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REGULARIZED SOLUTION TO THE FREDHOLM INTEGRAL EQUATION OF THE FIRST KIND WITH NOISY DATA†

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Abstract. In this paper, we use a modified Tikhonov regularization method to solve the Fredholm integral equation of the first kind. Under the assumption that measured data are contaminated with deterministic errors, we give two error estimates. The convergence rates can be obtained under the suitable choices of regularization parameters and the number of measured points. Some numerical experiments show that the proposed method is effective and stable.

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1. Introduction

Consider the Fredholm integral equation of the first kind

$$\int_0^1 K(t,s)f(s)ds = g(t), \tag{1}$$

where $K(t,s)$ is a given function called the kernel of an integral equation and $g(s)$ is measured discretely with deterministic errors. Define the following operator

$$\mathcal{K} : \mathcal{H} \to \mathcal{L}_2[0,1], \quad (\mathcal{K}f)(t) = \int_0^1 K(t,s)f(s)ds, \tag{2}$$

where $\mathcal{H}$ is either $\mathcal{L}_2[0,1]$ or a reproducing kernel Hilbert space $\mathcal{H}_R$ with reproducing kernel $R(t,s)$.
The Fredholm integral equation of the first kind is well known as an ill-posed

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problem by Hadamard's sense (see Ch.1 in [6] and [8, 14]). By the Riemann-Lebesgue lemma (see, e.g., [3, 4]), we know

$$\int_0^1 K(t,s)\sin\gamma s ds \to 0, \quad as \quad \gamma \to \infty,$$

which means that high frequency noise in a solution may be screened out by the integral operator, or it is equivalent to say that a very small change in the data g may lead to a large change in the solution f. Thus, we know the problem of Fredholm integral equation is ill-posed.

As a classical ill-posed problem, the Fredholm integral equations of the first kind have been investigated by many references, see [4]. One popular method is the Tikhonov regularization, but usually a continuous functional is used and the minimizer for the corresponding functional is difficult to be obtained. Many inverse problems can be formularized as a Fredholm integral equations of the first kind, for examples, the backward heat equation, harmonic continuation, and numerical differentiation [3].

In [7], the author discussed the equivalence of Tikhonov regularization and reproducing kernel Hilbert space approaches for the Fredholm integral equation of the first kind. In [11], Lukas gave a moment collocation method for the linear operator equations, and he also obtained the convergence rate results for the solutions of the Fredholm integral equations of the first kind. Recently, Li and Nashed gave a modified Tikhonov regularization method for the linear operator equations(see [9]), which is different from our method, and they obtained optimal convergence order of the regularized solutions by an *a priori* choice of the regularization parameter.

In this paper, we solve the Fredholm integral equation of the first kind by a modified Tikhonov regularization method based on the reproducing kernel Hilbert space, refer to [17, 18], and propose two *a priori* rules for choices of regularization parameters.

Find $f^\delta_{n,\alpha}$ in $\mathcal{H}$ to be the solution of the following minimization problem

$$\min_{h\in\mathcal{H}} \Phi(h) = \frac{1}{n}\sum_{j=1}^{n}\left[\left(\mathcal{K}h\right)(t_j) - g^\delta(t_j)\right]^2 + \alpha\|h\|^2_{\mathcal{H}}, \tag{3}$$

where $\mathcal{H}$ is a real Hilbert space mentioned in (2), $\{t_j\}_{j=1}^n$ are the measured points with $0 = t_1 < t_2 < \cdots < t_n = 1$ and $g^\delta(t_j)$ are the noisy data of function g at points t_j for $j = 1, 2, \cdots, n$.

In [10, 12, 15, 16, 17, 18], the noisy data $\{g^\delta(t_j)\}_{j=1}^n$ were assumed to contain some random errors. However, in practical applications, the reduplicated measurements are fairly difficult and even are impossible. Hence, in this paper, we consider the deterministic errors. Assume that the noisy data are given by

$$g^\delta(t_j) = g(t_j) + \varepsilon_j,$$

and root mean square error in the noisy data is bounded by a noise level δ as

$$\sqrt{\frac{1}{n}\sum_{j=1}^{n}\varepsilon_j^2} \leq \delta.$$

In Section 3, we will show that the solution $f_{n,\alpha}^{\delta}$ to the problem (3) can be expressed by

$$f_{n,\alpha}^{\delta} = (\eta_{t_1}, \eta_{t_2}, \cdots, \eta_{t_n})(Q_n + n\alpha I)^{-1}(g^{\delta}(t_1), g^{\delta}(t_2), \cdots, g^{\delta}(t_n))',$$

where $\{\eta_{t_i}\}_{i=1}^{n}$ and Q_n will be given in latter sections. The error estimate and convergence rate between the approximate solution $f_{n,\alpha}^{\delta}$ and one exact solution $\mathcal{K}^{\dagger}g$ are proved in Section 4, where $\mathcal{K}^{\dagger}g$ is that element f with minimal $\mathcal{H}$ norm satisfying $\mathcal{K}f = g$. In Section 5, we test two examples and numerical results show that our proposed method is effective and stable.

2. Preliminaries

In this section, some definitions and assumptions are reviewed as preparations for the proof of error estimates.

