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A META-SOFTWARE SYSTEM FOR ORTHOGONAL DESIGNS AND HADAMARD MATRICES

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Abstract. In this paper, we construct inequivalent Hadamard matrices based on several new and old full orthogonal designs, using circulant and symmetric block matrices. Not all orthogonal designs produce inequivalent Hadamard matrices, because the corresponding systems of equations do not possess solutions. The systems of equations arising when we search for inequivalent Hadamard matrices from full orthogonal designs using circulant and symmetric block matrices, can be concisely described using the periodic autocorrelation function of the generators of the block matrices. We use Maple, Magma, C and Unix tools to find many new inequivalent Hadamard matrices.

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1. Introduction

Definition 1.1. Let x_1, ..., x_t be commuting indeterminates. An orthogonal design X of order n and type $(s_1, \dots, s_t)$ denoted $\mathrm{OD}(n; s_1, \dots, s_t)$, where s_1, ..., s_t are positive integers, is a matrix of order n with entries from $\{0, \pm x_1, \dots, \pm x_t\}$, such that

$$XX^t = \left(\sum_{i=1}^{t} s_i x_i^2\right) I_n,$$

where X^t denotes the transpose of X and I_n denotes the identity matrix of order n.

Orthogonal designs are used in Combinatorics, Statistics, Coding Theory, Telecommunications and other areas. For more details on orthogonal designs see [5, 28] and on Hadamard matrices see [2].

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The concepts of the periodic (PAF) and non-periodic (NPAF) autocorrelation functions are described in [17].

2. Applications of Hadamard matrices

We give some references to works describing applications of Hadamard matrices. We do not aim to provide a comprehensive, or by all means complete, treatment of the subject, as this is not the purpose of the present paper. We are merely interesting in giving a flavor of the many different application areas involved, in order to exhibit that while Hadamard matrices are specialized types of orthogonal designs their applications are of a broader interest.

As first noted in [26], Hadamard matrices are used in Statistics where they generate optimal statistical designs used in weighing experiments. Hadamard matrices play an important role also in Coding Theory where they generated the so called Hadamard codes ([22]), i.e. error-correcting codes that correct the maximum number of errors. It is worthwhile to note that, a Hadamard code was used during the 1971 space probe Mariner 9 mission by NASA to correct for picture transmission error. The Mariner 9 mission and the Coding Theory used in that project are the subjects of [27] and [29].

Hadamard matrices are used in Telecommunications where they generate sequences used in digital communications and in Optics for the improvement of the quality and resolution of image scanners. More details, regarding their applications in communications and signal/image processing can be found in [30]. Last but not least, Hadamard matrices play an important role in Numerical Analysis in the study of the growth factor of Gaussian Elimination with Complete Pivoting ([3]). Hadamard matrices, are the only known matrices till today that achieve a growth factor equal to their dimension.

3. Sequences with zero periodic autocorrelation function and orthogonal designs

The classical Williamson array

$$H = \begin{pmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{pmatrix}$$

has been used to construct inequivalent Hadamard matrices [6]. Specifically, let U be the matrix of order n

$$U = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

which has the property $U^n = I_n$. The matrix U is used to define the block matrices of order n in the classical Williamson array, as polynomials in U with ± 1 coefficients. Then the block matrices will commute with each other. Moreover, by imposing symmetry conditions on the coefficients, the block matrices will be symmetric, in view of the fact that $U^T = U^{-1}$. The four matrices A, B, C, D are defined by polynomials in U as follows:

$$\begin{array}{ccccccccc} A & = & a_0 I_n & + & a_1 U & + & \cdots & + & a_{n-1}U^{n-1} \\ B & = & b_0 I_n & + & b_1 U & + & \cdots & + & b_{n-1}U^{n-1} \\ C & = & c_0 I_n & + & c_1 U & + & \cdots & + & c_{n-1}U^{n-1} \\ D & = & d_0 I_n & + & d_1 U & + & \cdots & + & d_{n-1}U^{n-1} \end{array} \tag{1}$$

where the $4n$ coefficients $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}, c_0, \ldots, c_{n-1}, d_0, \ldots, d_{n-1}$ satisfy the additional symmetry conditions

$$a_{n-i} = a_i, b_{n-i} = b_i, c_{n-i} = c_i, d_{n-i} = d_i, \; i = 1, \ldots, n-1. \tag{2}$$

