A RELATIONSHIP BETWEEN THE LIPSCHITZ CONSTANTS APPEARING IN TAYLOR’S FORMULA

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Abstract. Taylor’s formula is a powerful tool in analysis. In this study, we assume that an operator is m-times Fréchet-differentiable and satisfies a Lipschitz condition. We then obtain some Taylor formulas using only the Lipschitz constants. Applications are also provided.

1. Introduction

Taylor’s formula has been used for a long time as a powerful tool in analysis to study the convergence of iterative processes but also in other areas [1]–[4].

In this study we assume that operator $F$ is m-times (m a natural number) Fréchet-differentiable on a non-empty subset $D$ of a Banach space $X$ with values in a Banach space $Y$. Furthermore, we assume that operator $F^{(m)}$ is Lipschitz continuous on $D$. Then, the constant is used to relate the corresponding Lipschitz constants for operators $F^{(i)}$, $i = 1, 2, ..., m$. Applications are also provided in this study.

2. Taylor Formulas

We need the following results on Taylor’s formula for $m$-Fréchet-differentiable operators.

Theorem 2.1. Let $G : D \subseteq X \to Y$ be a m-times (m $\in$ N) Fréchet-differentiable operators defined on a non-empty subset $D$ of a Banach space $X$ with values in a Banach space $Y$.

Assume: (a) there exist a constant $e_{m+1} > 0$, and a convex subset $D_0$ of $D$ such that for all $x, y \in D_0$

$$
\|G^{(m)}(x) - G^{(m)}(y)\| \leq e_{m+1}\|x - y\|_X.
$$

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Then, for all \(x, y \in D_0\), the following estimate holds:

\[
\|G(x) - G(y) - \sum_{i=1}^{m} \frac{G^{(i)}(y)}{i!} (x - y)^i \|_Y \leq \frac{e_{m+1}}{(m+1)!} \|x - y\|_{X}^{m+1}.
\]

(b) If \(D_0 = U(x_0, R) = \{x \in X : \|x - x_0\| \leq R\} \subseteq D\) for some \(x_0 \in D\), \(R > 0\), and

\[
\|G^{(m)}(x) - G^{(m)}(x_0)\| \leq e_0 \|x - x_0\|_X \quad \text{for some } e_0 > 0
\]

holds true on \(D_0\), then

\[
\|G^{(k)}(x)\| \leq e_k \quad k = 0, 1, ..., m,
\]

and

\[
\|G^{(k-1)}(x) - G^{(k-1)}(y)\| \leq e_k \|x - y\|_X, \quad k = 1, ..., m,
\]

where, \(e_m = \|G^{(m)}(x_0)\| + e_0 R\),

\[
e_k = \|G^{(k)}(x_0)\| + e_{k+1} R, \quad k = 0, 1, ..., m - 1.
\]

(c) Under hypotheses of part (b) the following hold for all \(x, y \in D_0\), and \(k = 1, 2, ..., m\):

\[
\|G(x) - G(y) - \sum_{i=1}^{k} \frac{G^{(i)}(y)}{i!} (x - y)^i \|_Y \leq \frac{e_{k+1}}{(k+1)!} \|x - y\|_{X}^{k+1}.
\]

Proof. (a) Let us denote by \(\alpha \in Y\) the element given by

\[
\alpha = G(x) - G(y) - \sum_{i=1}^{m} \frac{G^{(i)}(y)}{i!} (x - y)^i.
\]

It is well known [3], [4] that there exists \(\beta \in L(Y, R)\) the space of bounded linear operators from \(Y\) into \(R\) so that

\[
\|\beta\|_X = 1, \quad \text{and } \beta(\alpha) = \|\alpha\|_Y.
\]

It then follows from (2.8), and (2.9) that

\[
\beta(\alpha) = |\beta(G(x)) - \beta(G(y)) - \sum_{i=1}^{m} \frac{\beta(G^{(i)}(y)(x - y)^i)}{i!}|.
\]

Let us define on \([0, 1]\) the real function

\[
\gamma(\theta) = \beta(G(y + \theta(x - y))).
\]
In view of the convexity of $D_0$, $y + \theta(x - y) \in D_0$ if $x, y \in D_0$. That is function $\gamma$ is well defined. It follows from the existence of the Fréchet-derivatives of operator $G$ that functions

\begin{equation}
\gamma^{(k)}(\theta) = \beta(G^{(k)}(y + \theta(x - y))(x - y)^k) \quad k = 0, 1, \ldots, m
\end{equation}

are well defined.

