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Fuzzy Pairwise β-(r, s)-continuous Mappings

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Abstract

We introduce the concepts of fuzzy pairwise β-(r,s)-continuous mappings and fuzzy pairwise β-(r,s)-open mappings in smooth bitopological spaces and then we investigate some of their characteristic properties.

Key words : fuzzy β-(r,s)-open sets, fuzzy β-(r,s)-closures, fuzzy pairwise β-(r,s)-continuous mappings

1. Introduction

After the introduction of fuzzy sets by Zadeh [9] in his classical paper, Chang [1] was the first to introduce the concept of a fuzzy topology on a set X by axiomatizing a collection T of fuzzy subsets of X, where he referred to each member of T as an open set. In his definition of fuzzy topology, fuzziness in the concept of openness of a fuzzy subset was absent. These spaces and its generalizations are later studied by several authors, one of which, developed by Šostak [8], used the idea of degree of openness. This type of generalization of fuzzy topological spaces was later rephrased by Chattopadhyay, Hazra, and Samanta [2], and by Ramadan [7]. Kandil [3] introduced and studied the notion of fuzzy bitopological spaces as a natural generalization of fuzzy topological spaces. Lee [4] introduced the concept of smooth bitopological spaces as a generalization of smooth topological spaces and Kandil's fuzzy bitopological spaces.

In this paper, we introduce the concepts of fuzzy pairwise β-(r,s)-continuous, fuzzy pairwise β-(r,s)-open and fuzzy pairwise β-(r,s)-closed mappings in smooth bitopological spaces and then we investigate some of their characteristic properties.

2. Preliminaries

Let I be the closed unit interval $[0,1]$ of the real line and let I_0 be the half open interval $(0,1]$ of the real line. For a set X, I^X denotes the collection of all mapping from X to I. A member μ of I^X is called a fuzzy set of X. By $\tilde{0}$ and $\tilde{1}$ we denote constant mappings on X with value 0 and 1, respectively. For any $\mu \in I^X$, μ^c denotes the complement $\tilde{1} - \mu$. All other notations are the standard notations of fuzzy set theory.

A *Chang's fuzzy topology* on X [1] is a family T of fuzzy sets in X which satisfies the following properties:

(1) $\tilde{0}, \tilde{1} \in T$.

(2) If $\mu_1, \mu_2 \in T$ then $\mu_1 \wedge \mu_2 \in T$.

(3) If $\mu_k \in T$ for all k, then $\bigvee \mu_k \in T$.

The pair (X, T) be called a *Chang's fuzzy topological space*. Members of T are called T-fuzzy open sets of X and their complements T-fuzzy closed sets of X.

A system (X, T_1, T_2) consisting of a set X with two Chang's fuzzy topologies T_1 and T_2 on X is called a *Kandil's fuzzy bitopological space*.

A *smooth topology* on X is a mapping $\mathcal{T} : I^X \to I$ which satisfies the following properties:

(1) $\mathcal{T}(\tilde{0}) = \mathcal{T}(\tilde{1}) = 1$.

(2) $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2)$.

(3) $\mathcal{T}(\bigvee \mu_i) \geq \bigwedge \mathcal{T}(\mu_i)$.

The pair $(X, \mathcal{T})$ is called a *smooth topological space*. For $r \in I_0$, we call μ a $\mathcal{T}$-fuzzy r-open set of X if $\mathcal{T}(\mu) \geq r$ and μ a $\mathcal{T}$-fuzzy r-closed set of X if $\mathcal{T}(\mu^c) \geq r$.

A system $(X, \mathcal{T}_1, \mathcal{T}_2)$ consisting of a set X with two smooth topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on X is called a *smooth bitopological space*. Throughout this paper the indices i, j take values in $\{1,2\}$ and $i = j$.

Let $(X, \mathcal{T})$ be a smooth topological space. Then it is easy to see that for each $r \not\in I_0$, an r-cut

$$\mathcal{T}_r = \{\mu \in I^X \mid \mathcal{T}(\mu) \geq r\}$$

is a Chang's fuzzy topology on X.

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Let (X, T) be a Chang's fuzzy topological space and $r \in I_0$. Then the mapping $T^r : I^X \to I$ is defined by

$$T^r(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ r & \text{if } \mu \in T - \{\tilde{0}, \tilde{1}\}, \\ 0 & \text{otherwise} \end{cases}$$

becomes a smooth topology.

Hence, we obtain that if $(X, \mathcal{T}_1, \mathcal{T}_2)$ is a smooth bitopological space and $r, s \in I_0$, then $(X, (\mathcal{T}_1)_r, (\mathcal{T}_2)_s)$ is a Kandil's fuzzy bitopological space. Also, if (X, T_1, T_2) is a Kandil's fuzzy bitopological space and $r, s \in I_0$, then $(X, (T_1)^r, (T_2)^s)$ is a smooth bitopological space.

