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STRONG CONVERGENCE OF A METHOD FOR VARIATIONAL INEQUALITY PROBLEMS AND FIXED POINT PROBLEMS OF A NONEXPANSIVE SEMIGROUP IN HILBERT SPACES

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Abstract. In this paper, we introduce a new iteration method based on the hybrid method in mathematical programming and the descent-like method for finding a common element of the solution set for a variational inequality and the set of common fixed points of a nonexpansive semigroup in Hilbert spaces. We obtain a strong convergence for the sequence generated by our method in Hilbert spaces. The result in this paper modifies and improves some well-known results in the literature for a more general problem.

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1. Introduction

Let H be a real Hilbert space with the scalar product and the norm denoted by the symbols $\langle ., . \rangle$ and $\|.\|$, respectively, and let C be a nonempty closed and convex subset of H. Denote by $P_C(x)$ the metric projection from $x \in H$ onto C. Let T be a nonexpansive mapping on C, i.e., $T : C \to C$ and $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We use $F(T)$ to denote the set of fixed points of T, i.e., $F(T) = \{x \in C : x = Tx\}$. We know that $F(T)$ is nonempty, if C is bounded, for more details see [3].

For finding a fixed point of a nonexpansive mapping T on C, Ishikawa [12] proposed the following method:

$$\begin{aligned} x_0 &\in C \quad \text{any element}, \\ y_k &= \alpha_k x_k + (1 - \alpha_k) T x_k, \\ x_{k+1} &= \beta_k x_k + (1 - \beta_k) T y_k, \end{aligned} \tag{1.1}$$

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where $\{\alpha_k\}$ and $\{\beta_k\}$ are two sequences of positive real numbers. When $\alpha_k = 1$ for all $k \geq 0$, we have the iterative process:

$$\begin{aligned} x_0 &\in C \quad \text{any element}, \\ x_{k+1} &= \beta_k x_k + (1-\beta_k)Tx_k, \end{aligned} \tag{1.2}$$

Introduced by Mann [15] in 1953. Both processes (1.1) and (1.2) have only weak convergence, in general (see [7] for an example). The formulation of process (1.2) is simpler than that of (1.1) and a convergence theorem for process (1.2) may possibly lead to a convergence theorem for (1.1) provided the sequence $\{\alpha_k\}$ satisfies certain appropriate conditions. However, the introduction of the process (1.1) has its own right. As a matter of fact, process (1.2) may fail to convergence while process (1.1) can still converge for a Lipschitz pseudocontractive mapping [5].

Recently, Alber [2] proposed the following descent-like iteration method:

$$x_{k+1} = P_C(x_k - \mu_k(x_k - Tx_k)), \quad \forall k \geq 0, x_0 \in C,$$

and proved that: if $\mu_k > 0, \sum_{k=0}^{\infty} \mu_k^2 < \infty$ and $\{x_k\}$ is bounded, then
(i) there exists a weak accumulation point $\tilde{x} \in C$ of $\{x_k\}$;
(ii) all weak accumulation points of $\{x_k\}$ belong to $F(T)$;
(iii) if $F(T)$ is a singleton, i.e., $F(T) = \{\tilde{x}\}$, then $\{x_k\}$ converges weakly to $\tilde{x}$.

Obviously, to find a fixed point of mapping T is equivalent to finding a zero for a mapping $I - T$ which is a monotone Lipschitz continuous mapping, where I denotes the identity mapping of H.

Recall that a mapping A in H is said to be:

(i) Lipschitz continuous with a Lipschitz constant $L > 0$ or L-Lipschitz continuous, if

$$\|Ax - Ay\| \leq L\|x - y\|;$$

(ii) monotone, if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in D(A),$$

the domain of A; and maximal monotone, if $G(A)$, the graph of A, is not properly contained in the graph of any other monotone mapping.

For finding a zero of the following inclusion

$$0 \in Ax \tag{1.3}$$

involving a maximal monotone mapping A, Rockafellar [24] considered the method:

$$c_n A x_{n+1} + x_{n+1} = x_n, \quad x_0 \in H, \tag{1.4}$$

where $c_n > c_0 > 0$, which is called the proximal point algorithm, and posed an open question whether (or not) the proximal algorithm (1.4) always converges strongly. This question was resolved in the negative by Güler [9]. To obtain strong convergence Solodov and Svaiter [27] proposed the following algorithm.

