SURFACES WITH POINTWISE 1-TYPE GAUSS MAP

DONG-SOO KIM

Abstract. In this article, we study generalized slant cylindrical surfaces (GSCS’s) with pointwise 1-type Gauss map of the first and second kinds. Our main results state that GSCS’s with pointwise 1-type Gauss map of the first kind coincide with surfaces of revolution with constant mean curvature; and the right cones are the only polynomial kind GSCS’s with pointwise 1-type Gauss map of the second kind.

1. Introduction and Preliminaries

The notion of finite type submanifolds in Euclidean or pseudo-Euclidean space, introduced by B.-Y. Chen during the late 1970’s, has become a useful tool for investigating and characterizing many important submanifolds (cf. [3, 4]). In [1, 2, 6] the notion of finite type was extended to differential maps, in particular, to Gauss map of submanifolds.

If a submanifold $M$ of Euclidean or pseudo-Euclidean space has 1-type Gauss map $G$, then $G$ satisfies $\Delta G = \lambda(G + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector $C$, where $\Delta$ is the Laplace operator corresponding to the induced metric on $M$ (cf [1, 2, 9]). However, the Laplacian of the Gauss map of several important surfaces such as helicoids, catenoids and right cones take a somewhat different form; namely,

$$\Delta G = f(G + C)$$

for some non-constant function $f$ and some constant vector $C$. For this reason, a submanifold is said to have pointwise 1-type Gauss map if its Gauss map satisfies (1.1) for some smooth function $f$ on $M$ and vector $C$. A submanifold with pointwise 1-type Gauss map is said to be of the first kind if the vector $C$ in (1.1) is the zero
vector. Otherwise, the pointwise 1-type Gauss map is said to be of the second kind ([5]).

Let $M$ be a surface of Euclidean 3-space $\mathbb{E}^3$. The map $G : M \to S^2 \subset \mathbb{E}^3$ which sends each point of $M$ to the unit normal vector to $M$ at the point is called the Gauss map of the surface $M$, where $S^2$ is the unit sphere in $\mathbb{E}^3$ centered at the origin.

For the matrix $g = (g_{ij})$ consisting of the components of the metric on $M$, we denote by $g^{-1} = (g^{ij})$ (resp. $G$ ) the inverse matrix (resp. the determinant) of the matrix $(g_{ij})$. The Laplacian $\Delta$ on $M$ is, in turn, given by

$$\Delta = -\frac{1}{\sqrt{G}} \sum_{i,j} \frac{\partial}{\partial x^i} \left( \sqrt{G} g^{ij} \frac{\partial}{\partial x^j} \right).$$

Here, we give an example of surfaces of revolution with pointwise 1-type Gauss map of the second kind.

**Example 1.1.** Consider the right cone $C_a$ which is parameterized by

$$x(u,v) = (v \cos u, v \sin u, av), \quad a \geq 0.$$

Then the Gauss map $G$ and its Laplacian $\Delta G$ are respectively given by

$$G = \frac{1}{\sqrt{1 + a^2}} (a \cos u, a \sin u, -1)$$

and

$$\Delta G = \frac{1}{v^2} \left( G + \left( 0, 0, \frac{1}{\sqrt{1 + a^2}} \right) \right).$$

It implies that the right cone has pointwise 1-type Gauss map of the second kind.

In [5], B.-Y. Chen, M. Choi and Y. H. Kim studied surfaces of revolution with pointwise 1-type Gauss map. In [7], U. Dursun studied flat surfaces in Euclidean 3-space with pointwise 1-type Gauss map.

The author and Y. H. Kim introduced the class of generalized slant cylindrical surfaces (GSCS’s) in [8]. This class includes surfaces of revolution and cylindrical surfaces as special cases. Thus, we need to consider the GSCS’s in $\mathbb{E}^3$ with pointwise 1-type Gauss map.

In this paper, we study the GSCS’s with pointwise 1-type Gauss map. In particular, we prove that GSCS’s with pointwise 1-type Gauss map of the first kind coincide with surfaces of revolution with constant mean curvature; and the right cones are the only polynomial kind GSCS’s with pointwise 1-type Gauss map of the second kind.
Hereafter, all objects are assumed to be connected and smooth unless mentioned otherwise.