Then the requirement $HH^t = (4n)I_{4n}$ gives rise to a system of polynomial equations in the $2n+2$ indeterminates (take n to be odd)

It is conceivable to use the same process with more general arrays that the Williamson array, i.e. with full orthogonal designs, to look for inequivalent Hadamard matrices.

Let m be a multiple of 4 and let $OD(m; a_1, \ldots, a_k)$ be a full orthogonal design ($a_1 + \cdots + a_k = m$) of order m with k variables ($k \leq \rho(m)$ where $\rho(m)$ denotes the Radon function [5]). Then we can replace each variable appearing in the full orthogonal design by a matrix of order n that we build with the matrix U and the necessary number of indeterminates. We wrote a meta-meta program (described in the next section) to implement this idea more systematically. After using our program with several different orthogonal designs, we give the following formalism which allows one to use the PAF concept [17] to provide a concise description of the systems of polynomial equations that arise in the search for inequivalent Hadamard matrices from orthogonal designs. This formalism enables one to determine whether certain blocks are suitable for the construction of a Hadamard matrix from a given orthogonal design as a system of linear equations.

In particular, consider n to be an odd integer such that $n \geq 3$ and set $p = \frac{n-1}{2}$. For all values of i from 1 to k, suppose that each of the a_i equal variables in the rows/columns of the orthogonal design is replaced by an $n \times n$ circulant and symmetric matrix of indeterminates A_i. When A_i is constructed as a polynomial via the matrix U, then a sequence of $p+1$ indeterminates $A_i^p = [a_i^0, \ldots, a_i^p]$ suffices to fully describe A_i. Denote by O the resulting $mn \times mn$ matrix. Then the relationship $OO^t = (mn)I_{mn}$ gives rise to the following system of p polynomial equations in $k(p+1)$ binary unknowns:

$$\begin{array}{lll} a_1 PAF_{A_1^p}(1) & + \cdots + a_k PAF_{A_k^p}(1) & + \dfrac{m}{2} = 0 \\ \vdots & \vdots & \vdots \\ a_1 PAF_{A_1^p}(p) & + \cdots + a_k PAF_{A_k^p}(p) & + \dfrac{m}{2} = 0 \end{array} \tag{3}$$

In addition, when $g = \gcd\left(a_1, \ldots, a_k, \dfrac{m}{2}\right) > 1$, these equations can be simplified by dividing throughout by g.

Example 3.1. To illustrate the application of the previous formalism, suppose we are given the full orthogonal design of order 16, $OD(16; 1, 2, 2, 2, 2, 2, 2, 3)$ in 8 variables a, b, c, d, e, f, g, h:

$$\begin{bmatrix}
a & b & b & b & c & c & d & d & e & e & f & f & g & g & h & h \\
-b & a & b & -b & c & -c & d & -d & e & -e & f & -f & g & -g & h & -h \\
b & b & -a & -b & -d & -d & c & c & -f & -f & e & e & -h & -h & g & g \\
b-b & b & -a & -d & d & c & -c & -f & f & e & -e & -h & h & g & -g & \\
-c & -c & -d & -d & a & b & b & b & g & g & -h & -h & -e & -e & f & f \\
-c & c & -d & d & -b & a & b & -b & g & -g & -h & h & -e & e & f & -f \\
d & d & -c & -c & b & b & -a & -b & h & h & g & g & -f & -f & -e & -e \\
d & -d & -c & c & b & -b & b & -a & h & -h & g & -g & -f & f & -e & e \\
-e & -e & -f & -f & -g & -g & h & h & a & b & b & b & c & c & -d & -d \\
-e & e & -f & f & -g & g & h & -h & -b & a & b & -b & c & -c & -d & d \\
f & f & -e & -e & -h & -h & -g & -g & b & b & -a & -b & d & d & c & c \\
f & -f & -e & e & -h & h & -g & g & b & -b & b & -a & d & -d & c & -c \\
-g & -g & -h & -h & e & e & -f & -f & -c & -c & d & d & a & b & b & b \\
-g & g & -h & h & e & -e & -f & f & -c & c & d & -d & -b & a & b & -b \\
h & h & -g & -g & f & f & e & e & -d & -d & -c & -c & b & b & -a & -b \\
h & -h & -g & g & f & -f & e & -e & -d & d & -c & c & b & -b & b & -a
\end{bmatrix}$$