2.1.Reproducing Kernel Hilbert Space and Reproducing Kernel. In [1, 13], authors discussed the properties of RKHS, which are restated below. A Hilbert space $\mathcal{H}_R$ of real-valued functions defined in a closed interval $[0, 1]$ is said to be a reproducing kernel Hilbert space (RKHS) if all the evaluation functionals $f \to f(s)$ for $f \in \mathcal{H}_R$ and $s \in [0, 1]$ are continuous. In this case there exists, by the Riesz representation theorem, a unique element in $\mathcal{H}_R$ (call it R_s) such that

$$\langle R_s, f\rangle_R = f(s), \quad f \in \mathcal{H}_R,$$

where $\langle\cdot,\cdot\rangle_R$ is the inner product in $\mathcal{H}_R$. The reproducing kernel (RK) is defined by

$$R(s, s') := \langle R_s, R_{s'}\rangle_R, \quad s, s' \in [0, 1].$$

Here, we take Sobolev space $H^m(0, 1)$ as an example of an RKHS, for an integer m, $m \geq 1$,

$$H^m(0,1) = \{f : f \in C^{m-1} \textit{ absolutely continuous}, f^{(m)} \in \mathcal{L}_2(0,1)\},$$

with the inner product

$$\langle\phi,\psi\rangle_{H^m(0,1)} = \langle\phi,\psi\rangle_{\mathcal{L}_2(0,1)} + \left\langle\phi^{(m)},\psi^{(m)}\right\rangle_{\mathcal{L}_2(0,1)}.$$

Then, the reproducing kernel is the Green's function for the Sturm-Liouville problem

$$(-1)^m u^{(2m)} + u = w, \quad in \quad (0, 1),$$
$$u^{(k)}(0) = u^{(k)}(1) = 0, \quad k = m, \cdots, 2m-1.$$

Assume that the linear functional $L_t : f \in \mathcal{H} \to (\mathcal{K}f)(t)$ is continuous for a fixed $t \in [0, 1]$, then by Riesz representation theorem, there exists a unique $\eta_t \in \mathcal{H}$ such that $L_t f = (\mathcal{K}f)(t) = \langle\eta_t, f\rangle$ for all $f \in \mathcal{H}$ in which $\langle\cdot,\cdot\rangle$ is the inner product in $\mathcal{H}$. In RKHS, the Riesz representation element of a continuous linear functional is generated by applying this linear operator onto the RK. Thus, if $\mathcal{H}$ is an RKHS, then we have $\eta_t(s) = \langle \eta_t, R_s \rangle = \mathcal{K}R_s = \int_0^1 K(t,u)R(s,u)du$, and if $\mathcal{H} = \mathcal{L}_2[0,1]$, then it is clear that $\eta_t(s) = K(t,s)$. Define $Q(t,s) = \langle \eta_t, \eta_s \rangle$ and note that $Q_t(s) = Q(t,s) = \mathcal{K}\eta_s$, then the function $Q(t,s)$ is given by

$$Q(t,s) = \begin{cases} \int_0^1 K(t,u)K(s,u)du, & if \quad \mathcal{H} = \mathcal{L}_2[0,1], \\ \int_0^1 \int_0^1 K(t,u)K(s,v)R(u,v)dudv, & if \quad \mathcal{H} = \mathcal{H}_R. \end{cases} \tag{4}$$

We suppose that $Q(t,s)$ is continuous on $[0,1]\times[0,1]$, then $Q(t,t)$ is well defined and finite for every fixed $t \in [0,1]$, and all the L_t are continuous. Moreover, if we require that $\mathcal{N}(\mathcal{K}^*)$, the null space of $\mathcal{K}^*$ (in $\mathcal{L}_2[0,1]$), is trivial, then the proposed RK is positive definite. Even though $\mathcal{N}(\mathcal{K}^*)$ is nontrivial, $\mathcal{H}$ should be replaced by the orthogonal complement of $\mathcal{N}(\mathcal{K}^*)$.

Let $\mathcal{H}_Q$ be the reproducing kernel Hilbert space with reproducing kernel $Q(t,s)$, $t,s \in [0,1]$ and inner product $\langle \cdot,\cdot \rangle_Q$. Here, by the isometric isomorphism between $\mathcal{H}$ and $\mathcal{H}_Q$, we give the definition of the inner product in RKHS $\mathcal{H}_Q$ as follows:

$$\langle \mathcal{K}f, \mathcal{K}g \rangle_Q = \langle f, g \rangle, for\ all\ f,h \in \mathcal{H}, \tag{5}$$

see [13] for more details.

2.2.Some assumptions. In this paper, the following assumptions should be used.

Assumption 1. *The errors in noisy data satisfy the following condition*

$$\frac{1}{n}\sum_{i=1}^{n} \varepsilon_i^2 \le \delta^2,$$

where the constant $\delta > 0$ is called the noise level.

Assumption 2. *The exact data $\{g(t_i)\}_{i=1}^n$ satisfy*

$$\frac{1}{n}\sum_{i=1}^{n} g^2(t_i) \le E^2,$$

where E is a constant.

Assumption 3. *Denote Q_n as an $n\times n$ matrix with ijth entry $\langle \eta_{t_i}, \eta_{t_j} \rangle = Q(t_i,t_j)$ which is given by (4) and λ_n is its smallest eigenvalue. Assume that λ_n satisfies one of the following conditions:*
(H_1) $\lambda_n \ge c(\frac{1}{n})^m$ for one $m \ge 1$, where $c > 0$ is a constant;
(H_2) $\lambda_n \ge c\beta^n$, where $c > 0$ and $0 < \beta < 1$ are both constants.

Remark. Note that constants m and β above depend on the choice of kernel function $K(t,s)$.