2. Generalized Slant Cylindrical Surfaces

For a fixed unit speed plane curve $X(s) = (x(s), y(s), 0)$, let $T(s) = X'(s)$ and $N(s) = (-y'(s), x'(s), 0)$ denote the unit tangent and principal normal vector, respectively. The curvature $\kappa(s)$ of $X(s)$ is defined by $T'(s) = \kappa(s)N(s)$ and we have $T(s) \times N(s) = V$, where $V$ denotes the unit vector $(0, 0, 1)$. For a constant $\theta$, we let $Y_\theta(s) = \cos \theta N(s) + \sin \theta V$. Then the ruled surface $M$ defined by

\begin{equation}
F(s, t) = X(s) + tY_\theta(s)
\end{equation}

is regular at $(s, t)$ where $1 - \cos \theta \kappa(s)t$ does not vanish. This ruled surface $M$ is called a slant cylindrical surface (SCS) over $X(s)$. For the unit normal vector $G = -\sin \theta N(s) + \cos \theta V$, $M$ satisfies

$\langle F_s, F_t \rangle = 0, \langle F_s, G \rangle = 0$.

This shows that the coordinate lines of $F$ are lines of curvature of $M$ with corresponding principal curvatures

\begin{equation}
k_1(s, t) = \frac{-\kappa(s) \sin \theta}{1 - \kappa(s)t \cos \theta}, k_2(s, t) = 0,
\end{equation}

respectively. The SCS with $\sin \theta = 0$ or $\cos \theta = 0$ is nothing but a parametrization of either a plane or a cylindrical surface.

In general, we consider another unit speed plane curve $W(t) = (z(t), w(t))$. If we let $Y_z(t) = z(t)N(s) + w(t)V$, then the parametrized surface defined by

\begin{equation}
H(s, t) = X(s) + Y_z(t)
\end{equation}

is regular at $(s, t)$ where $1 - \kappa(s)z(t)$ does not vanish. This parametrized surface $M$ is called a generalized slant cylindrical surface (GSCS) over $X(s)$. For the unit normal vector $G(s, t) = -w'(t)N(s) + z'(t)V$, $M$ satisfies

$\langle H_s, H_t \rangle = 0, \langle H_s, G \rangle = 0$.

This shows that $H(s, t)$ is a principal curvature coordinate system of $M$ with corresponding principal curvatures

\begin{equation}
k_1(s, t) = \frac{-\kappa(s)w'(t)}{1 - \kappa(s)z(t)}, k_2(s, t) = \kappa(t),
\end{equation}

respectively, where $\kappa(t) = z'(t)w''(t) - z''(t)w'(t)$ denotes the curvature of $W(t)$. 
If $W(t)$ is a straight line, then the GSCS $H(s, t)$ is nothing but a SCS. If $X(s)$ is a straight line, then the GSCS $H(s, t)$ is nothing but a cylindrical surface. Furthermore, we have the following ([8]).

**Proposition 2.1.** If $X(s)$ is a circle, then GSCS $M$ over $X(s)$ is a surface of revolution.

Therefore cylindrical surfaces and surfaces of revolution are special cases of GSCS’s.

Now we give the following:

**Proposition 2.2.** Let $M$ denote a GSCS given by (2.3). Then we have the following.

1. If the mean curvature $H$ is constant, then $M$ is a surface of revolution.
2. If the Gaussian curvature $K$ is constant, then $M$ is either a surface of revolution or an SCS.

