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A MULTIPLICITY RESULT FOR FOURTH-ORDER BOUNDARY VALUE PROBLEMS VIA CRITICAL POINTS THEOREM[†]

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Abstract. In this paper, using B.Ricceri's three critical points theorem, we prove the existence of at least three classical solutions for the problem

$$\begin{cases} u^{(4)}(t) = \lambda f(t, u(t)), \quad t \in (0,1), \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases}$$

under appropriate hypotheses.

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1. Introduction

In this work, we study the boundary value problem

$$\begin{cases} u^{(4)}(t) = \lambda f(t, u(t)), \quad t \in (0,1), \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases} \tag{1}$$

where $f : [a, b] \times R \to R$ is a continuous function and $\lambda > 0$. Since the problem (1) cannot transform into a system of second-order equation, the treatment method of second-order system does not apply to the problem (1). Thus, existing literature on the problem (1) is limited. In 1984, Agarwal and chow [1] firstly investigated the existence of the solutions of the problem (1) by contraction mapping and iterative methods, subsequently, Ma and Wu [2], Ma and Tisdel [3], Yao [4, 5] and Korman [6] studied the existence of positive solutions of this problem by the Krasnosel'skii fixed point theorem on cones, Leray-Schauder fixed point theorem and techniques of bifurcation theory.

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Recently, many papers have appeared in which the technical approach adopted is based on the three critical point theorem obtained by Ricceri [7]. We cite papers [8, 9, 10], where the authors, by using Ricceri's three critical point theorem, established the existence of at least three weak solutions to the Dirichlet boundary value problem.

In [8], Bonanno used the three-critical-points theorem to obtain three solutions of the two-point boundary value problem

$$\begin{cases} u'' + \lambda f(u) = 0, \\ u(0) = u(1) = 0, \end{cases}$$

where λ is a positive parameter and $f : R \to R$ is a continuous function.

In [9], Candito extended the main result of [8] to the nonautonomous case

$$\begin{cases} u'' + \lambda f(t, u) = 0, \\ u(0) = u(1) = 0, \end{cases}$$

where λ is a parameter and $f : [a, b] \times R \to R$ is a continuous function.

In [10], Ricceri's three critical point theorem has been successfully used to obtain multiple solutions for p-Laplacian type equations under Dirichlet boundary conditions. In [10], He and Ge extended the main results of [8, 9] to quasilinear differential equations, i.e.

$$\begin{cases} (|u'|^{p-2}u')' + \lambda f(t, u) = 0 \\ u(a) = u(b) = 0, \end{cases}$$

In this paper, we prove the existence of three solutions of problem (1). The technical approach is based on the three critical point theorem obtained by Ricceri [7] too. Our Theorem 1 under novel assumptions ensures the existence of an open interval $\Lambda \subset [0, +\infty)$ and a positive real number q, such that, for each $\lambda \in \Lambda$, problem (1) admits at least three classical solutions whose norms in H_0^2 are less than q. The aim of the present paper is to extended the main results of [8, 9, 10] to problem (1).

We recall here for the reader's convenience the three critical points Theorem of [7] and Proposition 3.1 of [11].

Theorem A. *Let X be a separable and reflexive real Banach space; $\Phi : X \to R$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi : X \to R$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that*

$$\lim_{\|u\| \to +\infty} (\Phi(u) + \lambda\Psi(u)) = +\infty$$

for all $\lambda \in [0, +\infty)$, and that there exists a continuous concave function $h : [0, +\infty) \to R$ such that

$$\sup_{\lambda \geq 0} \inf_{u \in X} (\Phi(u) + \lambda\Psi(u) + h(\lambda)) < \inf_{u \in X} \sup_{\lambda \geq 0} (\Phi(u) + \lambda\Psi(u) + h(\lambda))$$

Then, there exist an open interval $\Lambda \subset [0,+\infty)$ and a positive real number q such that, for each $\lambda \in \Lambda$, the equation

$$\Phi'(u) + \lambda\Psi'(u) = 0,$$

has at least three solutions in X whose norms are less than q.

Proposition B. *Let X be a non-empty set and Φ, J two real functions on X. Assume that there are $r > 0$ and $x_0, x_1 \in X$ such that*

$$\Phi(x_0) = J(x_0) = 0, \quad \Phi(x_1) > r,$$

$$\sup_{x\in\Phi^{-1}((\infty,r])} J(x) < r\frac{J(x_1)}{\Phi(x_1)}.$$

Then, for each ρ satisfying

$$\sup_{x\in\Phi^{-1}((\infty,r])} J(x) < \rho < r\frac{J(x_1)}{\Phi(x_1)},$$

one has

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u) + \lambda(\rho - J(x))) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u) + \lambda(\rho - J(x)))$$

2. Main results

Here and in the sequel, X will denote the Sobolev space H_0^2. The norm of H_0^2 is denoted by $\|\cdot\|$:

$$\|u\| = \{\int_0^1 |u^{''}(s)|^2 \mathrm{d}s\}^{\frac{1}{2}},$$

and H_0^2 is the completion of $C_0^\infty(0,1)$ with respect to this norm.