Using the integral form of Taylor’s formula [3], [4], we have:

\begin{equation}
\gamma(1) = \gamma(0) + \sum_{i=1}^{m} \gamma^{(i)}(0) + \frac{1}{(m - 1)!} \int_0^1 \gamma^{(m)}(\theta)(1 - \theta)^{m-1} d\theta.
\end{equation}

We also need the estimates:

\begin{equation}
\int_0^1 (1 - \theta)^{m-1} d\theta = \frac{1}{m},
\end{equation}

and

\begin{equation}
\int_0^1 \theta(1 - \theta)^{m-1} d\theta = \int_0^1 (1 - \theta)^{m-1} d\theta - \int_0^1 (1 - \theta)^m d\theta = \frac{1}{m} - \frac{1}{m + 1} = \frac{1}{m(m + 1)}.
\end{equation}

We then have:

\begin{equation}
\|\alpha\|_Y = \frac{1}{(m-1)!} \int_0^1 \gamma^{(m)}(\theta)(1 - \theta)^{m-1} d\theta - \int_0^1 \gamma^{(m)}(0)(1 - \theta)^{m-1} d\theta,
\end{equation}

but

\begin{equation}
|\gamma^{(m)}(\theta) - \gamma^{(m)}(0)| = |\beta(G^{(m)}(y + \theta(x - y))(x - y)^m) - \beta(G^{(m)}(y))(x - y)^m| \leq \|\beta\|\|G^{(m)}(y + \theta(x - y))(x - y)^m - (G^{(m)}(y))(x - y)^m\|_{X}^m
\end{equation}

and consequently,

\begin{equation}
\|\alpha\|_Y = \frac{e_{m+1}}{(m-1)!} \|x - y\|_X \int_0^1 \theta(1 - \theta)^{m-1} d\theta \leq \frac{e_{m+1}}{(m-1)!^{m}(m+1)} \|x - y\|_X^{m+1}.
\end{equation}

That completes the proof of part (a).

(b) It follows from (2.3)

\begin{equation}
\|G^{(m)}(x)\| \leq \|G^{(m)}(x_0)\| + e_{m+1} \|x - x_0\| = e_m,
\end{equation}

By Langrange’s theorem applied to $G^{(m-1)} : D_0 \to L(X^{m-1}, Y)$ we get

\begin{equation}
\|G^{(m-1)}(x) - G^{(m-1)}(y)\| \leq \|G^{(m)}(y + \theta(x - y))\| \|x - y\|_X \quad \theta \in (0, 1).
\end{equation}
If \( x, y \in D_0 \), and \( \theta \in (0, 1) \) we get \( y + \theta(x - y) \in D_0 \) and consequently
\[
\|G^{(m)}(y + \theta(x - y))\| \leq e_m,
\]
and
\[
\|G^{(m-1)}(x) - G^{(m-1)}(y)\| \leq e_m\|x - y\|_X.
\]
Estimates (2.3) and (2.4) are obtained by continuing the same way.

c) This part follows immediately from parts (a), and (b).

That completes the proof of the theorem. \( \square \)

3. Applications

The first application involves the most popular iterative process which is Newton’s method.

