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SOLUTION OF A NONLINEAR EQUATION WITH RIEMANN-LIOUVILLES FRACTIONAL DERIVATIVES BY HOMOTOPY PERTURBATION METHOD†

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Abstract. The aim of the paper is to apply Homotopy Perturbation Method (HPM) for the solution of a nonlinear fractional differential equation. Finally, the solution obtained by the Homotopy perturbation method has been numerically evaluated and presented in the form of tables and then compared with those obtained by truncated series method. A good agreement of the results is observed.

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1. Introduction

The fractional derivative has been occuring in many physical problems such as frequency dependent damping behavior of materials, motion of a large thin plate in a Newtonian fluid, creep and relaxation functions for viscoelastic materials, the $PI^{\lambda}D^{\mu}$ controller for the control of dynamical systems etc. [1-4]. Phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry and material science are also described by differential equations of fractional order [5-11]. The solution of the differential equation containing fractional derivative is much involved. Most recently, applications have included classes of nonlinear fractional differential equations (FDEs) [12] and their numerical solutions have been established by Diethelm and Ford [13]. Also, the solution of nonlinear fractional differential equation has been obtained through Adomian's decomposition method [12] and Variational iteration method [14].

The homotopy perturbation method (HPM) was first proposed by the Chinese mathematician Ji-Huan He [15-16]. Unlike classical techniques, the homotopy perturbation method leads to an analytical approximate and exact solutions of

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the nonlinear equations easily and elegantly without transforming the equation. or linearizing the problem and with high accuracy, minimal calculation and avoidance of physically unrealistic assumptions. As a numerical tool, the method provide us with numerical solution without discretization of the given equation, and therefore, it is not effected by computation round-off errors and one is not faced with necessity of large computer memory and time. In this paper, we will HPM for solving a nonlinear equation with Riemann-Liouville's fractional derivatives.

2. Mathematical definition

The mathematical definition of fractional calculus has been the main subject of many different approaches [17]. The left-sided Riemann-Liouville fractional integral of order $q > 0$ of a function $f(x)$ is defined as

$$\frac{d^{-q} f(x)}{dx^{-q}} = \frac{1}{\Gamma(q)} \int_0^x \frac{f(t)\, dt}{(x-t)^{1-q}}, \quad x > 0,$$

and the Riemann-Liouville's fractional derivative is defined as

$$\frac{d^q f(x)}{dx^q} = \frac{d^n}{dx^n} \left(\frac{d^{-(n-q) f(x)}}{dx^{-(n-q)}} \right) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dx^n} \int_0^x \frac{f(t)\, dt}{(x-t)^{1-n+q}},$$

where n is an integer that satisfies $n - 1 < q < n$.

3. Solution of a nonlinear fractional differential equation

As an illustration of the present analysis, let us consider the HPM for solving the following nonlinear fractional differential equation:

$$\frac{du}{dt} + \frac{d^{1/2}u}{dt^{1/2}} - 2u^2 = 0, \tag{1}$$

According to HPM, we construct the following homotopy

$$\frac{du}{dt} + p \left\{ \frac{d^{1/2}u}{dt^{1/2}} - 2u^2 \right\} = 0, \tag{2}$$

Assume the solution of Eq. (2) to be in the form:

$$u = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + ..., \tag{3}$$

Substituting (3) into (2) and equating the coefficients of like powers p, we get the following set of differential equations

$$p^0 : \quad \frac{du_0}{dt} = 0,$$

$$p^1 : \quad \frac{du_1}{dt} + \frac{d^{1/2}u_0}{dt^{1/2}} - 2u_0^2 = 0,$$

$$p^2: \quad \frac{du_2}{dt} + \frac{d^{1/2}u_1}{dt^{1/2}} - 4u_0u_1 = 0,$$

$$\vdots$$

Consequently, the first few terms of the HPM series solution are as follows:

$$u_0(t) = c,$$

$$u_1(t) = \frac{2c\hbar\sqrt{t}}{\sqrt{\pi}} - 2c^2\hbar t,$$

$$u_2(t) = \frac{2c\hbar\sqrt{t}}{\sqrt{\pi}} - 2c^2\hbar t + c\hbar^2\left(2\frac{\sqrt{t}}{\sqrt{\pi}} - 2ct + t - \frac{8ct^{3/2}}{\sqrt{\pi}} + 4c^2t^2\right),$$

$$\vdots$$

and so on. Hence, the HPM series solution is

$$u(t) = c + \left(2c^2t - 2c\frac{\sqrt{t}}{\sqrt{\pi}}\right) + \left(4c^3t^2 - \frac{8c^2t^{3/2}}{\sqrt{\pi}} + ct\right) + ... \tag{4}$$