and that we are replacing each variable by a symmetric and circulant matrix of order n. Then we can write down directly the system of polynomial equations that arise in an efficient fashion for hard computation in a machine.

- $n = 3, m = 16, p = 1$

$$a_0 a_1 + 3b_0 b_1 + 2c_0 c_1 + 2d_0 d_1 + 2e_0 e_1 + 2f_0 f_1 + 2g_0 g_1 + 2h_0 h_1 + 8 = 0$$

The above equation has 4096 solutions, when all variables take ± 1 values. One solution (in the format $[a_0 a_1 b_0 b_1 c_0 c_1 d_0 d_1 e_0 e_1 f_0 f_1 g_0 g_1 h_0 h_1]$) is given by:

```
[-1, -1, -1, -1, -1, 1, -1, 1, -1, 1, 1, -1, -1, 1, 1, -1]
```

These solutions give rise to 4096 Hadamard matrices of order $16 \cdot 3 = 48$. Now we search for inequivalent Hadamard matrices of order 48 using Magma, within this set of 4096 Hadamard matrices.

- $n = 5, m = 16, p = 2$

$$a_0 a_2 + a_1 a_2 + 3b_0 b_2 + 3b_1 b_2 + 2c_0 c_2 + 2c_1 c_2 + 2d_0 d_2 + 2d_1 d_2 +$$

$$2e_0 e_2 + 2e_1 e_2 + 2f_0 f_2 + 2f_1 f_2 + 2g_0 g_2 + 2g_1 g_2 + 2h_0 h_2 + 2h_1 h_2 + 8 = 0$$

$$a_0a_1 + a_1a2 + 3b_0b_1 + 3b_1b_2 + 2c_0c_1 + 2c_1c_2 + 2d_0d_1 + 2d_1d_2 +$$

$$2e_0e_1 + 2e_1e_2 + 2f_0f_1 + 2f_1f_2 + 2g_0g_1 + 2g_1g_2 + 2h_0h_1 + 2h_1h_2 + 8 = 0$$

The above equations have 92160 solutions, when all variables take ± 1 values. One solution (in the format $[a_0a_1a_2b_0b_1b_2c_0c_1c_2d_0d_1d_2e_0e_1e_2f_0f_1f_2g_0g_1g_2h_0h_1h_2]$) is given by:

```
[-1, -1, 1, -1, -1, 1, -1, -1, -1, -1, -1, 1, -1, 1, -1, -1,
 1, -1, -1, 1, 1, -1, 1, -1]
```

These solutions give rise to 92160 Hadamard matrices of order $16 \cdot 5 = 80$. Now we search for inequivalent Hadamard matrices of order 80 using Magma, within this set of 92160 Hadamard matrices.