3. The regularized solution $f^\delta_{n,\alpha}$ to the problem (3)

In this paper, we assume that $\{\eta_{t_i}\}_{i=1}^n$ are linearly independent. Then $\{\eta_{t_i}\}_{i=1}^n$ span a subspace of $\mathcal{H}$, and we denote $V_n := span\{\eta_{t_1}, \eta_{t_2}, \cdots, \eta_{t_n}\}$. Thus, for every function $f \in \mathcal{H}$, f can be expressed as $f = \phi + f_0$ with $\phi \in V_n$ and $f_0 \in V_n^\perp$. In the paper [17], the author had stated that the solution $f^\delta_{n,\alpha}$ to the problem (3) is a function in space V_n given by

$$f^\delta_{n,\alpha} = (\eta_{t_1}, \eta_{t_2}, \cdots, \eta_{t_n})(Q_n + n\alpha I)^{-1}(g^\delta(t_1), g^\delta(t_2), \cdots, g^\delta(t_n))', \qquad (6)$$

where Q_n is an $n\times n$ matrix with ijth entry $\langle \eta_{t_i}, \eta_{t_j}\rangle = Q(t_i, t_j)$. In the following we give a proof in Theorem 1.

Define $g^\delta_{n,\alpha} \equiv \mathcal{K} f^\delta_{n,\alpha}$, then by the definitions of η_t and $Q(t,s)$, we have

$$g^\delta_{n,\alpha} = (Q_{t_1}, Q_{t_2}, \cdots, Q_{t_n})(Q_n + n\alpha I)^{-1}(g^\delta(t_1), g^\delta(t_2), \cdots, g^\delta(t_n))', \qquad (7)$$

where Q_t is given by $Q_t(s) = Q(t,s)$.

Theorem 1. *Under the assumptions in Section 2.1, the solution to problem (3) is given by (6).*

Proof. Let $f \in \mathcal{H}$ and write it as $f = \phi + \psi$ with $\phi \in V_n$ and $\psi \in V_n^\perp$. Then, for $i = 1, 2, \cdots, n$,

$$(\mathcal{K}f)(t_i) = \langle \eta_{t_i}, f\rangle_{\mathcal{H}} = \langle \eta_{t_i}, \phi\rangle_{\mathcal{H}} + \langle \eta_{t_i}, \psi\rangle_{\mathcal{H}} = \langle \eta_{t_i}, \phi\rangle_{\mathcal{H}} = (\mathcal{K}\phi)(t_i). \qquad (8)$$

By Pythagoras' theorem, $\|f\|^2_{\mathcal{H}} = \|\phi+\psi\|^2_{\mathcal{H}} = \|\phi\|^2_{\mathcal{H}} + \|\psi\|^2_{\mathcal{H}}$. Combining this with (8), we obtain that

$$\Phi(f) = \frac{1}{n}\sum_{j=1}^n \left[(\mathcal{K}\phi)(t_j) - g^\delta(t_j)\right]^2 + \alpha\|\phi\|^2_{\mathcal{H}} + \alpha\|\psi\|^2_{\mathcal{H}}, \qquad (9)$$

and minimizing this over $\psi \in V_n^\perp$ gives that $\psi = 0$.

Thus, the minimizer of Φ can be written as $f^\delta_{n,\alpha} = \sum_{j=1}^n c_j \eta_{t_j}$, and substituting this into functional Φ gives

$$\Phi(f^\delta_{n,\alpha}) = \frac{1}{n}\sum_{i=1}^n \left[\sum_{j=1}^n c_j \langle \eta_{t_i}, \eta_{t_j}\rangle - g^\delta(t_i)\right]^2 + \alpha \sum_{p=1}^n\sum_{q=1}^n c_p c_q \langle \eta_{t_p}, \eta_{t_q}\rangle. \quad (10)$$

Differentiating (10) with respect to c_j, $j = 1, 2, \cdots, n$ and setting them all equal to zero, we have

$$\frac{2}{n}\sum_{i=1}^n \left[\sum_{q=1}^n c_q \langle \eta_{t_i}, \eta_{t_q}\rangle - g^\delta(t_i)\right]\langle \eta_{t_i}, \eta_{t_j}\rangle + 2\alpha\sum_{i=1}^n c_i \langle \eta_{t_i}, \eta_{t_j}\rangle = 0,$$

for $j = 1, 2, \cdots, n$. In matrix-vector notation, this may be written as

$$(c_1, c_2, \cdots, c_n)(Q_n' + n\alpha I)Q_n = (g^\delta(t_1), g^\delta(t_2), \cdots, g^\delta(t_n))Q_n.$$

Since $\{\eta_{t_i}\}_{i=1}^n$ are linearly independent, Q_n is nonsingular and positive definite. Furthermore, $Q_n' = Q_n$. Thus, Q_n and $Q_n + n\alpha I$ for $\alpha > 0$ are invertible. Then

$$(c_1, c_2, \cdots, c_n) = (g^\delta(t_1), g^\delta(t_2), \cdots, g^\delta(t_n))(Q_n + n\alpha I)^{-1}.$$

The proof of Theorem 1 is completed. □

4. Error estimates and convergence rates

For the convergence rates of the regularization solution, in [4, 5, 10, 17], the authors had already obtained some results where they assumed that the errors $\{\varepsilon(t_i)\}_{i=1}^n$ were uncorrelated random variables with zero mean ("white noise") and gave some convergence rates on the mathematical expectation of the computed error. In this section, we will discuss the convergence rates by using the deterministic noisy data under suitable choice for the regularization parameter and the number of measured points.