Now, consider the initial value problem for the nonlinear fractional differential equation

$$\frac{du}{dt} + \frac{d^{1/2}u}{dt^{1/2}} - 2u^2 = 0, \quad u(0) = 1. \tag{5}$$

Therefore, the solution of the Eq.(4) is

$$u(t) = 1 + \left(2t - 2\frac{\sqrt{t}}{\sqrt{\pi}}\right) + \left(4t^2 - \frac{8t^{3/2}}{\sqrt{\pi}} + t\right) + ... \tag{6}$$

4. Verification of the solution

We can look for the solution $u(t)$ of the Eq.(5) in the form of the fractional power series:

$$u(t) = \sum_{i=0}^{\infty} u_i t^{1/2}, \tag{7}$$

with $u_0 = c$, where c is a constant. By substituting (7) into (5) and comparing the coefficients of the results fractional power series, we obtain [12]:

$$u(t) = c - \frac{2c\sqrt{t}}{\sqrt{\pi}} + \left(c + 2c^2\right)t - \left(\frac{8c^2}{\sqrt{\pi}} + \frac{4c}{3\sqrt{\pi}}\right)t^{3/2} + \left[\frac{c}{2} + \left(5 + \frac{4}{\pi}\right)c^2 + 4c^3\right]t^2 - \left[\frac{8c}{15\sqrt{\pi}} + \left(\frac{32}{3\sqrt{\pi}} + \frac{64}{15\pi\sqrt{\pi}}\right)c^2 + \frac{352c^3}{15\sqrt{\pi}}\right]t^{5/2} + ... \quad (8)$$

5. Numerical examples and discussions

Assuming $c = 1$ and retaining upto 5th power of $t$in Eq.(8), the truncated fractional power series becomes

$$u(t) = 1 - \frac{2\sqrt{t}}{\sqrt{\pi}} + 3t - \left(\frac{28}{3\sqrt{\pi}}\right)t^{3/2} + \left[\frac{19}{2} + \frac{4}{\pi}\right]t^2 - \left[\frac{104}{3\sqrt{\pi}} + \frac{64}{15\pi\sqrt{\pi}}\right]t^{5/2} + ... \quad (9)$$

We compare the solution (4) by HPM with that of Eq.(8) by fractional power series method and the results are given below in Tables 1-3.

Table 1

Comparison between HPM-2nd order approximation and fractional power series solution (truncated upto 13-terms)

t(time)	HPM-2nd	Power series solution(13 terms)	Absolute error
0	1	1	0
0.01	0.913049	0.912804	0.000245
0.02	0.889257	0.888937	0.000320
0.03	0.874706	0.874525	0.000181
0.04	0.864616	0.864758	0.000142
0.05	0.857224	0.857839	0.000615
0.06	0.851670	0.852877	0.001207
0.07	0.847467	0.849361	0.001894
0.08	0.844317	0.846971	0.002654
0.09	0.842021	0.845495	0.003474
0.1	0.840445	0.844794	0.004349

Table 2

Comparison between HPM-3rd order approximation and fractional power series solution (truncated upto 13-terms)

t(time)	HPM-3rd	Power series solution(13 terms)	Absolute error
0	1	1	0
0.01	0.912799	0.913142	0.000005
0.02	0.888954	0.889583	0.000017
0.03	0.874595	0.874525	0.000070
0.04	0.864911	0.864758	0.000153
0.05	0.858096	0.857839	0.000257
0.06	0.853251	0.852839	0.000374
0.07	0.849854	0.849361	0.000493
0.08	0.847574	0.846971	0.000603
0.09	0.846183	0.845495	0.000688
0.1	0.845522	0.844794	0.000728