Definition 2.1. [4] Let $(X, \mathcal{T})$ be a smooth topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *$\mathcal{T}$-fuzzy r-closure* is defined by

$$\mathcal{T}\text{-Cl}(\mu, r) = \bigwedge\{\rho \in I^X \mid \mu \le \rho, \mathcal{T}(\rho^c) \ge r\}$$

and the *$\mathcal{T}$-fuzzy r-interior* is defined by

$$\mathcal{T}\text{-Int}(\mu, r) = \bigvee\{\rho \in I^X \mid \mu \ge \rho, \mathcal{T}(\rho) \ge r\}.$$

Lemma 2.2. [4] Let μ be a fuzzy set of a smooth topological space $(X, \mathcal{T})$ and let $r \in I_0$. Then we have:

(1) $\mathcal{T}\text{-Cl}(\mu, r)^c = \mathcal{T}\text{-Int}(\mu^c, r)$.

(2) $\mathcal{T}\text{-Int}(\mu, r)^c = \mathcal{T}\text{-Cl}(\mu^c, r)$.

Definition 2.3. [6] Let μ be a fuzzy set of a smooth bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $r, s \in I_0$. Then μ is said to be

(1) a *$(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r, s)-open* set if

$\mu \le \mathcal{T}_j\text{-Cl}(\mathcal{T}_i\text{-Int}(\mathcal{T}_j\text{-Cl}(\mu, s), r), s)$,

(2) a *$(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r, s)-closed* set if

$\mathcal{T}_j\text{-Int}(\mathcal{T}_i\text{-Cl}(\mathcal{T}_j\text{-Int}(\mu, s), r), s) \le \mu$.

Definition 2.4. [4, 5] Let $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{U}_1, \mathcal{U}_2)$ be a mapping from a smooth bitopological space X to a smooth bitopological space Y and $r, s \in I_0$. Then f is said to be

(1) a *fuzzy pairwise (r, s)-continuous* mapping if the induced mapping $f : (X, \mathcal{T}_1) \to (Y, \mathcal{U}_1)$ is a fuzzy r-continuous mapping and the induced mapping $f : (X, \mathcal{T}_2) \to (Y, \mathcal{U}_2)$ is a fuzzy s-continuous mapping,

(2) a *fuzzy pairwise (r, s)-semicontinuous* mapping if $f^{-1}(\mu)$ is a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy (r, s)-semiopen set of X for each $\mathcal{U}_1$-fuzzy r-open set μ of Y and $f^{-1}(\nu)$ is a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy (s, r)-semiopen set of X for each $\mathcal{U}_2$-fuzzy s-open set ν of Y,

(3) a *fuzzy pairwise (r, s)-precontinuous* mapping if $f^{-1}(\mu)$ is a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy (r, s)-preopen set of X for each $\mathcal{U}_1$-fuzzy r-open set μ of Y and $f^{-1}(\nu)$ is a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy (s, r)-preopen set of X for each $\mathcal{U}_2$-fuzzy s-open set ν of Y.

3. Fuzzy pairwise β-(r, s)-continuous mappings

Definition 3.1. Let $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{U}_1, \mathcal{U}_2)$ be a mapping from a smooth bitopological space X to a smooth bitopological space Y and $r, s \in I_0$. Then f is called

(1) a *fuzzy pairwise β-(r, s)-continuous* mapping if $f^{-1}(\mu)$ is a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy β-(r, s)-open set of X for each $\mathcal{U}_1$-fuzzy r-open set μ of Y and $f^{-1}(\nu)$ is a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy β-(s, r)-open set of X for each $\mathcal{U}_2$-fuzzy s-open set ν of Y,

(2) a *fuzzy pairwise β-(r, s)-open* mapping if $f(\rho)$ is a $(\mathcal{U}_1, \mathcal{U}_2)$-fuzzy β-(r, s)-open set of Y for each $\mathcal{T}_1$-fuzzy r-open set ρ of X and $f(\lambda)$ is a $(\mathcal{U}_2, \mathcal{U}_1)$-fuzzy β-(s, r)-open set of Y for each $\mathcal{T}_2$-fuzzy s-open set λ of X,

(3) a *fuzzy pairwise β-(r, s)-closed* mapping if $f(\rho)$ is a $(\mathcal{U}_1, \mathcal{U}_2)$-fuzzy β-(r, s)-closed set of Y for each $\mathcal{T}_1$-fuzzy r-closed set ρ of X and $f(\lambda)$ is a $(\mathcal{U}_2, \mathcal{U}_1)$-fuzzy β-(s, r)-closed set of Y for each $\mathcal{T}_2$-fuzzy s-closed set λ of X.

Remark 3.2. It is clear that every fuzzy pairwise (r, s)-semicontinuous mapping is a fuzzy pairwise β-(r, s)-continuous mapping and every fuzzy pairwise (r, s)-precontinuous mapping is a fuzzy pairwise β-(r, s)-continuous mapping. However, the following example show that all of the converses need not be true.