- $n = 7, m = 16, p = 3$

$a_0a_2+a_1a_3+a_2a_3+3b_0b_2+3b_1b_3+3b_2b_3+2c_0c_2+2c_1c_3+2c_2c_3+2d_0d_2+2d_1d_3 + 2d_2d_3 + 2e_0e_2 + 2e_1e_3 + 2e_2e_3 + 2f_0f_2 + 2f_1f_3 + 2f_2f_3 + 2g_0g_2 + 2g_1g_3 + 2g_2g_3 + 2h_0h_2 + 2h_1h_3 + 2h_2h_3 + 8 = 0$

$a_2a_3 + a_0a_1 + a_1a_2 + 3b_2b_3 + 2c_2c_3 + 2d_2d_3 + 2e_2e_3 + 2f_2f_3 + 2g_2g_3 + 2h_2h_3 + 3b_0b_1 + 3b_1b_2 + 2c_0c_1 + 2c_1c_2 + 2d_0d_1 + 2d_1d_2 + 2e_0e_1 + 2e_1e_2 + 2f_0f_1 + 2f_1f_2 + 2g_0g_1 + 2g_1g_2 + 2h_0h_1 + 2h_1h_2 + 8 = 0$

$a_1a_3 + a_1a_2 + a_0a_3 + 3b_1b_3 + 2c_1c_3 + 2d_1d_3 + 2e_1e_3 + 2f_1f_3 + 2g_1g_3 + 2h_1h_3 + 3b_1b_2 + 2c_1c_2 + 2d_1d_2 + 2e_1e_2 + 2f_1f_2 + 2g_1g_2 + 2h_1h_2 + 3b_0b_3 + 2c_0c_3 + 2d_0d_3 + 2e_0e_3 + 2f_0f_3 + 2g_0g_3 + 2h_0h_3 + 8 = 0$

The above equations have 1105920 solutions, when all variables take ± 1 values. One solution (in the format $[a_0a_1a_2a_3b_0b_1b_2b_3c_0c_1c_2c_3d_0d_1d_2d_3e_0e_1e_2e_3f_0f_1f_2f_3g_0g_1g_2g_3h_0h_1h_2h_3]$) is given by:

```
[-1, -1, -1, 1, -1, -1, -1, 1, -1, -1, -1, 1, -1, -1, -1, 1,
 -1, -1, 1, -1, -1, -1, 1, -1, 1, -1, 1, -1, -1, 1, -1, 1]
```

These solutions give rise to 1105920 Hadamard matrices of order $16 \cdot 7 = 112$. Now we search for inequivalent Hadamard matrices of order 112 using Magma, within this set of 1105920 Hadamard matrices.

4. Meta-meta programming for orthogonal designs

Meta-programming is not a new concept, and has been successfully employed before in cases where software reuse, the process of creating software systems from existing software rather than building software systems from scratch, was neeeded [19]. Before continuing we will list some uses of meta-programming:

- Generation - metacode that generates code
- Transformation - metacode that modifies code (similar to generation)
- Translation - transformation into another language
- Analysis - metacode that analyzes code

The metasoftware we have developed for orthogonal designs and Hadamard matrices in order to search for inequivalent Hadamard matrices makes efficient use of all previous four methods of meta-programming. In particular, the meta-program is using bash shell as its metalanguage whilst the object-language that each program is manipulated are the Computer Algebra Systems, Maple and

Magma. Maple provides an excellent way for performing symbolic and numerical computations, especially when we have to interpret methods that are based on combinatorial mathematics. We implemented a Maple package containing the necessary constructions for the generation of Hadamard matrices from orthogonal designs in Maple, in order to achieve the best possible flexibility in terms of portability with other Computational Algebra Systems, such as Magma. We finally used Magma to automatically perform the searches for inequivalent Hadamard matrices. Earlier attempts of software systems capable to perform searches for inequivalent Hadamard matrices could be found in [4], [11], [12], [13], [15] and [16]. A recent application of the presented metasoftware can be found in [18]. The metasoftware given below, could be seen as a unification and further expansion of the aforementioned works which leads to the production of large databases of new inequivalent Hadamard matrices. All details of the presented metasoftware from an algorithmic perspective are presented in this Section, in Algorithm 1.

We wrote a meta-meta program that accepts as input a text file with an orthogonal design and produces a Maple file that produces a C file that can be compiled and executed to solve exhaustively the system of equations corresponding to this orthogonal design. The meta-meta program is using bash shell, and the `CodeGeneration` Maple package. Some of the principal difficulties in the design of this program lie in the dynamic production of the values of the variables that capture the characteristics of the orthogonal design in the input text file, i.e. the order, the number of variables and the list of different variables. We give below in pseudo-code the meta-meta program we have used.