For the simplicity of proof, hereafter we suppose that the $\{t_i\}_{i=1}^n$ are equally distributed points, $t_i = (i-1)/(n-1)$, $i = 1, 2, \cdots, n$.

From paper [13], we know $\mathcal{K}(\mathcal{H}) = \mathcal{H}_Q$, and for $g \in \mathcal{H}_Q$, $\|\mathcal{K}^\dagger g - f_{n,\alpha}^\delta\|_{\mathcal{H}}^2 = \|\mathcal{K}(\mathcal{K}^\dagger g) - \mathcal{K} f_{n,\alpha}^\delta\|_Q^2 = \|g - g_{n,\alpha}^\delta\|_Q^2$, where $\|\cdot\|_Q$ is the norm in $\mathcal{H}_Q$, see [13] for details.

Define an orthogonal projection operator $P_n : \mathcal{H}_Q \to \mathcal{H}_n$, where $\mathcal{H}_n := span\{Q_{t_1}, Q_{t_2}, \cdots, Q_{t_n}\}$. Obviously, $\mathcal{H}_n \subset \mathcal{H}_Q$. By $g_{n,\alpha}^\delta, P_n g \in \mathcal{H}_n$, $P_n g - g_{n,\alpha}^\delta \in \mathcal{H}_n$, and $g - P_n g \in \mathcal{H}_n^\perp$ (the orthogonal complement of subspace $\mathcal{H}_n$), we have

$$\|g - g_{n,\alpha}^\delta\|_Q^2 = \|g - P_n g\|_Q^2 + \|P_n g - g_{n,\alpha}^\delta\|_Q^2.$$

In order to obtain the convergence rates of the regularization solution, we first list the following theorem about $\|g - P_n g\|_Q$ in [16].

Theorem 2. *Let g have a representation*

$$g(t) = \int_0^1 Q(t,s)\rho(s)ds$$

for some $\rho \in \mathcal{L}_2[0,1]$ and suppose that $Q(t,s)$ satisfies:

(i) $(\partial^l/\partial t^l)Q(t,s)$ exists and is continuous on $[0,1]\times[0,1]$ for $t \neq s$, $l = 0, 1, 2, \cdots, 2q$, $(\partial^l/\partial t^l)Q(t,s)$ exists and is continuous on $[0,1]\times[0,1]$ for $l = 0, 1, 2, \cdots, 2q-2$;

(ii) $\lim_{t\uparrow s}(\partial^{2q-1}/\partial t^{2q-1})Q(t,s)$ and $\lim_{t\downarrow s}(\partial^{2q-1}/\partial t^{2q-1})Q(t,s)$ exist and are bounded for all $s \in [0,1]$.

Then $g \in \mathcal{H}_Q$ and

$$\|g - P_n g\|_Q \le (6q)^q (C_1(t_n - t_1) + C_2)^{1/2} \left[\int_0^1 \rho^2(t)dt\right]^{1/2} (n-1)^{-q},$$

where

$$C_1 = (1 + 2q\Theta_q) \sup_{t \neq s, t, s \in [0,1]} \left| \frac{1}{(2q)!} \frac{\partial^{2q}}{\partial t^{2q}} Q(t,s) \right|,$$
$$C_2 = 2(1 + 2q\Theta_q) \sup_{t,s \in [0,1]} \left| \frac{1}{(2q-1)!} \frac{\partial^{2q-1}}{\partial t^{2q-1}} Q(t,s) \right|,$$
$$\Theta_q = [3(2q-1)]^{2q-1},$$

and it is understood that if $\frac{\partial^{2q-1}}{\partial t^{2q-1}} Q(t,s)$ *is undefined the maximum of the left and right absolute derivative is taken.*

In the following, we will give an error bound for $\|P_n g - g^{\delta}_{n,\alpha}\|^2_Q$.

By the definition of orthogonal projection operator $P_n : \mathcal{H}_Q \to \mathcal{H}_n$, it is easy to obtain

$$P_n g = (Q_{t_1}, Q_{t_2}, \cdots, Q_{t_n}) Q_n^{-1} (g(t_1), g(t_2), \cdots, g(t_n))'. \tag{11}$$

Using (7) and (11), we further have

$$P_n g - g^{\delta}_{n,\alpha} = (Q_{t_1}, Q_{t_2}, \cdots, Q_{t_n}) \left[n\alpha (Q_n + n\alpha I)^{-1} Q_n^{-1} \mathbf{g} - (Q_n + n\alpha I)^{-1} \boldsymbol{\varepsilon} \right], \tag{12}$$

where $\mathbf{g}$ and $\boldsymbol{\varepsilon}$ are defined as $\mathbf{g} = (g(t_1), g(t_2), \cdots, g(t_n))'$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)'$, respectively. By (12), we have

$$\begin{aligned} \|P_n g - g^{\delta}_{n,\alpha}\|^2_Q &= \|(Q_{t_1}, Q_{t_2}, \cdots, Q_{t_n}) \left[n\alpha (Q_n + n\alpha I)^{-1} Q_n^{-1} \mathbf{g} - (Q_n + n\alpha I)^{-1} \boldsymbol{\varepsilon} \right] \|^2_Q \\ &= (n\alpha)^2 \mathbf{g}' Q_n^{-1} (Q_n + n\alpha I)^{-1} Q_n (Q_n + n\alpha I)^{-1} Q_n^{-1} \mathbf{g} \\ &\quad - 2n\alpha \mathbf{g}' Q_n^{-1} (Q_n + n\alpha I)^{-1} Q_n (Q_n + n\alpha I)^{-1} \boldsymbol{\varepsilon} \\ &\quad + \boldsymbol{\varepsilon}' (Q_n + n\alpha I)^{-1} Q_n (Q_n + n\alpha I)^{-1} \boldsymbol{\varepsilon}. \end{aligned}$$