Example 3.3. Let $X = \{x, y\}$ and μ_1, μ_2, μ_3 and μ_4 be fuzzy sets of X defined as

$$\mu_1(x) = 0.4, \quad \mu_1(y) = 0.7;$$

$$\mu_2(x) = 0.1, \quad \mu_2(y) = 0.2;$$

$$\mu_3(x) = 0.8, \quad \mu_3(y) = 0.5;$$

and

$$\mu_4(x) = 0.7, \quad \mu_4(y) = 0.6.$$

Define $\mathcal{T}_1 : I^X \to I$ and $\mathcal{T}_2 : I^X \to I$ by

$$\mathcal{T}_1(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ 0 & \text{otherwise;} \end{cases}$$

and

$$\mathcal{T}_2(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{3} & \text{if } \mu = \mu_2, \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly $(\mathcal{T}_1, \mathcal{T}_2)$ is a smooth bitopology on X. Define $\mathcal{U}_1 : I^X \to I$ and $\mathcal{U}_2 : I^X \to I$ by

$$\mathcal{U}_1(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_3, \\ 0 & \text{otherwise;} \end{cases}$$

and

$$\mathcal{U}_2(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly $(\mathcal{U}_1, \mathcal{U}_2)$ is a smooth bitopology on X. Consider the identity mapping $1_X : (X, \mathcal{T}_1, \mathcal{T}_2) \to (X, \mathcal{U}_1, \mathcal{U}_2)$. Then it is a fuzzy pairwise β-$(\frac{1}{2}, \frac{1}{3})$-continuous mapping which is not a fuzzy pairwise $(\frac{1}{2}, \frac{1}{3})$-semicontinuous mapping.

Define $\mathcal{V}_1 : I^X \to I$ and $\mathcal{V}_2 : I^X \to I$ by

$$\mathcal{V}_1(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_4, \\ 0 & \text{otherwise;} \end{cases}$$

and

$$\mathcal{V}_2(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly $(\mathcal{V}_1, \mathcal{V}_2)$ is a smooth bitopology on X. Consider the identity mapping $1_X : (X, \mathcal{T}_1, \mathcal{T}_2) \to (X, \mathcal{V}_1, \mathcal{V}_2)$. Then it is a fuzzy pairwise β-$(\frac{1}{2}, \frac{1}{3})$-continuous mapping which is not a fuzzy pairwise $(\frac{1}{2}, \frac{1}{3})$-precontinuous mapping.

Definition 3.4. Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be a smooth bitopological space and $r, s \in I_0$. For each $\mu \in I^X$, the *$(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r, s)-closure* is defined by

$$(\mathcal{T}_i, \mathcal{T}_j)\text{-}\beta\text{Cl}(\mu, r, s) = \bigwedge\{\rho \in I^X \mid \mu \le \rho,\ \rho \text{ is } (\mathcal{T}_i, \mathcal{T}_j)\text{-fuzzy } \beta\text{-}(r, s)\text{-closed}\}$$

and the *$(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r, s)-interior* is defined by

$$(\mathcal{T}_i, \mathcal{T}_j)\text{-}\beta\text{Int}(\mu, r, s) = \bigvee\{\rho \in I^X \mid \mu \ge \rho,\ \rho \text{ is } (\mathcal{T}_i, \mathcal{T}_j)\text{-fuzzy } \beta\text{-}(r, s)\text{-open}\}.$$

Lemma 3.5. For a fuzzy set μ of a smooth bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ and let $r, s \in I_0$, we have:

(1) $(\mathcal{T}_i, \mathcal{T}_j)$-$\beta$Cl$(\mu, r, s)^c = (\mathcal{T}_i, \mathcal{T}_j)$-$\beta$Int$(\mu^c, r, s)$.

(2) $(\mathcal{T}_i, \mathcal{T}_j)$-$\beta$Int$(\mu, r, s)^c = (\mathcal{T}_i, \mathcal{T}_j)$-$\beta$Cl$(\mu^c, r, s)$.

Proof. (1) Since $(\mathcal{T}_i, \mathcal{T}_j)$-$\beta$Int$(\mu, r, s)$ is a $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r, s)-open set and $(\mathcal{T}_i, \mathcal{T}_j)$-$\beta$Int$(\mu, r, s) \le \mu$, we have $(\mathcal{T}_i, \mathcal{T}_j)$-$\beta$Int$(\mu, r, s)^c$ is $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r, s)-closed set of X and $\mu^c \le (\mathcal{T}_i, \mathcal{T}_j)$-$\beta$Int$(\mu, r, s)^c$. Thus

$$\begin{aligned}&(\mathcal{T}_i, \mathcal{T}_j)\text{-}\beta\text{Cl}(\mu^c, r, s)\\ &\le (\mathcal{T}_i, \mathcal{T}_j)\text{-}\beta\text{Cl}((\mathcal{T}_i, \mathcal{T}_j)\text{-}\beta\text{Int}(\mu, r, s)^c, r, s)\\ &= (\mathcal{T}_i, \mathcal{T}_j)\text{-}\beta\text{Int}(\mu, r, s)^c.\end{aligned}$$