Table 3
Comparison between HPM-4th order approximation and fractional power series solution (truncated upto 13-terms)

t(time)	HPM-4th	Power series solution(13 terms)	Absolute error
0	1	1	0
0.01	0.912803	0.913142	0.000001
0.02	0.888934	0.889583	0.000003
0.03	0.874517	0.874525	0.000008
0.04	0.864746	0.864758	0.000012
0.05	0.857827	0.857839	0.000012
0.06	0.852871	0.852839	0.000006
0.07	0.849369	0.849361	0.000008
0.08	0.846998	0.846971	0.000027
0.09	0.845539	0.845495	0.000044
0.1	0.844840	0.844794	0.000046

From the above three tables we observe that approximate solution is in good agreement with truncated series solution. Of course the accuracy can be improved by computing more terms in the HPM.

6. Conclusion

The HPM is straightforward, without restrictive assumptions and the components of the series solution can be easily computed using any mathematical symbolic package. Moreover, this method does not change the problem into a convenient one for the use of linear theory. It, therefore, provides more realistic series solutions that generally converge very rapidly in real physical problems. When solutions are computed numerically, the rapid convergence is obvious. Moreover, no linearization or perturbation is required.

References

1. I. Podlubny, Fractional Differential Equations, Academic Press, 1999.
2. R.L. Bagley, P.J. Torvik, On the appearance of the fractional derivative in the behavior of real materials, J. Appl. Mech. 51 (1984) 294–298.
3. M. Caputo, Elasticita‘ e Dissipazione, Zanichelli, Bologna, 1969.
4. L.E. Suarez, A. Shokooh, An eigenvector expansion method for the solution of motion containing fractional derivatives, ASME J. Appl. Mech. 64 (1997) 629–635.
5. M.W. Michalski, Derivatives of non integer order and their applications, Dissertationes Mathematical (Habilitationsschrift), Polska Akademia Nauk, Instytut Matematyczuy Warszawa,1993.
6. W.G. Glockle, T.F. Nonnenmacher, Fractional integral operators and Foxfunctions in the theory of viscoelasticity, Macromolecules 24 (1991) 6424–6434.
7. Y. Babenko, Non integer differential equation, Conference Bordeaux 3–8 July (1994a).
8. Y. Babenko, Non integer differential equation in Engineering: Chemical Engineering, Conference Bordeaux 3–8 July (1994b).
9. F. Mainardi, Fractional relaxation and fractional diffusion equations: mathematical aspects in: Proceedings of the 14th IMACS World Congress, vol. 1, 1994. pp. 329–332.
10. L. Gaul, P. Klein, S. Kempfle, Damping description involving fractional operators, Mech.Syst. Signal Process. 5 (1991) 81–88.
11. M. Ochmann, S. Makarov, Representation of the absorption of nonlinear waves by fractional derivatives, J. Acoust. Soc. Am. 94 (6) (1993).
12. K. Diethelm, A.D. Ford, On the solution of nonlinear fractional-order differential Equations used in the modeling of viscoplasticity, in: F. Keil, W. Mackens, H. Voß, J. Werther (Eds.), Scientific Computing in Chemical Engineering II. Computational Fluid Dynamics, Reaction Engineering, and Molecular Properties, Springer-Verlag, Heidelberg, 1999, pp. 217–224. 570 S. Saha Ray, R.K. Bera / Appl. Math. Comput. 167 (2005) 561–571.
13. K. Diethelm, N.J. Ford, The numerical solution of linear and nonlinear fractional differential equations involving fractional derivatives of several orders, Numerical analysis Report 379, Manchester Centre for Computational Mathematics, 2001.
14. S. Abbasbandy, An approximation solution of a nonlinear equation with Riemann-Liouville's fractional derivative by He's variational iteration method
15. J.H. He, Homotopy perturbation technique, Comput. Methods Appl. Mech. Engrg. 178 (1999) 257-262.
16. J.H. He, A coupling method of a homotopy technique and a perturbation technique for non-linear problems, Internat. J. Non-Linear Mech. 35 (2000) 37-43.
17. K.B. Oldham, J. Spanier, The Fractional Calculus, Academic Press, New York, 1974.

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