It is known that the positive definite matrix Q_n has the decomposition $Q_n = \Gamma D \Gamma'$, where Γ is an orthogonal matrix and D is diagonal with $\nu\nu$th entry λ_ν, thus we further obtain

$$\begin{aligned} \|P_n g - g^{\delta}_{n,\alpha}\|^2_Q &= (n\alpha)^2 \mathbf{g}' \Gamma D^{-1} (D + n\alpha I)^{-1} D (D + n\alpha I)^{-1} D^{-1} \Gamma' \mathbf{g} \\ &\quad - 2n\alpha \mathbf{g}' \Gamma D^{-1} (D + n\alpha I)^{-1} D (D + n\alpha I)^{-1} \Gamma' \boldsymbol{\varepsilon} \\ &\quad + \boldsymbol{\varepsilon}' \Gamma (D + n\alpha I)^{-1} D (D + n\alpha I)^{-1} \Gamma' \boldsymbol{\varepsilon} \\ &= (n\alpha)^2 \mathbf{g}' \Gamma (D + n\alpha I)^{-1} D^{-1} (D + n\alpha I)^{-1} \Gamma' \mathbf{g} \\ &\quad - 2n\alpha \mathbf{g}' \Gamma (D + n\alpha I)^{-1} (D + n\alpha I)^{-1} \Gamma' \boldsymbol{\varepsilon} \\ &\quad + \boldsymbol{\varepsilon}' \Gamma (D + n\alpha I)^{-1} D (D + n\alpha I)^{-1} \Gamma' \boldsymbol{\varepsilon}. \end{aligned}$$

Furthermore, by Assumptions 1–2, note that Γ is orthogonal matrix, we get the following estimate:

$$\begin{aligned}\|P_n g - g^\delta_{n,\alpha}\|^2_Q &\le (n\alpha)^2\|\mathbf{g}'\|_2\|(D+n\alpha I)^{-2}D^{-1}\|_2\|\mathbf{g}\|_2\\ &\quad + 2n\alpha\|\mathbf{g}'\|_2\|(D+n\alpha I)^{-1}\|^2_2\|\boldsymbol{\varepsilon}\|_2\\ &\quad + \|\boldsymbol{\varepsilon}'\|_2\|(D+n\alpha I)^{-2}D\|_2\|\boldsymbol{\varepsilon}\|_2\\ &\le \frac{n^3\alpha^2E^2}{\lambda_n(\lambda_n+n\alpha)^2} + \frac{2n^2\alpha E\delta}{(\lambda_n+n\alpha)^2} + \frac{\delta^2}{4\alpha}\\ &\le \frac{n^2\alpha E^2}{4\lambda_n^2} + \frac{nE\delta}{2\lambda_n} + \frac{\delta^2}{4\alpha},\end{aligned} \tag{13}$$

where $\|\cdot\|_2$ is the 2-norm of a vector or a matrix and λ_n is the smallest eigenvalue of matrix Q_n, and the last term $\frac{\delta^2}{4\alpha}$ follows from

$$\frac{\lambda_i}{(\lambda_i+n\alpha)^2} = \frac{1}{(\sqrt{\lambda_i}+n\alpha/\sqrt{\lambda_i})^2} \le \frac{1}{4n\alpha},$$

for all $i = 1, 2, \cdots, n$.

Next, we will discuss convergence rates of the regularized solution and the generalized inverse $\mathcal{K}^\dagger g$ through the assumptions (H_1) and (H_2) in Assumption 3, respectively.

In the case of (H_1), namely, $\lambda_n \ge c(\frac{1}{n})^m$, we have the following

$$\textit{the right hand side of } (13) \le \frac{\alpha n^{2m+2}E^2}{4c^2} + \frac{n^{m+1}E\delta}{2c} + \frac{\delta^2}{4\alpha}. \tag{14}$$

Then, we can choose a regularization parameter $\alpha = k_1\delta^{\frac{3}{2}}$ by an *a priori* rule and choose the total number of the measured points n such that $\frac{1}{n} \approx \frac{1}{k_2}\delta^{\frac{1}{2m+2}}$. Thus, substituting $n = [k_2\delta^{-\frac{1}{2m+2}}]$ and $\alpha = k_1\delta^{\frac{3}{2}}$ into (14) yields

$$\|P_n g - g^\delta_{n,\alpha}\|^2_Q \le \frac{k_1k_2^{2m+2}E^2\delta^{\frac{1}{2}}}{4c^2} + \frac{k_2^{m+1}E\delta^{\frac{1}{2}}}{2c} + \frac{\delta^{\frac{1}{2}}}{4k_1} = O(\delta^{\frac{1}{2}}),\ as \quad \delta\to 0.$$

If the smallest eigenvalue of matrix Q_n satisfies condition (H_2) in Assumption 3, namely, we have $\lambda_n \ge c\beta^n$ for $0<\beta<1$, then we can obtain the following estimate

$$\textit{the right hand side of } (13) \le \frac{n^2\alpha E^2}{4c^2\beta^{2n}} + \frac{nE\delta}{2c\beta^n} + \frac{\delta^2}{4\alpha}. \tag{15}$$