Conversely, $(\mathcal{T}_i, \mathcal{T}_j)$-$\beta$Cl$(\mu^c, r, s)^c$ is a $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r, s)-open set of X and $(\mathcal{T}_i, \mathcal{T}_j)$-$\beta$Cl$(\mu^c, r, s)^c \le \mu$. Thus

$$\begin{aligned}&(\mathcal{T}_i, \mathcal{T}_j)\text{-}\beta\text{Cl}(\mu^c, r, s)^c\\ &= (\mathcal{T}_i, \mathcal{T}_j)\text{-}\beta\text{Int}((\mathcal{T}_i, \mathcal{T}_j)\text{-}\beta\text{Cl}(\mu^c, r, s)^c, r, s)\\ &\le (\mathcal{T}_i, \mathcal{T}_j)\text{-}\beta\text{Int}(\mu, r, s)\end{aligned}$$

and hence

$$(\mathcal{T}_i, \mathcal{T}_j)\text{-}\beta\text{Int}(\mu, r, s)^c \le (\mathcal{T}_i, \mathcal{T}_j)\text{-}\beta\text{Cl}(\mu^c, r, s).$$

(2) Similar to (1). □

Theorem 3.6. Let $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{U}_1, \mathcal{U}_2)$ be a mapping and $r, s \in I_0$. Then the following statements are equivalent:

(1) f is a fuzzy pairwise β-(r, s)-continuous mapping.

(2) $f^{-1}(\mu)$ is a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy β-(r, s)-closed set of X for each $\mathcal{U}_1$-fuzzy r-closed set μ of Y and $f^{-1}(\nu)$ is a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy β-(s, r)-closed set of X for each $\mathcal{U}_2$-fuzzy s-closed set ν of Y.

(3) For each fuzzy set ρ of X,

$$f((\mathcal{T}_1, \mathcal{T}_2)\text{-}\beta\text{Cl}(\rho, r, s)) \le \mathcal{U}_1\text{-Cl}(f(\rho), r)$$

and

$$f((\mathcal{T}_2, \mathcal{T}_1)\text{-}\beta\text{Cl}(\rho, s, r)) \le \mathcal{U}_2\text{-Cl}(f(\rho), s).$$

(4) For each fuzzy set μ of Y,

$$(\mathcal{T}_1, \mathcal{T}_2)\text{-}\beta\text{Cl}(f^{-1}(\mu), r, s) \le f^{-1}(\mathcal{U}_1\text{-Cl}(\mu, r))$$

and

$$(\mathcal{T}_2, \mathcal{T}_1)\text{-}\beta\text{Cl}(f^{-1}(\mu), s, r) \le f^{-1}(\mathcal{U}_2\text{-Cl}(\mu, s)).$$

(5) For each fuzzy set μ of Y,

$$f^{-1}(\mathcal{U}_1\text{-Int}(\mu, r)) \le (\mathcal{T}_1, \mathcal{T}_2)\text{-}\beta\text{Int}(f^{-1}(\mu), r, s)$$

and

$$f^{-1}(\mathcal{U}_2\text{-Int}(\mu, s)) \le (\mathcal{T}_2, \mathcal{T}_1)\text{-}\beta\text{Int}(f^{-1}(\mu), s, r).$$

Proof. (1) $\Rightarrow$ (2) Let μ be any $\mathcal{U}_1$-fuzzy r-closed set and ν any $\mathcal{U}_2$-fuzzy s-closed set of Y. Then μ^c is a $\mathcal{U}_1$-fuzzy r-open set and ν^c is a $\mathcal{U}_2$-fuzzy s-open set of Y. Since f is a fuzzy pairwise β-(r, s)-continuous mapping, $f^{-1}(\mu^c)$ is a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy β-(r, s)-open set and $f^{-1}(\nu^c)$ is a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy β-(s, r)-open set of X. Thus $f^{-1}(\mu)$ is a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy β-(r, s)-closed set and $f^{-1}(\nu)$ is a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy β-(s, r)-closed set of X.

(2) $\Rightarrow$ (3) Let ρ be any fuzzy set of X. Then $\mathcal{U}_1$-Cl$(f(\rho), r)$ is a $\mathcal{U}_1$-fuzzy r-closed set and $\mathcal{U}_2$-Cl$(f(\rho), s)$ is a $\mathcal{U}_2$-fuzzy s-closed set of Y. By (2),

$f^{-1}(\mathcal{U}_1\text{-Cl}(f(\rho),r))$ is a $(\mathcal{T}_1,\mathcal{T}_2)$-fuzzy β-(r,s)-closed set and $f^{-1}(\mathcal{U}_2\text{-Cl}(f(\rho),s))$ is a $(\mathcal{T}_2,\mathcal{T}_1)$-fuzzy β-(s,r)-closed set of X. Since $f(\rho) \le \mathcal{U}_1\text{-Cl}(f(\rho),r)$ and $f(\rho) \le \mathcal{U}_2\text{-Cl}(f(\rho),s)$, we have

$$\begin{aligned}&(\mathcal{T}_1,\mathcal{T}_2)\text{-}\beta\text{Cl}(\rho,r,s)\\ &\le (\mathcal{T}_1,\mathcal{T}_2)\text{-}\beta\text{Cl}(f^{-1}f(\rho),r,s)\\ &\le (\mathcal{T}_1,\mathcal{T}_2)\text{-}\beta\text{Cl}(f^{-1}(\mathcal{U}_1\text{-Cl}(f(\rho),r)),r,s)\\ &= f^{-1}(\mathcal{U}_1\text{-Cl}(f(\rho),r))\end{aligned}$$

and

$$\begin{aligned}&(\mathcal{T}_2,\mathcal{T}_1)\text{-}\beta\text{Cl}(\rho,s,r)\\ &\le (\mathcal{T}_2,\mathcal{T}_1)\text{-}\beta\text{Cl}(f^{-1}f(\rho),s,r)\\ &\le (\mathcal{T}_2,\mathcal{T}_1)\text{-}\beta\text{Cl}(f^{-1}(\mathcal{U}_2\text{-Cl}(f(\rho),s)),s,r)\\ &= f^{-1}(\mathcal{U}_2\text{-Cl}(f(\rho),s)).\end{aligned}$$