**Proof.** It follows from (2.4) that

$$2H = \kappa(t) + \frac{-\kappa(s)w'(t)}{1 - \kappa(s)z(t)}, \quad K = \frac{\kappa(s)\kappa(t)w'(t)}{1 - \kappa(s)z(t)}.$$

Hence we have

$$\kappa(t) - 2H = \kappa(s)\{\kappa(t)z(t) - 2Hz(t) + w'(t)\},$$

and

$$K = \kappa(s)\{Kz(t) - \kappa(t)w'(t)\}.$$

Suppose that $H$ is constant. If $\kappa(t) - 2H \neq 0$, then (2.6) shows that $\kappa(s)$ is a nonzero constant, and hence $M$ is a surface of revolution. If $\kappa(t) - 2H = 0$, then (2.5) implies $\kappa(s)w'(t) = 0$. In case $\kappa(s_0) \neq 0$ for some $s_0$, $w'(t)$ vanishes identically, and hence $M$ is a part of a plane. Otherwise, $\kappa(s)$ vanishes identically. Hence $X(s)$ is a straight line. Thus $M$ is a part of a plane ($H = 0$) or a circular cylinder ($H \neq 0$).

Now suppose that $K$ is constant. If $K \neq 0$, it follows from (2.7) that $\kappa(s)$ is a nonzero constant, and hence $M$ is a surface of revolution. In case $K = 0$ and $\kappa(s_0) \neq 0$, (2.7) shows that $\kappa(t)$ vanishes identically, and hence $M$ is an SCS. In case $K = 0$ and $\kappa(s)$ vanish identically, then $M$ is a cylindrical surface. \qed
3. GSCS’s with Pointwise 1-type Gauss Map of the First Kind

Let \( X(s) = (x(s), y(s), 0) \) be a unit speed plane curve with the Frenet frame \( \{T(s), N(s)\} \). We consider GSCS’s parametrized by

\[
H(s, t) = X(s) + Y_s(t),
\]

where \( W(t) = (z(t), w(t)) \) is a unit speed plane curve, \( Y_s(t) = z(t)N(s) + w(t)V \), and \( V = (0, 0, 1) \). Then \( H(s, t) \) is regular at \( (s, t) \) where \( Q(s, t) = 1 - \kappa(s)z(t) \) does not vanish and we get

\[
H_s = Q(s, t)T(s), \quad H_t = z'(t)N(s) + w'(t)V,
\]

\[
G(s, t) = -w'(t)N(s) + z'(t)V.
\]

The Laplacian \( \Delta \) on \( M \) is given by

\[
\Delta f = -Q^{-3}\{\kappa'(s)z(t)f_s + Qf_{ss} - Q^2\kappa(s)z'(t)f_t + Q^3f_{tt}\}.
\]

Hence it follows from (3.2) and (3.3) that

\[
-Q^3 \Delta G = \kappa'(s)w'(t)T(s) + Q\{\kappa(s)^2w'(t) + \kappa(s)z'(t)w''(t)
- \kappa^2w'''(t)N(s) + Q^2\{-\kappa(s)z'(t)z''(t) + Qz'''(t)\}V.
\]

Now suppose that \( M \) has the pointwise 1-type Gauss map \( G \) which satisfies (1.1). Then, letting \( C = C_1(s)T(s) + C_2(s)N(s) + C_3V \), we have the following.

\[
\kappa'(s)w'(t) = -Q^4C_1(s)f(s, t),
\]

\[
\kappa(s)^2w'(t) + Q\kappa(s)z'(t)w''(t) - Q^2w'''(t) = Q^2f(s, t)\{w'(t) - C_2(s)\},
\]

and

\[
\kappa(s)z'(t)z''(t) - Qz'''(t) = Qf(s, t)\{z'(t) + C_3\}.
\]

Using above, we get the following:

**Theorem 3.1.** Let \( M \) be a GSCS given by (3.1). Suppose that \( M \) has pointwise 1-type Gauss map \( G \) of the first kind. Then \( M \) is a surface of revolution.

**Proof.** Since \( C = C_1(s)T(s) + C_2(s)N(s) + C_3V = 0 \), it follows from (3.5) that \( \kappa'(s)w'(t) = 0 \). In case \( \kappa'(s_0) \neq 0 \) for some \( s_0 \), \( w(t) \) is constant, and hence \( M \) is a part of a plane. Otherwise, \( \kappa \) is constant. If \( \kappa \) is nonzero, then \( M \) is a surface of revolution. If \( \kappa = 0 \), then it follows from (3.6) and (3.7) that

\[
z'''(t) + f(s, t)z'(t) = 0, \quad w'''(t) + f(s, t)w'(t) = 0.
\]

This shows that \( \kappa'(t) = 0 \). Thus \( M \) is a plane or a circular cylinder. \( \square \)
Combining Theorem 3.1 in [5] and Proposition 2.2, Theorem 3.1 shows directly the following.