For $\delta < 1$, take $\beta^n \approx \sqrt{\delta}$, *i.e.*, $n = [\frac{\ln\delta}{2\ln\beta}]$, and choose a regularization parameter $\alpha = k_1\delta^{\frac{3}{2}}|\ln\delta|^{-1}$, then (15) becomes

$$\|P_n g - g^\delta_{n,\alpha}\|^2_Q \le \frac{k_1E^2\delta^{\frac{1}{2}}|\ln\delta|}{16c^2|\ln\beta|^2} + \frac{E\delta^{\frac{1}{2}}|\ln\delta|}{4c|\ln\beta|} + \frac{\delta^{\frac{1}{2}}|\ln\delta|}{4k_1} = O(\delta^{\frac{1}{2}}|\ln\delta|),\ as \quad \delta\to 0.$$

Note that, $\delta^{\frac{1}{2}}|\ln\delta| \to 0$, as $\delta \to 0$, thus we have the following convergence estimates.

Theorem 3. *Suppose that Assumptions 1–2 and the conditions in Theorem 2 hold, then we have the following convergence rates:*

(i) If condition (H_1) *in assumption 3 is satisfied, then choose* $\alpha = k_1\delta^{\frac{3}{2}}$ *and* $n = [k_2\delta^{-\frac{1}{2m+2}}]$*, we obtain the Hölder-type estimate*

$$\|\mathcal{K}^\dagger g - f_{n,\alpha}^\delta\|_{\mathcal{H}}^2 = O(\delta^{\frac{q}{m+1}}) + O(\delta^{\frac{1}{2}}), \quad as \quad \delta \to 0;$$

(ii) If condition (H_2) *in assumption 3 is satisfied, choose* $\alpha = k_1\delta^{3/2}|\ln\delta|^{-1}$ *and* $n = [\frac{\ln\delta}{2\ln\beta}]$*, then we obtain the log-type estimate*

$$\|\mathcal{K}^\dagger g - f_{n,\alpha}^\delta\|_{\mathcal{H}}^2 = O((\frac{2\ln\beta}{\ln\delta})^{2q}) + O(\delta^{\frac{1}{2}}|\ln\delta|), \quad as \quad \delta \to 0.$$

5. Numerical verification

In this section, we test two numerical examples. To verify the computational accuracy of numerical solutions, we calculate the root mean square error (RMSE) by

$$E(f_{n,\alpha}^\delta) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(f_{n,\alpha}^\delta(\bar{t}_i) - \mathcal{K}^\dagger g(\bar{t}_i))^2}.$$

where $\{\bar{t}_i\}$ are test points and N is the total number of uniformly distributed points on $[0,1]$. In our computations, we always take $N = 201$.

Example 1. *Let the Fredholm integral equation of first kind be*

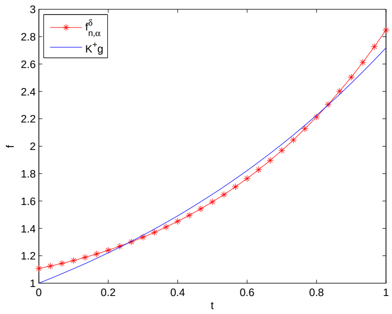
$$(\mathcal{K}f)(t) = \int_0^1 \exp(ts)f(s)ds = \frac{\exp(t+1)-1}{t+1}, \quad t \in [0,1],$$

where the exact solution is $f(s) = \exp(s)$.

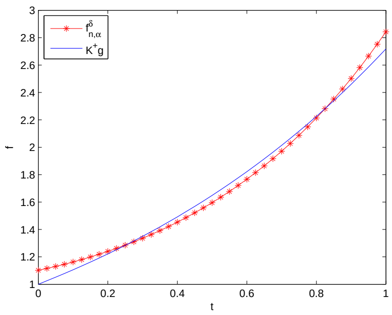
In this example, kernel function is $K(t,s) = \exp(ts)$, $\mathcal{K} : \mathcal{L}_2[0,1] \to \mathcal{H}_Q$, $(\mathcal{K}f)(t) = g(t)$, where $g(t) = \frac{\exp(t+1)-1}{t+1}$. In our computations, the noisy data $g^\delta(t_i)$ are generated by $g^\delta(t_i) = g(t_i) + \delta\cos(4t_i\pi)$ where $t_i = \frac{i-1}{n-1}$ for $i = 1, 2, \cdots, n$. The subspace of $\mathcal{L}_2[0,1]$ is $V_n := span\{1, \exp(\frac{s}{n-1}), \exp(\frac{2s}{n-1}), \cdots, \exp(s)\}$ and the reproducing kernel of $\mathcal{H}_Q$ is $Q(t,s) = \frac{1}{t+s}(\exp(t+s)-1)$ for $t+s \neq 0$ and $Q(t,s) = 1$ for $t = s = 0$. Note that $Q(t,s)$ is infinitely differentiable with respect to t, convergence rate for $\|g - P_n g\|_Q$ is much higher than the one for $\|g_{n,\alpha}^\delta - P_n g\|_Q$ in Theorem 3. In this example, the ijth entry of matrix Q_n is given by follows:

$$Q(t_i, t_j) = \begin{cases} 1, & if \quad i = j = 1, \\ \frac{n-1}{i+j-2}(\exp(\frac{i+j-2}{n-1}) - 1), & otherwise. \end{cases}$$

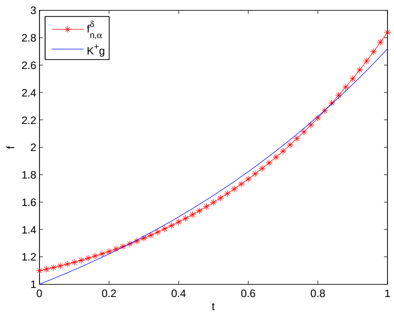
In numerical experiments, we use the *a priori* choice rule for the regularization parameter $\alpha = k_1\delta^{\frac{3}{2}}$, where k_1 is fixed at 0.05. Because the condition H_1 or H_2 can not be verified for this example, the best number n can not be given by our proposed method. In our test, we compute some numerical results by using various numbers $n = 31, 41, 51$. The computed results are shown in Figs. $1-3$.