In particular, the systems of equations arising from the search for inequivalent Hadamard matrices from full orthogonal designs using circulant and symmetric block matrices, can be solved using high-performance computing. Not all orthogonal designs produce inequivalent Hadamard matrices, because the corresponding systems of equations may (or may not) posses solutions, as it was shown in [10]. We used Maple to automatically generate C programs that we subsequently parallelized via a bash/sed/awk script. In detail, two-phase metaprogram that accepts as input a text file with an orthogonal design and produces a Maple file that produces a C file that can be compiled and executed to solve exhaustively the system of equations corresponding to this orthogonal design. The meta-meta program is using bash shell, and the `CodeGeneration` Maple package. The use of scripting and meta-programming with Maple's automatic code generation functionalities, ensure efficient and correct prototyping. During the second phase, the program uses Magma to build a Hadamard matrix corresponding to each solution found and then uses the buckets algorithm, given in [8], to locate inequivalent Hadamard matrices. The constructive bounds on the number of inequivalent Hadamard matrices are constantly updated on the Web. Another important outcome of this on-going project is the interaction with Coding Theory, via self-dual code constructions based on Hadamard matrices for instance.

Algorithm 1 HMOD

procedure HMOD(OD, n) ▷ n is the size of block matrices
Require: n odd
Ensure: Generation of binary vectors corresponding to Hadamard matrices
validate that given OD is an orthogonal design
do awk to detect the order and the variables of the OD
assign variable names to the variables of the OD
calculate the order of Hadamard matrices
assign a list to the k variables of the OD
set p equal to $(n-1)/2$
transform in Maple format given inputs
create Maple input file from OD and the variables of the OD ▷ Begin Maple phase
    begin for loop from 1 to k
 create the sequence of $p+1$ indeterminates
 replace the OD variables with symmetric circulant matrices
    end for loop
    begin for loop from 1 to p
 compute PAF equations from relation (3)
    end for loop
do sed to create a Maple file containing the polynomial equations
call procedure `Maple2C` to convert the equations in C format ▷ End Maple phase
compile the C file ▷ Begin C phase
execute the C executable ▷ End C phase
do sed/awk to convert the output into solutions in Magma format
return solutions as binary vectors representing the Hadamard matrices
end procedure

procedure MAPLE2C(PolEqs) ▷ PolEqs are the polynomial equations in Maple format
Require: Polynomial equations in Maple format
Ensure: Conversion to C format
call `CodeGeneration` Maple package
declare types of variables in C and procedures to be converted
execute C conversion of polynomial equations
return the polynomial equations in C format
end procedure

5. Specific full orthogonal designs

Using algorithm 1 we searched for inequivalent Hadamard matrices with the 41 full orthogonal designs listed below. The first orthogonal design in the list below is the classical Williamson array, that we included in our search simply for consistency check purposes. The remaining 40 full orthogonal designs appear in the book [5] or were found in the papers [7], [9], [14].