Hence

$$\begin{aligned}f((\mathcal{T}_1,\mathcal{T}_2)\text{-}\beta\text{Cl}(\rho,r,s)) &\le ff^{-1}(\mathcal{U}_1\text{-Cl}(f(\rho),r))\\ &\le \mathcal{U}_1\text{-Cl}(f(\rho),r)\end{aligned}$$

and

$$\begin{aligned}f((\mathcal{T}_2,\mathcal{T}_1)\text{-}\beta\text{Cl}(\rho,s,r)) &\le ff^{-1}(\mathcal{U}_2\text{-Cl}(f(\rho),s))\\ &\le \mathcal{U}_2\text{-Cl}(f(\rho),s).\end{aligned}$$

(3) $\Rightarrow$ (4) Let μ be any fuzzy set of Y. Then $f^{-1}(\mu)$ is a fuzzy set of X. By (3),

$$\begin{aligned}&f((\mathcal{T}_1,\mathcal{T}_2)\text{-}\beta\text{Cl}(f^{-1}(\mu),r,s))\\ &\le \mathcal{U}_1\text{-Cl}(ff^{-1}(\mu),r)\\ &\le \mathcal{U}_1\text{-Cl}(\mu,r)\end{aligned}$$

and

$$\begin{aligned}&f((\mathcal{T}_2,\mathcal{T}_1)\text{-}\beta\text{Cl}(f^{-1}(\mu),s,r))\\ &\le \mathcal{U}_2\text{-Cl}(ff^{-1}(\mu),s)\\ &\le \mathcal{U}_2\text{-Cl}(\mu,s).\end{aligned}$$

Thus

$$\begin{aligned}&(\mathcal{T}_1,\mathcal{T}_2)\text{-}\beta\text{Cl}(f^{-1}(\mu),r,s)\\ &\le f^{-1}f((\mathcal{T}_1,\mathcal{T}_2)\text{-}\beta\text{Cl}(f^{-1}(\mu),r,s))\\ &\le f^{-1}(\mathcal{U}_1\text{-Cl}(\mu,r))\end{aligned}$$

and

$$\begin{aligned}&(\mathcal{T}_2,\mathcal{T}_1)\text{-}\beta\text{Cl}(f^{-1}(\mu),s,r)\\ &\le f^{-1}f((\mathcal{T}_2,\mathcal{T}_1)\text{-}\beta\text{Cl}(f^{-1}(\mu),s,r))\\ &\le f^{-1}(\mathcal{U}_2\text{-Cl}(\mu,s)).\end{aligned}$$

(4) $\Rightarrow$ (5) Let μ be any fuzzy set of Y. Then μ^c is a fuzzy set of Y. By (4),

$$\begin{aligned}&(\mathcal{T}_1,\mathcal{T}_2)\text{-}\beta\text{Cl}(f^{-1}(\mu)^c,r,s)\\ &= (\mathcal{T}_1,\mathcal{T}_2)\text{-}\beta\text{Cl}(f^{-1}(\mu^c),r,s)\\ &\le f^{-1}(\mathcal{U}_1\text{-Cl}(\mu^c,r))\end{aligned}$$

and

$$\begin{aligned}&(\mathcal{T}_2,\mathcal{T}_1)\text{-}\beta\text{Cl}(f^{-1}(\mu)^c,s,r)\\ &= (\mathcal{T}_2,\mathcal{T}_1)\text{-}\beta\text{Cl}(f^{-1}(\mu^c),s,r)\\ &\le f^{-1}(\mathcal{U}_2\text{-Cl}(\mu^c,s)).\end{aligned}$$

By Lemma 3.5,

$$\begin{aligned}&f^{-1}(\mathcal{U}_1\text{-Int}(\mu,r))\\ &\quad= f^{-1}(\mathcal{U}_1\text{-Cl}(\mu^c,r))^c\\ &\quad\le (\mathcal{T}_1,\mathcal{T}_2)\text{-}\beta\text{Cl}(f^{-1}(\mu^c),r,s)^c\\ &\quad= (\mathcal{T}_1,\mathcal{T}_2)\text{-}\beta\text{Int}(f^{-1}(\mu),r,s)\end{aligned}$$

and

$$\begin{aligned}&f^{-1}(\mathcal{U}_2\text{-Int}(\mu,s))\\ &\quad= f^{-1}(\mathcal{U}_2\text{-Cl}(\mu^c,s))^c\\ &\quad\le (\mathcal{T}_2,\mathcal{T}_1)\text{-}\beta\text{Cl}(f^{-1}(\mu^c),s,r)^c\\ &\quad= (\mathcal{T}_2,\mathcal{T}_1)\text{-}\beta\text{Int}(f^{-1}(\mu),s,r).\end{aligned}$$