$(a)\quad \delta = 0.001, \quad n = 31;$



$(b)\quad \delta = 0.001, \quad n = 41;$



$(c)\quad \delta = 0.001, \quad n = 51.$

Fig. 1: Solid lines represent the exact solution data and stars are computed approximate data.

Fig. 1 shows numerical results for $n = 31, 41, 51$ respectively when the noise level $\delta = 0.001$. The RMSEs for $n = 31, 41, 51$ are 0.0529, 0.0506, 0.0492, respectively.

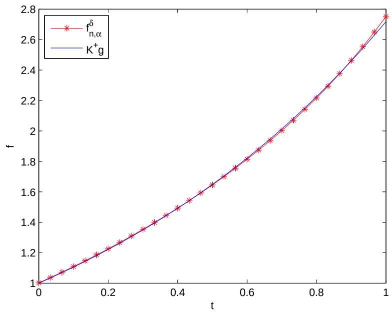
Fig. 2 displays numerical results for $n = 31, 41, 51$, respectively when the noise level $\delta = 0.0001$. The RMSEs when $n = 31, 41, 51$ are all 0.0084. It is observed that the results for $\delta = 0.0001$ are much better than $\delta = 0.001$.

Fig. 3 shows the plots of RMSEs with respect to n for $\delta = 0.001$ and $\delta = 0.0001$ respectively. It can be observed that there is no large change for RMSEs while $20 < n < 200$.

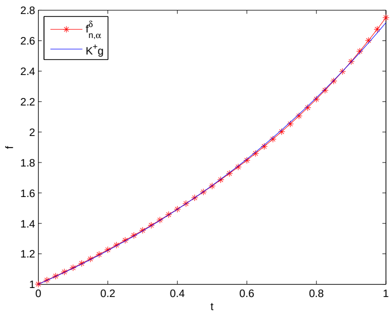
Example 2. *The second example is chosen as following:*

$$\int_0^1 \exp(-ts) f(s) ds = \frac{3 - 3\exp(-t)\cos(3) - \exp(-t)t\sin(3)}{t^2 + 9}, \quad t \in [0, 1],$$

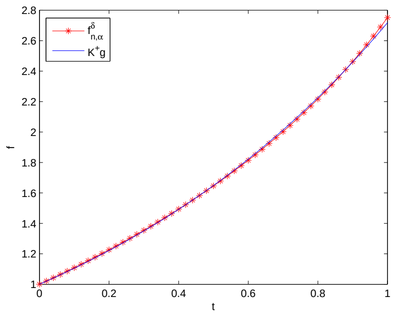
where the exact solution is $f(s) = \sin(3s)$.



$(a) \quad \delta = 0.0001, \quad n = 31;$



$(b) \quad \delta = 0.0001, \quad n = 41;$



$(c) \quad \delta = 0.0001, \quad n = 51.$

Fig. 2: Solid lines represent the exact solution data and stars are computed approximate data.

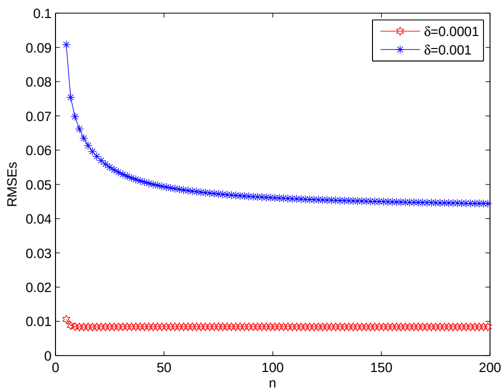


Fig. 3: RMSEs with respect to n.

In this example, $K(t,s) = \exp(-ts)$, $g(t) = \frac{3-3\exp(-t)\cos(3)-\exp(-t)t\sin(3)}{t^2+9}$ and reproducing kernel $Q(t,s) = \frac{1}{t+s}(1-\exp(-t-s))$ for $t+s \neq 0$ and $Q(t,s) = 1$ for $t = s = 0$. In our computations, we also give the noisy data $g^\delta(t_i)$ by $g^\delta(t_i) = g(t_i) + \delta \sin(4(0.5-t_i)\pi)$ where $t_i = \frac{i-1}{n-1}$ for $i = 1, 2, \cdots, n$. In this example, $V_n := span\{1, \exp(-\frac{s}{n-1}), \exp(-\frac{2s}{n-1}), \cdots, \exp(-s)\}$ and the ijth entry of Q_n is given below:

$$Q(t_i, t_j) = \begin{cases} 1, & if \quad i = j = 1, \\ \frac{n-1}{i+j-2}(1-\exp(-\frac{i+j-2}{n-1})), & otherwise. \end{cases}$$

In numerical experiments, we also use the *a priori* choice rule for the regularization parameter α as the same as the Example 1, but k_1 is fixed at 0.02. Because the condition H_1 or H_2 can not be verified for this example, either, the best number n can not be given by our proposed method. In our test, we compute some numerical results by using various numbers $n = 31, 41, 51$. The computed results are illustrated in Figs. $4-6$.