Let $k > 2$ be a positive constant, and define the real function $g(t,\xi)$ by

$$g(t,\xi) = \int_0^\xi f(t,u)du, \quad \text{for all} \quad (t,\xi) \in [0,1]\times R.$$

Our main results fully depend on the following Lemma 1.

Lemma 1. *Assume that there exist two positive constants d, c, with $c < \frac{4kd\sqrt{2k}}{\pi}$, such that*

(i) $g(t,\xi) \geq 0$ for each $(t,\xi) \in [0,\frac{1}{k}] \cup [1-\frac{1}{k},1] \times [0,d]$,

(ii) $\max_{(t,\xi)\in[0,1]\times[-c,c]} g(t,\xi) < \frac{c^2\pi^2}{32k^3d^2}\int_{\frac{1}{k}}^{1-\frac{1}{k}} g(t,d)dt$.

Then there exist $r > 0$ and $u \in X$ such that

$$2r < \|u\|^2$$

and

$$\max_{(t,\xi)\in[0,1]\times[-c,c]} g(t,\xi) \leq 2r\frac{\int_0^1 g(t,u(t))dt}{\|u\|^2}$$

Proof. We define the function

$$u(t)=\begin{cases} 2k^2dt^2, & 0\le t\le \frac{1}{2k},\\ -2k^2d(t-\frac{1}{k})^2+d, & \frac{1}{2k}\le t\le \frac{1}{k},\\ d, & \frac{1}{k}\le t\le 1-\frac{1}{k},\\ -2k^2d(t-1+\frac{1}{k})^2+d, & 1-\frac{1}{k}\le t\le 1-\frac{1}{2k},\\ 2k^2d(t-1)^2, & 1-\frac{1}{2k}\le t\le 1,\end{cases}$$

and $2r=\pi^2c^2$. It is clear that $u\in H_0^2$ and $\|u\|^2=32k^3d^2$. Hence, taking into account that $c<\frac{4kd\sqrt{2k}}{\pi}$, one has

$$2r=\pi^2c^2<32k^3d^3=\|u\|^2.$$

Moreover, owing to our assumptions, we have

$$\begin{aligned}\frac{\int_0^1 g(t,u(t))dt}{\|u\|^2}2r &\ge \frac{\int_{\frac{1}{k}}^{1-\frac{1}{k}} g(t,u(t))dt}{32k^3d^2}2\left(\frac{c^2\pi^2}{2}\right)\\ &=\frac{c^2\pi^2}{32k^3d^2}\int_{\frac{1}{k}}^{1-\frac{1}{k}} g(t,d)dt>\max_{(t,\xi)\in[0,1]\times[-c,c]} g(t,\xi)\end{aligned}$$

i.e.,

$$\max_{(t,\xi)\in[0,1]\times[-c,c]} g(t,\xi)\le 2r\frac{\int_0^1 g(t,u(t))dt}{\|u\|^2}.$$

Then the proof is completed. □

Our main result is the following theorem.

Theorem 1. *Suppose that there exist four positive constants c,d,μ,s with $s<2$ and with $c<\frac{4kd\sqrt{2k}}{\pi}$, such that*

(i) conditions (i) and (ii) in Lemma 1 hold,

(ii) $g(t,\xi)\le\mu(1+|\xi|^s)$ for each $t\in[0,1]$ and $\xi\in R$.

Then there exist an open interval $\Lambda\subset[0,+\infty)$ and a positive real number q, such that, for each $\lambda\in\Lambda$, problem (1) admits at least three classical solutions belonging to $C^2[0,1]$ whose norms in H_0^2 are less than q.

Proof. For each $u\in X$, we define

$$\Phi(u)=\frac{1}{2}\|u\|^2,\quad \Psi(u)=-\int_0^1\left(\int_0^{u(t)} f(t,x)dx\right)dt,\quad J(u)=\Phi(u)+\lambda\Psi(u).$$

It is well known that the critical points of J are the classical solutions of (6). So, our end is to verify that Φ and Ψ satisfy the assumptions of Theorem **A**. It is easy to see that Φ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, and Ψ is a continuous Gâteaux differentiable functional whose Gâteaux derivative is compact.