**Corollary 3.2.** Let M be a GSCS given by (3.1). Then the following are equivalent.

1. M has pointwise 1-type Gauss map $G$ of the first kind.
2. M has constant mean curvature.
3. M is a surface of revolution with constant mean curvature.

**Remark 3.3.** Surfaces of revolution with constant mean curvature are also known as surfaces of Delaunay (cf. [10, p.115]).

4. **GSCS’s with Pointwise 1-type Gauss Map of the Second Kind**

Consider a GSCS $M$ parametrized by (3.1). If $M$ is not cylindrical, then $W(t)$ can be parametrized by $W(t) = (t, g(t))$ for some function $g = g(t)$. Hence $M$ is given by

\begin{equation}
H(s, t) = X(s) + tN(s) + g(t)V.
\end{equation}

If $g(t)$ is a polynomial in $t$, then $M$ is said to be of polynomial kind ([5]). $H(s, t)$ is regular at $(s, t)$ where $Q(s, t) = 1 - t\kappa(s) \neq 0$ and we get

\begin{equation}
G(s, t) = \frac{1}{P(t)} \{ -g'(t)N(s) + V \}, P(t) = \sqrt{1 + g'(t)^2}.
\end{equation}

The Laplacian $\Delta$ on $M$ is given by

\begin{equation}
\Delta f = -P^{-4}Q^{-3} \left\{ \kappa'(s)T + P^4 f_s + P^4 Q f_{ss} \right\} - (P^2 Q^2 \kappa(s) + Q^2 g'g'')f_t + P^2 Q^3 f_{tt} \right\}.
\end{equation}

Hence it follows from (4.2) and (4.3) that

\begin{equation}
\Delta G = -\kappa'(s)g'P^{-1}Q^{-3}T(s) - P^{-7}Q^{-2} \left\{ \kappa(s)^2 g'P^6 + \kappa(s)g''P^2Q \right\} + g'(g'')^2 Q^2 - g''P^2Q^2 + 3g'(g'')^2Q^2 \right\} N(s) - P^{-7}Q^{-1} \left\{ (3g'(g'')^2 - (g'')^2 - g'g'' - (g')^3g'' + \kappa(s)g'g''P^2 \right\} V.
\end{equation}

Suppose that the Gauss map $G$ satisfies (1.1) with nonzero constant vector $C$. Then, letting $C = C_1(s)T(s) + C_2(s)N(s) + C_3V$, we have the following,

\begin{equation}
PQ^3C_1(s)f(s, t) + \kappa'(s)g'(t) = 0,
\end{equation}
\[ P^6Q^2f(s,t)\{-g'(t) + PC_2(s)\} + \kappa(s)^2g' P^6 \]
\[ + \kappa(s)g'' P^2Q + g'(g'')^2Q^2 - g'' P^2Q^2 + 3g'(g'')^2Q^2 = 0, \]
and
\[ P^6Qf(s,t)\{1 + C_3P\} + \{3(g')^2(g'')^2 \]
\[ - (g'')^2 - g' g''' - (g')^3 g'''\}Q + \kappa(s)g'' P^2 = 0. \]

It follows from (4.5) and (4.7) that
\[ C_3\kappa'(s)g' P^6 + \kappa'(s)g' P^5 \]
\[ = C_1(s)Q^3\{3(g')^2(g'')^2 - (g')^3 g'''\} + C_1(s)\kappa(s)g' g'' P^2Q^2 \]
\[ - C_1(s)Q^3\{(g'')^2 + g' g'''\}. \]

Suppose that \( M \) is a GSCS of polynomial kind, that is, \( g(t) \) is a polynomial in \( t \).

Denote by \( \text{deg}(g(t)) \) the degree of \( g(t) \).