(5) $\Rightarrow$ (1) Let μ be any $\mathcal{U}_1$-fuzzy r-open set and ν any $\mathcal{U}_2$-fuzzy s-open set of Y. Then $\mathcal{U}_1\text{-Int}(\mu,r) = \mu$ and $\mathcal{U}_2\text{-Int}(\nu,s) = \nu$. By (5),

$$\begin{aligned}f^{-1}(\mu) &= f^{-1}(\mathcal{U}_1\text{-Int}(\mu,r))\\ &\le (\mathcal{T}_1,\mathcal{T}_2)\text{-}\beta\text{Int}(f^{-1}(\mu),r,s)\\ &\le f^{-1}(\mu)\end{aligned}$$

and

$$\begin{aligned}f^{-1}(\nu) &= f^{-1}(\mathcal{U}_2\text{-Int}(\nu,s))\\ &\le (\mathcal{T}_2,\mathcal{T}_1)\text{-}\beta\text{Int}(f^{-1}(\nu),s,r)\\ &\le f^{-1}(\nu).\end{aligned}$$

So $f^{-1}(\mu) = (\mathcal{T}_1,\mathcal{T}_2)\text{-}\beta\text{Int}(f^{-1}(\mu),r,s)$ and $f^{-1}(\nu) = (\mathcal{T}_2,\mathcal{T}_1)\text{-}\beta\text{Int}(f^{-1}(\nu),s,r)$. Hence $f^{-1}(\mu)$ is a $(\mathcal{T}_1,\mathcal{T}_2)$-fuzzy β-(r,s)-open set and $f^{-1}(\nu)$ is a $(\mathcal{T}_2,\mathcal{T}_1)$-fuzzy β-(s,r)-open set of X. Thus f is a fuzzy pairwise β-(r,s)-continuous mapping. □

Theorem 3.7. Let $f : (X,\mathcal{T}_1,\mathcal{T}_2) \to (Y,\mathcal{U}_1,\mathcal{U}_2)$ be a bijection and $r,s \in I_0$. Then f is a fuzzy pairwise β-(r,s)-continuous mapping if and only if $\mathcal{U}_1\text{-Int}(f(\rho),r) \le f((\mathcal{T}_1,\mathcal{T}_2)\text{-}\beta\text{Int}(\rho,r,s))$ and $\mathcal{U}_2\text{-Int}(f(\rho),s) \le f((\mathcal{T}_2,\mathcal{T}_1)\text{-}\beta\text{Int}(\rho,s,r))$ for each fuzzy set ρ of X.

Proof. Let f be a fuzzy pairwise β-(r,s)-continuous mapping and ρ any fuzzy set of X. Then $\mathcal{U}_1\text{-Int}(f(\rho),r)$ is a $\mathcal{U}_1$-fuzzy r-open set and $\mathcal{U}_2\text{-Int}(f(\rho),s)$ is a $\mathcal{U}_2$-fuzzy s-open set of Y. Since f is a fuzzy pairwise β-(r,s)-continuous mapping, we have $f^{-1}(\mathcal{U}_1\text{-Int}(f(\rho),r))$ is a $(\mathcal{T}_1,\mathcal{T}_2)$-fuzzy β-(r,s)-open set and $f^{-1}(\mathcal{U}_2\text{-Int}(f(\rho),s))$ is a $(\mathcal{T}_2,\mathcal{T}_1)$-fuzzy β-(s,r)-open set of X. Since f is fuzzy pairwise β-(r,s)-continuous and one-to-one, we have

$$\begin{aligned}&f^{-1}(\mathcal{U}_1\text{-Int}(f(\rho),r))\\ &\quad\le (\mathcal{T}_1,\mathcal{T}_2)\text{-}\beta\text{Int}(f^{-1}f(\rho),r,s)\\ &\quad= (\mathcal{T}_1,\mathcal{T}_2)\text{-}\beta\text{Int}(\rho,r,s)\end{aligned}$$

and
$$\begin{aligned}&f^{-1}(\mathcal{U}_2\text{-Int}(f(\rho),s))\\&\quad\le(\mathcal{T}_2,\mathcal{T}_1)\text{-}\beta\text{Int}(f^{-1}f(\rho),s,r)\\&\quad=(\mathcal{T}_2,\mathcal{T}_1)\text{-}\beta\text{Int}(\rho,s,r).\end{aligned}$$

Since f is onto,
$$\begin{aligned}&\mathcal{U}_1\text{-Int}(f(\rho),r)\\&=ff^{-1}(\mathcal{U}_1\text{-Int}(f(\rho),r))\\&\le f((\mathcal{T}_1,\mathcal{T}_2)\text{-}\beta\text{Int}(\rho,r,s))\end{aligned}$$

and
$$\begin{aligned}&\mathcal{U}_2\text{-Int}(f(\rho),s)\\&=ff^{-1}(\mathcal{U}_2\text{-Int}(f(\rho),s))\\&\le f((\mathcal{T}_2,\mathcal{T}_1)\text{-}\beta\text{Int}(\rho,s,r))\end{aligned}$$