Fig. 4 illustrates the results about $n = 31, 41, 51$ respectively when the noise level $\delta = 0.001$. And the RMSEs when $n = 31, 41, 51$ are 0.0432, 0.0428, 0.0425, respectively.

Fig. 5 shows the results about $n = 31, 41, 51$ respectively when the noise level $\delta = 0.0001$. And the RMSEs when $n = 31, 41, 51$ are 0.0129, 0.0143, 0.0154, respectively. They also show that the results when $\delta = 0.0001$ are much better than $\delta = 0.001$.

Fig. 6 illustrates the plots of RMSEs with respect to n for $\delta = 0.001$ and $\delta = 0.0001$ respectively. It can also be observed that there is no large change for RMSEs while n becomes larger and larger, but there is a minimum at about $n = 20$ when $\delta = 0.0001$.

6. Conclusions

In this paper, based on reproducing kernel Hilbert space, we solve the Fredholm integral equation of the first kind by a modified Tikhonov regularization method when the errors are deterministic. Under the suitable choices of the regularization parameter and the number of measured points, we obtain two error estimates and convergence rates for the regularized solution. Numerical results for two examples show the effectiveness and stability of the proposed method.

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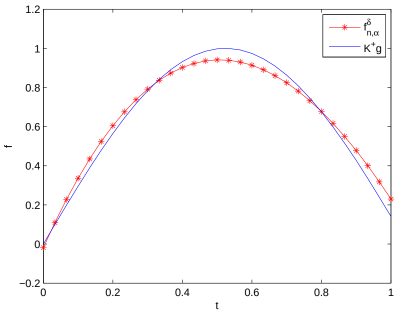
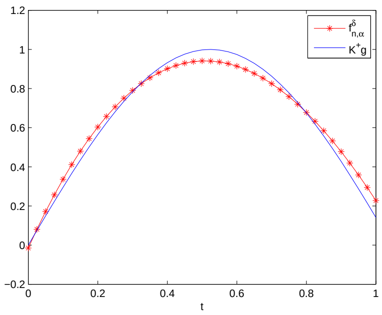
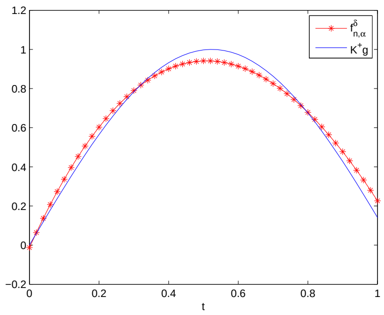


$(a) \quad \delta = 0.001, \quad n = 31;$



$(b) \quad \delta = 0.001, \quad n = 41;$



$(c) \quad \delta = 0.001, \quad n = 51.$

Fig. 4: Solid lines represent the exact solution data and stars are computed approximate data.

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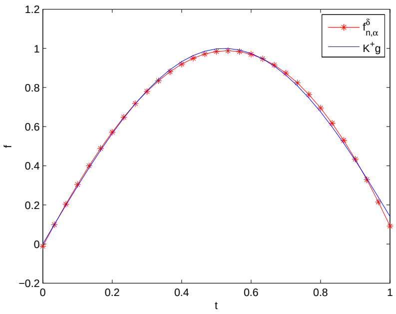
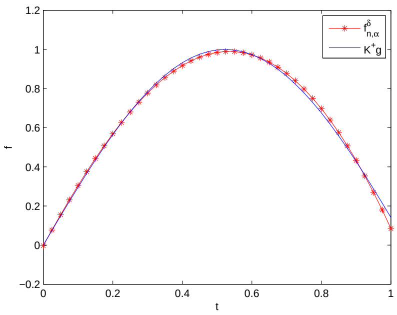
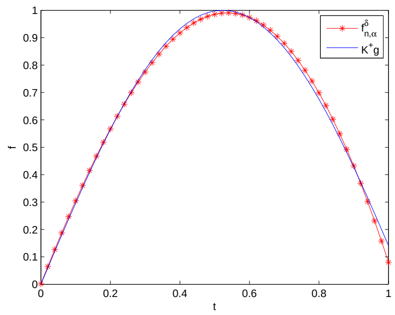


$(a)\quad \delta = 0.0001, \quad n = 31;$



$(b)\quad \delta = 0.0001, \quad n = 41;$



$(c)\quad \delta = 0.0001, \quad n = 51.$

Fig. 5: Solid lines represent the exact solution data and stars are computed approximate data.

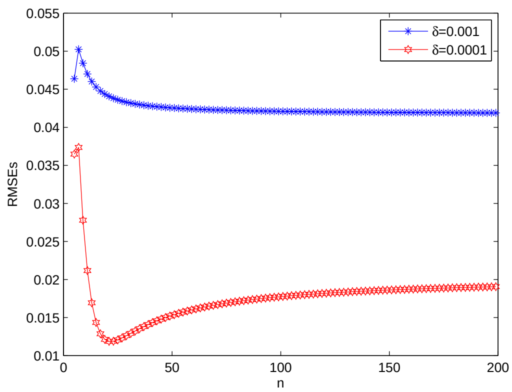


Fig. 6: RMSEs with respect to n.

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