Table 1. List of orthogonal designs used by algorithm 1

OD(4;1,1,1,1)	OD(16;1,1,1,1,3,3,3,3)
OD(16;1,2,2,2,2,2,2,3)	OD(24;1,1,1,1,2,2,8,8)
OD(24;1,1,1,1,5,5,5,5)	OD(24;1,1,2,2,3,3,6,6)
OD(24;1,4,4,15)	OD(24;2,2,2,2,4,4,4,4)
OD(24;3,3,3,3,3,3,3,3)	OD(32;4,4,4,4,4,4,4,4)
OD(32;4,4,4,4,8,8)	OD(40;1,1,1,1,9,9,9,9)
OD(40;1,1,1,2,9,26)	OD(40;1,1,2,2,17,17)
OD(40;2,2,2,2,8,8,8,8)	OD(40;2,2,5,5,5,5,8,8)
OD(40;5,5,5,5,5,5,5,5)	OD(56;1,1,2,2,25,25)
OD(56;4,4,4,4,10,10,10,10)	OD(56;7,7,7,7,7,7,7,7)
OD(64;16,16,16,16)	OD(64;4,4,4,4,12,12,12,12)
OD(64;4,4,8,8,8,8,8,8,8)	OD(64;4,8,8,8,8,8,8,12)
OD(64;8,8,8,8,16,16)	OD(64;8,8,8,8,8,8,8,8)
OD(96;4,4,4,4,20,20,20,20)	OD(96;8,8,8,8,16,16,16,16)
OD(128;16,16,16,16,16,16,16,16)	OD(128;8,16,16,16,16,16,16,24)
OD(128;8,8,16,16,16,16,16,16,16)	OD(128;8,8,8,8,24,24,24,24)
OD(144;12,12,12,12,24,24,24,24)	OD(144;4,4,4,4,32,32,32,32)
OD(112;14,14,14,14,14,14,14,14)	OD(80;4,4,10,10,10,10,16,16)
OD(112;8,8,8,8,20,20,20,20)	OD(80;4,4,4,4,16,16,16,16)
OD(144;18,18,18,18,18,18,18,18)	OD(96;12,12,12,12,12,12,12,12)
OD(80;10,10,10,10,10,10,10,10)	

6. Inequivalent Hadamard matrices

The importance of using orthogonal designs to look for inequivalent Hadamard matrices of several orders, is exhibited by the fact that the larger the order of the orthogonal design and the fewer the number of variables in it, the less equations and variables we need to look for inequivalent Hadamard matrices of large orders. To illustrate this point, consider the full orthogonal design

$$OD(144; 4, 4, 4, 4, 32, 32, 32, 32)$$

of order 144 in 8 variables. When we use algorithm HMOD with $n = 5$, then we are constructing Hadamard matrices of order 432 by solving an easy system of 2 equations in 32 binary variables. However, if we were to use the classical Williamson array for instance, to construct Hadamard matrices of order 432, then we would need to consider block matrices of order $n = 108$, which is currently entirely outside the scope of any known algorithm.

The computations are using Magma V 2.13 and are still on-going. With N_n in the following tables we denote the number of inequivalent Hadamard matrices found for each order n.

Table 2. Constructive lower bounds for inequivalent Hadamard matrices of order 40 to 168

n	40	48	72	80	96	112	120	160	168
N_n	8	10	238	30	3	122	5161	8	6760

Table 3. Constructive lower bounds for inequivalent Hadamard matrices of order 192 to 448

n	192	200	240	280	320	400	448
N_n	14	7246	148	2940	34	818	10

Remark 6.1. We note that the presented Hadamard matrices in the Tables, that are constructed using orthogonal designs, are inequivalent since for each order we found that the corresponding 4-profiles are different.

In addition, one could check the generated Hadamard matrices from orthogonal designs for inequivalence using the graph isomorphism criterion, which is more time consuming [1, 23], but locates much more inequivalent Hadamard matrices.

The complete classification for Hadamard matrices of order n is well known for $n \equiv 0 \pmod 4$, $n \leq 28$. For $n = 32, 36$ there are at least $3,578,006$ and $4,745,357$ inequivalent Hadamard matrices respectively, see [25]. On the other hand, there are some theoretical results which provide huge lower bounds, see [20, 21, 24] but these matrices are not available. From this perspective we believe that our lower bounds on the number of inequivalent Hadamard matrices, which are presented in this section have value for practitioners in the fields of Combinatorics and Statistics (see Section 2) since these are constructive. Moreover, the given inequivalent Hadamard matrices are only a sample; considering that the list of the 41 full orthogonal designs given in Section 5, is by no means complete. The main aim of the presented metasoftware is to establish a computational framework for the automated construction of inequivalent Hadamard matrices, whenever this is desired by the specialists of Design Theory and its applications.

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