Conversely, let μ be any $\mathcal{U}_1$-fuzzy r-open set and ν any $\mathcal{U}_2$-fuzzy s-open set of Y. Then $\mathcal{U}_1\text{-Int}(\mu,r)=\mu$ and $\mathcal{U}_2\text{-Int}(\nu,s)=\nu$. Since f is onto,
$$\begin{aligned}&f((\mathcal{T}_1,\mathcal{T}_2)\text{-}\beta\text{Int}(f^{-1}(\mu),r,s))\\&\quad\ge\mathcal{U}_1\text{-Int}(ff^{-1}(\mu),r)\\&\quad=\mathcal{U}_1\text{-Int}(\mu,r)\\&\quad=\mu\end{aligned}$$

and
$$\begin{aligned}&f((\mathcal{T}_2,\mathcal{T}_1)\text{-}\beta\text{Int}(f^{-1}(\nu),s,r))\\&\quad\ge\mathcal{U}_2\text{-Int}(ff^{-1}(\nu),s)\\&\quad=\mathcal{U}_2\text{-Int}(\nu,s)\\&\quad=\nu.\end{aligned}$$

Since f is one-to-one, we have
$$\begin{aligned}f^{-1}(\mu)&\le f^{-1}f((\mathcal{T}_1,\mathcal{T}_2)\text{-}\beta\text{Int}(f^{-1}(\mu),r,s))\\&=(\mathcal{T}_1,\mathcal{T}_2)\text{-}\beta\text{Int}(f^{-1}(\mu),r,s)\\&\le f^{-1}(\mu)\end{aligned}$$

and
$$\begin{aligned}f^{-1}(\nu)&\le f^{-1}f((\mathcal{T}_2,\mathcal{T}_1)\text{-}\beta\text{Int}(f^{-1}(\nu),s,r))\\&=(\mathcal{T}_2,\mathcal{T}_1)\text{-}\beta\text{Int}(f^{-1}(\nu),s,r)\\&\le f^{-1}(\nu).\end{aligned}$$

So $f^{-1}(\mu)=(\mathcal{T}_1,\mathcal{T}_2)\text{-}\beta\text{Int}(f^{-1}(\mu),r,s)$ and $f^{-1}(\nu)=(\mathcal{T}_2,\mathcal{T}_1)\text{-}\beta\text{Int}(f^{-1}(\nu),s,r)$. Hence $f^{-1}(\mu)$ is a $(\mathcal{T}_1,\mathcal{T}_2)$-fuzzy β-(r,s)-open set and $f^{-1}(\nu)$ is a $(\mathcal{T}_2,\mathcal{T}_1)$-fuzzy β-(s,r)-open set of X. Therefore f is a fuzzy pairwise β-(r,s)-continuous mapping. □

Theorem 3.8. Let $f:(X,\mathcal{T}_1,\mathcal{T}_2)\to(Y,\mathcal{U}_1,\mathcal{U}_2)$ be a mapping and $r,s\in I_0$. Then the following statements are equivalent:

(1) f is a fuzzy pairwise β-(r,s)-open mapping.

(2) For each fuzzy set ρ of X,
$$f(\mathcal{T}_1\text{-Int}(\rho,r))\le(\mathcal{U}_1,\mathcal{U}_2)\text{-}\beta\text{Int}(f(\rho),r,s)$$
and
$$f(\mathcal{T}_2\text{-Int}(\rho,s))\le(\mathcal{U}_2,\mathcal{U}_1)\text{-}\beta\text{Int}(f(\rho),s,r).$$

(3) For each fuzzy set μ of Y,
$$\mathcal{T}_1\text{-Int}(f^{-1}(\mu),r)\le f^{-1}((\mathcal{U}_1,\mathcal{U}_2)\text{-}\beta\text{Int}(\mu,r,s))$$
and
$$\mathcal{T}_2\text{-Int}(f^{-1}(\mu),s)\le f^{-1}((\mathcal{U}_2,\mathcal{U}_1)\text{-}\beta\text{Int}(\mu,s,r)).$$

Proof. (1) ⇒ (2) Let ρ be any fuzzy set of X. Clearly $\mathcal{T}_1\text{-Int}(\rho,r)$ is a $\mathcal{T}_1$-fuzzy r-open set and $\mathcal{T}_2\text{-Int}(\rho,s)$ is a $\mathcal{T}_2$-fuzzy s-open set of X. Since f is a fuzzy pairwise β-(r,s)-open mapping, $f(\mathcal{T}_1\text{-Int}(\rho,r))$ is a $(\mathcal{U}_1,\mathcal{U}_2)$-fuzzy β-(r,s)-open set and $f(\mathcal{T}_2\text{-Int}(\rho,s))$ is a $(\mathcal{U}_2,\mathcal{U}_1)$-fuzzy β-(s,r)-open set of Y. Thus
$$\begin{aligned}&f(\mathcal{T}_1\text{-Int}(\rho,r))\\&=(\mathcal{U}_1,\mathcal{U}_2)\text{-}\beta\text{Int}(f(\mathcal{T}_1\text{-Int}(\rho,r)),r,s)\\&\le(\mathcal{U}_1,\mathcal{U}_2)\text{-}\beta\text{Int}(f(\rho),r,s)\end{aligned}$$

and
$$\begin{aligned}&f(\mathcal{T}_2\text{-Int}(\rho,s))\\&=(\mathcal{U}_2,\mathcal{U}_1)\text{-}\beta\text{Int}(f(\mathcal{T}_2\text{-Int}(\rho,s)),s,r)\\&\le(\mathcal{U}_2,\mathcal{U}_1)\text{-}\beta\text{Int}(f(\rho),s,r)\end{aligned}$$