We need a result on majorizing sequences for Newton’s method
\[
\begin{align*}
  x_{n+1} & = x_n - F'(x_n)^{-1}F(x_n) \quad (n \geq 0), \quad (x_0 \in D) \\
  \end{align*}
\]
for generating a sequence \( \{x_n\} \) approximating a solution \( x^* \) of equation
\[
F(x) = 0.
\]

**Lemma 3.1** ([3, Lemma 1.1.2, p. 14]). Assume:
there exist constants \( L_0 \geq 0, L \geq 0, \) with \( L_0 \leq L, \) and \( \eta \geq 0, \) such that:
\[
\begin{align*}
  q_0 = L \eta \left\{ \begin{array}{ll}
    \leq \frac{1}{2}, & \text{if } L_0 \neq 0, \\
    < \frac{1}{2}, & \text{if } L_0 = 0,
  \end{array} \right.
\end{align*}
\]
where,
\[
L = \frac{1}{8} \left( L + 4 L_0 + \sqrt{L^2 + 8 L_0 L} \right).
\]

Then, sequence \( \{t_k\} \) \( (k \geq 0) \) given by
\[
\begin{align*}
  t_0 &= 0, \quad t_1 = \eta, \quad t_{k+1} = t_k + \frac{L (t_k - t_{k-1})^2}{2 (1 - L_0 t_k)} \quad (k \geq 1),
\end{align*}
\]
is well defined, nondecreasing, bounded from above by \( t^{**} \) and converges to its unique least upper bound \( t^* \in [0, t^{**}] \), where
\[
\begin{align*}
  t^{**} &= \frac{2 \eta}{2 - \delta}, \\
  1 \leq \delta &= \frac{4 L}{L + \sqrt{L^2 + 8 L_0 L}} < 2 \quad \text{for } L_0 \neq 0.
\end{align*}
\]
Moreover, the following estimates hold:

\[(3.8)\quad L_0 \ t^* \leq 1,\]

\[(3.9)\quad 0 \leq t_{k+1} - t_k \leq \left(\frac{\delta}{2}\right)^k (t_k - t_{k-1}) \leq \cdots \leq \left(\frac{\delta}{2}\right)^k \eta, \quad (k \geq 1),\]

\[(3.10)\quad t_{k+1} - t_k \leq \left(\frac{\delta}{2}\right)^k \left(2q_0\right)^{2^{k-1}} \eta, \quad (k \geq 0),\]

\[(3.11)\quad 0 \leq t^* - t_k \leq \left(\frac{\delta}{2}\right)^k \left(\frac{2q_0}{1 - (2q_0)^{2^k}}\right) \eta, \quad (2q_0 < 1), \quad (k \geq 0).\]

We also need the result related to Lemma 3.1.

**Lemma 3.2.** Let \( m \geq 2 \) be a natural number; \( \alpha_i \) non-negative numbers, \( i = 2, \ldots, m + 1, \eta > 0 \) and define functions \( P, L_0, L, H \) on \((0, +\infty)\) by

\[(3.12)\quad P(r) = \frac{\alpha_{m+1}}{(m+1)!} r^{m+1} + \frac{\alpha_m}{m!} r^m + \ldots + \frac{\alpha_2}{2!} r^2 - r + \eta,\]

\[(3.13)\quad L_0(r) = \frac{1 + p'(r)}{r} = \frac{\alpha_{m+1}}{m!} r^{m-1} + \frac{\alpha_m}{(m-1)!} r^{m-2} + \ldots + \alpha_2,\]

\[(3.14)\quad L(r) = P''(r) = \frac{\alpha_{m+1}}{(m-1)!} r^{m-1} + \frac{\alpha_m}{(m-2)!} r^{m-2} + \ldots + \alpha_2\]

and

\[(3.15)\quad H(r) = (L(r) + 4L_0(r) + \sqrt{L^2(r) + 8L_0(r)L(r)})\eta - 4.\]

Assume:

\[(3.16)\quad H(\eta) < 0.\]

Then, function \( H \) has a unique positive zero \( r_0 \) such that

\[(3.17)\quad r_0 > \eta.\]

Moreover, for a fixed \( r^* \in (\eta, r_0) \), set

\[(3.18)\quad L_0 = L_0(r^*), \quad \text{and} \quad L = L(r^*).\]

Then, the conclusions of Lemma 3.1 hold for iteration \( \{t_n\} \).
Proof. Function $H$ is well defined, since $L_0$, and $L$ are positive functions. Moreover, $H$ is increasing, since $H'(r) > 0$ for $r > 0$. It then follows from (3.12) that $H(r) > 0$ for sufficiently large $r > 0$. The existence, uniqueness of $r_0$ follows from the intermediate value theorem, and the monotonicity of function $H$, respectively. Using the definition of $L_0$, $L$, and $H$, we deduce that estimate (3.17) holds.