If \( \text{deg}(g(t)) = n \geq 2 \), then \( P^2 \) is a polynomial of degree \( 2n - 2 \). By comparing the degree of both sides of (4.8), we see that \( C_3\kappa'(s) = 0 \), and hence we get
\[ \kappa'(s)g' P^5 = C_1(s)Q^3\{3(g')^2(g'')^2 - (g')^3 g'''\} + C_1(s)\kappa(s)g' g'' P^2Q^2 \]
\[ - C_1(s)Q^3\{(g'')^2 + g' g'''\}. \]

By comparing the degree of both sides of (4.9), we see that \( \kappa'(s) = 0 \). Thus, if \( \kappa \neq 0 \), \( M \) is a surface of revolution. If \( \kappa = 0 \), then \( T, N \) are constant vectors and \( M \) is a cylindrical surface over a plane curve \( W(t) \). Since \( Q = 1 \), we have from (4.4)
\[ \Delta G = -P^{-7}\{g'(g'')^2 - g''^2 P^2 + 3g'(g'')^2\}N \]
\[ - P^{-7}\{3(g')^2(g'')^2 - (g'')^2 - g' g''' - (g')^3 g''\}V. \]

Using (1.1), we get \( C_1 = 0, C_2' = C_3' = 0 \), and
\[ (1 + (g')^2)\{C_2 A - C_2 B - C_3 D\}^2 = \{g' A - g' B + D\}^2, \]
where
\[ A = 3(g')^2(g'')^2 - (g')^3 g''' \]
\[ B = (g'')^2 + g' g'' \]
\[ D = 4g'(g''^2 - g''' - (g')^2 g'''). \]

By comparing the coefficient of highest degree of both sides of (4.11), we get \( C_2' = 1 \), and hence again we get \( C_3 = 0 \). This shows that the coefficient of highest degree of \( g' A D \) becomes zero, which is a contradiction.

If \( \text{deg}(g(t)) = 1 \), then \( M \) is a slant cylindrical (non-cylindrical) surface. Note that
\[ P = \sqrt{1 + a^2}, \]
where \( g'(t) = a \neq 0 \). By applying (4.5) and (4.6), we get
\[ PQ^3C_1(s)f(s,t) + a\kappa'(s) = 0, \]
where
\[ \kappa'(s) = \frac{d}{ds}\kappa(s). \]
and

\begin{equation}
Q^2 f(s,t)\{PC_2(s) - a\} + a\kappa(s)^2 = 0.
\end{equation}

Suppose that \(\kappa'(s_0) \neq 0\) for some \(s_0\). Then on an interval \(I\), we have \(\kappa'(s) \neq 0\). On \(I\), \(f(s,t)\) is given by

\begin{equation}
f(s,t) = \frac{-a\kappa'(s)}{PQ^3C_1(s)}.
\end{equation}

Hence, by applying \(Q = 1 - \kappa(s)t\), it follows from (4.13) and (4.14) that

\begin{equation}
aP\kappa(s)^2C_1(s) - aP\kappa'(s)C_2(s) + a^2\kappa'(s) - aP\kappa(s)^3C_1(s)t = 0.
\end{equation}

The coefficient of \(t\) in (4.16) must vanish, and hence \(C_1(s) = 0\) on \(I\), which contradicts to (4.13). This contradiction shows that \(\kappa(s)\) is a constant. Therefore \(M\) is a plane or a right circular cone.

Summarizing above, we obtain

**Theorem 4.1.** Suppose that a GSCS \(M\) of polynomial kind has pointwise 1-type Gauss map \(G\) of the second kind. Then \(M\) is a surface of revolution.

Hence, combining Theorem 4.1 in [5], we get

**Corollary 4.2.** A GSCS \(M\) of polynomial kind has the pointwise 1-type Gauss map \(G\) of the second kind if and only if it is a plane or a right circular cone.

**References**

4. ______: *Finite type submanifolds and generalizations.* University of Rome (1985).


Department of Mathematics, Chonnam National University, Kwangju 500-757, Korea
Email address: dosokim@chonnam.ac.kr