{b}(T(X))-\mathrm{b}(T(Y))|$. Hence $((T(X), T(Y)) \in \mathcal{R}_{SD}(\mathbb{B}_k)$. Therefore T preserves $\mathcal{R}_{SD}(\mathbb{B}_k)$.

Conversely, assume that T preserves $\mathcal{R}_{SD}(\mathbb{B}_k)$. Since T is a surjective linear operator, there exists permutation σ on $\{(i,j) | i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\}$ such that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$ by Theorem 2.1. Hence T maps any line to a line by Lemma 3.1. Therefore T is a (P, Q)-operator by Lemma 2.3 since all the entries of B are 1. □

4. Linear preservers of $\mathcal{R}_{MA}(\mathbb{B}_k)$

Recall that

$$\mathcal{R}_{MA}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_n(\mathbb{B}_k)^2 |\ \mathrm{b}(XY) = \mathrm{b}(X) + \mathrm{b}(Y) - n\}.$$

Consider $X = I_n$ and $Y = E_{1,1}$. Then $\mathrm{b}(XY) = \mathrm{b}(E_{1,1}) = 1$ and hence $\mathrm{b}(XY) = \mathrm{b}(X) + \mathrm{b}(Y) - n$. That is $(X, Y) \in \mathcal{R}_{MA}(\mathbb{B}_k)$. Thus $\mathcal{R}_{MA}(\mathbb{B}_k)$ is not an empty set.

Theorem 4.1. *Let $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$, $n > 4$, be a surjective linear operator. Then T preserves $\mathcal{R}_{MA}(\mathbb{B}_k)$ if and only if there exists a permutation matrix P such that $T(X) = PXP^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$.*

Proof. $\Leftarrow$) If $T(X) = PXP^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$ and some permutation matrix P, then $T(XY) = P(XY)P^T = PXP^T PYP^T = T(X)T(Y)$. Thus $\mathrm{b}(T(X)T(Y)) = \mathrm{b}(T(XY)) = \mathrm{b}(PXYP^T) = \mathrm{b}(XY) = \mathrm{b}(X) + \mathrm{b}(Y) - n = \mathrm{b}(PXP^T) + \mathrm{b}(PYP^T) - n = \mathrm{b}(T(X)) + \mathrm{b}(T(Y)) - n$. Hence T preserves $\mathcal{R}_{MA}(\mathbb{B}_k)$.

$\Rightarrow$) Now we assume that T preserves $\mathcal{R}_{MA}(\mathbb{B}_k)$. Since T is surjective, by Theorem 2.1 we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for some permutation σ. If $\mathrm{b}(A) = n$, then $T(E_{i,j}, A) \in \mathcal{R}_{MA}(\mathbb{B}_k)$ and since $\mathrm{b}(T(E_{i,j})) = 1$ and T preserves $\mathcal{R}_{MA}(\mathbb{B}_k)$, it follows that $\mathrm{b}(T(A)) = n$. Therefore T maps the set of matrices with Boolean rank n to itself. If the preimage of a row is not dominated by any line, then there are cells $E_{r,s}$ and $E_{p,q}$ such that $T(E_{r,s} + E_{p,q}) \le E_{i,h} + E_{i,l}$ and $r \ne p, s \ne q$. By extending $E_{r,s} + E_{p,q}$ to a permutation matrix by adding $n-2$ cells, we find a matrix which is the image of a permutation matrix but is dominated by $n-1$ lines; a contradiction since T maps the set of matrices with Boolean rank n to itself. Thus the preimage of every row is a row or column and, similarly, the preimage of every column is a row or a column. Hence T maps any line to a line. By Lemma 2.3, we have that T is a (P, Q, B)-operator with $B = J$. i.e., T is a (P, Q)-operator. Since $(E_{1,1}, E_{2,1} + E_{3,2} + \ldots + E_{n,n-1}) \in \mathcal{R}_{MA}(\mathbb{B}_k)$ while $(E_{1,1}, E_{1,2}+E_{2,3}+\ldots+E_{n-1,n}) \notin \mathcal{R}_{MA}(\mathbb{B}_k)$, we have that the transpose operator does not preserve $\mathcal{R}_{MA}(\mathbb{B}_k)$, thus there exist permutation matrices P and Q such that $T(X) = PXQ$. Without loss of generality, we may assume that $P = I$. If $Q \ne I$, we assume that Q corresponds to the permutation π and $\pi(1) \ne 1$. Without loss of generality, $T(E_{1,1}) = E_{1,2}$. Then $(E_{1,1}, E_{2,2} + E_{3,3} + \ldots + E_{n,n}) \in \mathcal{R}_{MA}(\mathbb{B}_k)$, while $(T(E_{1,1}), T(E_{2,2}+E_{3,3}+\ldots+E_{n,n})) \notin \mathcal{R}_{MA}(\mathbb{B}_k)$ since $(E_{1,2})(E_{2,\pi(2)} + E_{3,\pi(3)} + \ldots + E_{n,\pi(n)}) = E_{1,2}E_{2,\pi(2)} \ne O$. This contradiction gives that $Q = P^T$ and hence $T(X) = PXP^T$. $\square$

5. Linear preservers of $\mathcal{R}_{M3}(\mathbb{B}_k)$

Recall that
$\mathcal{R}_{M3}(\mathbb{B}_k) = \{(X, Y, Z) \in \mathbb{M}_n(\mathbb{B}_k)^3 |\ \mathrm{b}(XYZ)+\mathrm{b}(Y) = \mathrm{b}(XY) + \mathrm{b}(YZ)\}$.