That completes the proof of lemma. \□

Hypothesis (3.16) can be replaced by the weaker, and more general

\begin{equation}
\text{Function } H \text{ has a minimal positive zero } r_0. \tag{3.19}
\end{equation}

We can show the following semilocal convergence result for Newton’s method (3.1).

\textbf{Proposition 3.3.} \textit{Let } $F : D \subseteq X \to Y$ \textit{be a } $m$\textit{-times Fréchet-differentiable operator defined on a non-empty, open and convex subset } $D$ \textit{of a Banach space } $X$ \textit{with values in a Banach space } $Y$. \textit{Assume there exists } $x_0 \in D$, $e_{m+1} > 0$ \textit{such that}

\begin{align}
&F'(x_0)^{-1} \in D; \tag{3.20} \\
&\|F'(x_0)^{-1}(F'(x) - F'(y))\| \leq e_{m+1}\|x - y\| \text{ for all } x, y \in D; \tag{3.21}
\end{align}

\textit{hypotheses of Lemmas 2.1, 2.2 hold for}

\begin{align}
&G = F'(x_0)^{-1}F, \quad \alpha_i = e_i \quad i = 1, 2, \ldots, m + 1; \tag{3.22} \\
&\mathcal{U}(x_0, \alpha^*) \subseteq D \tag{3.23}
\end{align}

\textit{and}

\begin{align}
&\alpha^* = t^* \text{ or } t^{**} < r_0; \tag{3.24}
\end{align}

\textit{where } $t^*$, $t^{**}$ \textit{are given in Lemma 3.1, and } $r_0$ \textit{is in Lemma 3.2.}

\textit{Then, sequence } $\{x_n\}$ \textit{generated by Newton’s method (3.1) is well-defined, remains in } $\mathcal{U}(x_0, \alpha^*)$ \textit{for all } $n \geq 0$, \textit{and converges to a unique solution } $x^* \in \mathcal{U}(x_0, \alpha^*)$ \textit{of equation } $F(x) = 0$.

\textit{Moreover, the following estimates hold for all } $n \geq 0$:

\begin{align}
&\|x_{n+1} - x_n\| \leq t_{n+1} - t_n \tag{3.25} \\
&\|x_n - x^*\| \leq t^* - t_n. \tag{3.26}
\end{align}
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Proof. Simply repeat the proof of Theorem by Argyros in [2], but use (2.2) for $x = x_{n+1}, y = x_n$ to obtain:

\[ \|F'(x_n)^{-1}F'(x_0)\| \leq \frac{1}{1 - L}\|x_n - x_0\| \leq \frac{1}{1 - Lt_n}, \]

(3.27)

\[ \|F'(x_0)^{-1}F'(x_n)\| \leq \frac{L}{2}\|x_{n+1} - x_n\|^2 \leq \frac{L}{2}(t_{n+1} - t_n)^2, \]

(3.28)

and by (3.1)

\[ \|x_{n+1} - x_n\| \leq \|F'(x_n)^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_n)\| \leq \frac{L(t_{n+1} - t_n)^2}{2(1 - Lo t_n)} = t_{n+1} - t_n. \]

(3.29)

That completes the proof of the proposition. □

As a second application, consider $X = Y = \mathbb{R}, D = \mathbb{U}(0, 1)$, and define function $F$ on $D$ by

\[ F(x) = e^x. \]

(3.30)

Then, for any $m \geq 1$,

\[ a_{m+1} = e, \quad \|F'(x_0)^{-1}F^{(m)}(x_0)\| = 1, \quad a_0^{m+1} = e - 1, \quad a_m = 1 + (e - 1)R, \]

(3.31)

and

\[ a_k = 1 + eR, \quad k = 0, 1, 2, \ldots, m - 1. \]

(3.32)

Estimates (2.2) and (2.7) can now be obtained with these choices.

References


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