Moreover, thanks to (ii) and to Poincaré inequality, one has

$$\lim_{\|u\|\to+\infty} \Phi(u) + \lambda\Psi(u) = +\infty,$$

for all $\lambda \in (0, +\infty)$.

Now, taking into account that

$$\max_{t\in[0,1]} |u(t)| \leq \frac{1}{\pi}\|u\|.$$

It follows that

$$\Phi^{-1}((-\infty, r]) \subset \left\{ u \in X : \ |u(t)| \leq \frac{\sqrt{2r}}{\pi} \right\},$$

for each $r > 0$.

On the other hand, we have

$$\sup_{x\in\Phi^{-1}((\infty,r])} -\Psi(u) = \sup_{\|u\|^2\leq 2r} \int_0^1 g(t, u(t))dt \leq \max_{(t,\xi)\in[0,1]\times[-c,c]} g(t, \xi).$$

Now, owing to Lemma 1, there exists $r > 0$ and $u \in X$ such that

$$\max_{(t,\xi)\in[0,1]\times[-c,c]} g(t, \xi) < 2r \frac{\int_0^1 g(t, u(t))dt}{\|u\|^2} = r\frac{(-\Psi(u))}{\Phi(u)}$$

So, there exists $\rho > 0$, such that

$$\sup_{x\in\Phi^{-1}((\infty,r])} -\Psi(u) < \rho < r\frac{(-\Psi(u))}{\Phi(u)}.$$

Finally, owing to Proposition **B**, choosing $h(\lambda) = \rho\lambda$ for $\lambda \geq 0$, then we obtain

$$\sup_{\lambda\geq 0} \inf_{u\in X} (\Phi(u) + \lambda\Psi(u) + h(\lambda)) < \inf_{u\in X} \sup_{\lambda\geq 0}(\Phi(u) + \lambda\Psi(u) + h(\lambda))$$

Hence, by an application of Theorem **A**, we complete the proof. □

Now, our conclusion follows from Theorem 1.

Let $l \in C[0, 1]$ and $h \in C(R)$ be two nonnegative functions. Put

$$L(t) = \int_0^t l(\tau)d\tau, \qquad H(\xi) = \int_0^\xi h(\tau)d\tau.$$

We consider the special case of problem

$$\begin{cases} u^{(4)}(t) + \lambda l(t)h(u) = 0, \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases} \tag{2}$$

Corollary 1. *Suppose that there exist four positive constants c, d, σ, s with $s < 2$ and with $c < \frac{4kd\sqrt{2k}}{\pi}$, such that*

(i) $\max_{t\in[0,1]} l(t) < \frac{c^2\pi^2}{32k^3d^2}\frac{H(d)}{H(c)}[L(1-\frac{1}{k}) - L(\frac{1}{k})]$,

(ii)$H(\xi) \leq \sigma(1+|\xi|^s)$ *for each* $t \in [0,1]$ *and* $\xi \in R$.

Then there exist an open interval $\Lambda \subset [0,+\infty)$ and a positive real number q, such that, for each $\lambda \in \Lambda$, problem (2) admits at least three classical solutions belonging to $C^2[0,1]$ whose norms in H_0^2 are less than q.

Proof. Let

$$f(t,u) = l(t)h(u), \quad \text{for each} \quad (t,u) \in [0,1]\times R,$$

and we have

$$\max_{(t,\xi)\in[0,1]\times[-c,c]} g(t,\xi) = \max_{(t,\xi)\in[0,1]\times[-c,c]} \int_0^\xi f(t,x)dx = \max_{t\in[0,1]} l(t)H(c).$$

Taking $\mu = \sigma \max_{t\in[0,1]} l(t)$, it is easy to verify that all the assumptions of Theorem 1 are satisfied. So the proof is finished. □

Finally, we give an example to illustrate our main result.

Example 1. We consider (2) with $f(t,u) = t\cdot h(u)$, where

$$h(u) = \begin{cases} e^{-u}u^{12}(13-u), & u \in [0,13], \\ 0, & u > 13. \end{cases}$$

In this case, one has $L(t) = \frac{t^2}{2}$, and

$$H(\xi) = \begin{cases} e^{-u}u^{13}, & u \in [0,13], \\ (13e)^{-13}, & u > 13. \end{cases}$$

It is easy to verify that with $c = \frac{1}{2}, d = 2, k = 8, s = 1, \sigma = (13e)^{-13}$, all conditions of Corollary 1 are satisfied. Therefore there exist an open interval $\Lambda \subset [0,+\infty)$ and a positive real number q, such that, for each $\lambda \in \Lambda$, problem (2) admits at least three classical solutions belonging to $C^2[0,1]$ whose norms in H_0^2 are less than q.

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