Lemma 5.1. *Let $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$, $n > 4$, be a surjective linear operator. If T preserves $\mathcal{R}_{M3}(\mathbb{B}_k)$, then there exists a permutation matrix P such that $T(X) = PXP^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$.*

Proof. By Theorem 2.1, we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for a certain permutation σ on $\{(i,j)|1 \le i, j \le n\}$. It can be easily proved that $(E_{i,j}, E_{j,h}, E_{h,l}) \in \mathcal{R}_{M3}(\mathbb{B}_k)$ for all l and arbitrary fixed i, j and h. Thus

$$\begin{aligned} &\mathrm{b}(T(E_{i,j})T(E_{j,h})) + \mathrm{b}(T(E_{j,h})T(E_{h,l})) \\ &= \mathrm{b}(T(E_{i,j})T(E_{j,h})T(E_{h,l})) + \mathrm{b}(T(E_{j,h})). \end{aligned} \tag{3.9.1}$$

It follows from Theorem 2.1 that $T(E_{i,j}) = E_{p,q}$, $T(E_{j,h}) = E_{r,s}$, $T(E_{h,l}) = E_{u,v}$ for subscripts p, q, r, s, u, and v. Since $\mathrm{b}(E_{r,s}) = 1 \ne 0$, it follows from equality (3.9.1) that either $q = r$ or $s = u$ or both. If, for all $l = 1, \ldots, n$, both equalities hold, then for fixed i, j, and h, all matrices $T(E_{h,l})$, $l = 1, \ldots, n$, have their nonzero elements lying in one row. Thus T maps rows to rows. Similarly, it is easy to see that T maps columns to columns. Assume now that there exists an index l such that only one of the above equalities holds for the triple $(E_{i,j}, E_{j,h}, E_{h,l})$. Without loss of generality, assume that $s = u$ and $q \ne r$. Thus for arbitrary m, $1 \le m \le n$, one has that $(E_{i,j}, E_{j,h}, E_{h,m}) \in$

$\mathcal{R}_{M3}(\mathbb{B}_k)$. By Theorem 2.1, $T(E_{h,m}) = E_{w,z}$ for certain w and z depending on h and m. In the above notation, $(E_{p,q}, E_{r,s}, E_{w,z}) \in \mathcal{R}_{M3}(\mathbb{B}_k)$. Since $q \neq r$, it follows that $w = s$ for any w. Thus, in this case, we also obtain that rows are transformed to rows. By the same arguments with the first matrix, it is easy to see that columns are transformed to columns. In the other case ($s \neq u$ and $q = r$), one obtains similarly that rows are transformed to rows and columns to columns. By Lemma 2.3, if follows that there exist permutation matrices P and Q such that $T(X) = P(X \circ B)Q$ with $B = J$. (i.e., $T(X) = PXQ$) for all $X \in \mathbb{M}_n(\mathbb{B}_k)$.) In order to show that $Q = P^T$ it suffices to note that $(E_{i,j}, E_{j,j}, E_{j,i}) \in \mathcal{R}_{M3}(\mathbb{B}_k)$. Let π be the permutation corresponding to P and τ be the permutation corresponding to Q^T. Therefore $(T(E_{i,j}), T(E_{j,j}), T(E_{j,i})) = (PE_{i,j}Q, PE_{j,j}Q, PE_{j,i}Q) = (E_{\pi(i),\tau(j)}), E_{\pi(j),\tau(j)}, E_{\pi(j),\tau(i)}) \in \mathcal{R}_{M3}(\mathbb{B}_k)$. Thus $\pi \equiv \tau$ and $Q = P^T$. □

Theorem 5.2. *Let $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$ be a surjective linear operator, for $n > 4$. Then T preserves $\mathcal{R}_{M3}(\mathbb{B}_k)$ if and only if $T(X) = PXP^T$ for all $X \in \mathbb{M}_n(\mathbb{B}_k)$, where P is a permutation matrix.*

Proof. $\Leftarrow$) If $(X, Y, Z) \in \mathcal{R}_{M3}(\mathbb{B}_k)$, then
$\mathrm{b}(XYZ) + \mathrm{b}(Y) = \mathrm{b}(XY) + \mathrm{b}(YZ)$.
Thus
$\mathrm{b}(T(X)T(Y)T(Z)) + \mathrm{b}(T(Y))$
$= \mathrm{b}(PXP^TPYP^TPZP^T) + \mathrm{b}(PYP^T)$
$= \mathrm{b}(PXYZP^T) + \mathrm{b}(PYP^T)$
$= \mathrm{b}(XYZ) + \mathrm{b}(Y)$.
Similarly
$\mathrm{b}(T(X)T(Y)) + \mathrm{b}(T(Y)T(Z))$
$= \mathrm{b}(XY) + \mathrm{b}(YZ)$.
Hence
$\mathrm{b}(T(X)T(Y)T(Z)) + \mathrm{b}(T(Y))$
$= \mathrm{b}(T(X)T(Y)) + \mathrm{b}(T(Y)T(Z))$.
i.e., $(T(X), T(Y), T(Z)) \in \mathcal{R}_{M3}(\mathbb{B}_k)$. Therefore T preserves $\mathcal{R}_{M3}(\mathbb{B}_k)$.
$\Rightarrow$) Assume that T preserves $\mathcal{R}_{M3}(\mathbb{B}_k)$. Then, by Lemma 5.1, T has the form $T(X) = PXP^T$ for some permutation matrix P. □

As a concluding remark, we have characterized the linear operators that preserve the extreme sets of matrix pairs over nonbinary Boolean algebra which come from certain Boolean rank inequalities.

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