(2) ⇒ (3) Let μ be any fuzzy set of Y. Then $f^{-1}(\mu)$ is a fuzzy set of X. By (2),
$$\begin{aligned}&f(\mathcal{T}_1\text{-Int}(f^{-1}(\mu),r))\\&\quad\le(\mathcal{U}_1,\mathcal{U}_2)\text{-}\beta\text{Int}(ff^{-1}(\mu),r,s)\\&\quad\le(\mathcal{U}_1,\mathcal{U}_2)\text{-}\beta\text{Int}(\mu,r,s)\end{aligned}$$

and
$$\begin{aligned}&f(\mathcal{T}_2\text{-Int}(f^{-1}(\mu),s))\\&\quad\le(\mathcal{U}_2,\mathcal{U}_1)\text{-}\beta\text{Int}(ff^{-1}(\mu),s,r)\\&\quad\le(\mathcal{U}_2,\mathcal{U}_1)\text{-}\beta\text{Int}(\mu,s,r).\end{aligned}$$

Thus we have
$$\begin{aligned}&\mathcal{T}_1\text{-Int}(f^{-1}(\mu),r)\\&\le f^{-1}f(\mathcal{T}_1\text{-Int}(f^{-1}(\mu),r))\\&\le f^{-1}((\mathcal{U}_1,\mathcal{U}_2)\text{-}\beta\text{Int}(\mu,r,s))\end{aligned}$$

and
$$\begin{aligned}&\mathcal{T}_2\text{-Int}(f^{-1}(\mu),s)\\&\le f^{-1}f(\mathcal{T}_2\text{-Int}(f^{-1}(\mu),s))\\&\le f^{-1}((\mathcal{U}_2,\mathcal{U}_1)\text{-}\beta\text{Int}(\mu,s,r)).\end{aligned}$$

(3) $\Rightarrow$ (1) Let ρ be any $\mathcal{T}_1$-fuzzy r-open set and λ any $\mathcal{T}_2$-fuzzy s-open set of X. Then $\mathcal{T}_1$-Int$(\rho, r) = \rho$ and $\mathcal{T}_2$-Int$(\lambda, s) = \lambda$. By (3),

$$\begin{aligned} \rho &= \mathcal{T}_1\text{-Int}(\rho, r) \\ &\leq \mathcal{T}_1\text{-Int}(f^{-1}f(\rho), r) \\ &\leq f^{-1}((\mathcal{U}_1, \mathcal{U}_2)\text{-}\beta\text{Int}(f(\rho), r, s)) \end{aligned}$$

and

$$\begin{aligned} \rho &= \mathcal{T}_2\text{-Int}(\lambda, s) \\ &\leq \mathcal{T}_2\text{-Int}(f^{-1}f(\lambda), s) \\ &\leq f^{-1}((\mathcal{U}_2, \mathcal{U}_1)\text{-}\beta\text{Int}(f(\lambda), s, r)). \end{aligned}$$

Hence we have

$$\begin{aligned} f(\rho) &\leq ff^{-1}((\mathcal{U}_1, \mathcal{U}_2)\text{-}\beta\text{Int}(f(\rho), r, s)) \\ &\leq (\mathcal{U}_1, \mathcal{U}_2)\text{-}\beta\text{Int}(f(\rho), r, s) \\ &\leq f(\rho) \end{aligned}$$

and

$$\begin{aligned} f(\lambda) &\leq ff^{-1}((\mathcal{U}_2, \mathcal{U}_1)\text{-}\beta\text{Int}(f(\lambda), s, r)) \\ &\leq (\mathcal{U}_2, \mathcal{U}_1)\text{-}\beta\text{Int}(f(\lambda), s, r) \\ &\leq f(\lambda) \end{aligned}$$

Thus $f(\rho) = (\mathcal{U}_1, \mathcal{U}_2)$-$\beta$Int$(f(\rho), r, s)$ and $f(\lambda) = (\mathcal{U}_2, \mathcal{U}_1)$-$\beta$Int$(f(\lambda), s, r)$. Hence $f(\rho)$ is a $(\mathcal{U}_1, \mathcal{U}_2)$-fuzzy β-(r, s)-open set and $f(\lambda)$ is a $(\mathcal{U}_2, \mathcal{U}_1)$-fuzzy β-(s, r)-open set of Y. Therefore f is a fuzzy pairwise β-(r, s)-open mapping. □

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