Algorithm: Choose any $x_0 \in H$ and $\sigma \in [0,1)$. At iteration k, having x_k, choose $\mu_k > 0$ and

$$\begin{aligned} v_k \in Ay_k, v_k + \mu_k(y_k - x_k) = \varepsilon_k, \\ \|\varepsilon_k\| \le \sigma \max\{\|v_k\|, \mu_k\|y_k - x_k\|\}, \\ H_k = \{z \in H : \langle z - y_k, v_k\rangle \le 0\}, \\ W_k = \{z \in H : \langle z - x_k, x_0 - x_k\rangle \le 0\}, \\ x_{k+1} = P_{H_k \cap W_k}(x_0). \end{aligned} \tag{1.5}$$

They proved that if the sequence of regularization parameters $\{\mu_k\}$ is bounded from above, then $\{x_k\}$ converges strongly to $z_0 = P_{\tilde{S}}(x_0)$, where $\tilde{S}$ denotes the set of solutions of (1.3). Moreover, basing on an important property that H_k and W_k in (1.5) are two halfspaces, they showed that

$$x_{k+1} = x_0 + \lambda_1 v_k + \lambda_2(x_0 - x_k), \tag{1.6}$$

where λ_1, λ_2 is the solution of the linear system of two equations with two unknowns

$$\begin{aligned} \lambda_1\|v_k\|^2 + \lambda_2\langle v_k, x_0 - x_k\rangle = -\langle x_0 - x_k, v_k\rangle \\ \lambda_1\langle v_k, x_0 - x_k\rangle + \lambda_2\|x_0 - x_k\|^2 = -\|x_0 - x_k\|^2. \end{aligned} \tag{1.7}$$

Further, for finding a fixed point for a nonexpansive mapping on C, motivated by the Solodov and Svaiter's algorithm, Nakajo and Takahashi [21] considered the sequence $\{x_k\}$ generated by

$$\begin{aligned} x_0 \in C, \\ y_k = \alpha_k x_k + (1 - \alpha_k)Tx_k, \\ C_k = \{z \in C : \|z - y_k\| \le \|z - x_k\|\}, \\ Q_k = \{z \in C : \langle z - x_k, x_0 - x_k\rangle \le 0\}, \\ x_{k+1} = P_{C_k \cap Q_k}(x_0), k \ge 0, \end{aligned} \tag{1.8}$$

where $\{\alpha_k\} \subset [0, a]$ for some $a \in [0,1)$. They showed that $\{x_k\}$ converges strongly to $P_{F(T)}(x_0)$. Process (1.8) is called [11] a CQ method for the Mann iteration process because at each step the Mann iterate (denoted by y_k in (1.8)) is used to construct the sets C_k and Q_k which are in turn used to construct the next iterate x_{k+1} and hence the name. Yanes and Xu [17] extended Nakajo and Takahash's iteration process (1.8) to the Ishikawa iteration process and (1.8) with $y_k = \alpha_k x_0 + (1 - \alpha_k)Tx_k$. Marino and Xu [16] further suggested the following modified Mann's algorithm based on the CQ method which has been studied by some others (see, [1, 10, 13, 28-30]):

$$\begin{aligned} x_0 \in C, \\ y_k = \alpha_k x_k + (1 - \alpha_k)Tx_k, \\ C_k = \{z \in C : \|z - y_k\|^2 \le \|z - x_k\|^2 + (1 - \alpha_k)(\gamma - \alpha_k)\|x_k - Tx_k\|^2\}, \\ Q_k = \{z \in C : \langle z - x_k, x_0 - x_k\rangle \le 0\}, \\ x_{k+1} = P_{C_k \cap Q_k}(x_0), \end{aligned} \tag{1.9}$$

for finding a fixed point of γ-strictly pseudocontractive mapping T, i.e., T satisfies the following condition

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \gamma\|(I - T)x - (I - T)y\|^2 \tag{1.10}$$

for all $x, y \in C$ and $\gamma \in [0, 1)$ is a fixed number. Recently, to find a fixed point of a γ-trictly pseudocontractive mapping T on C, Yao and Chen [32] proposed the following algorithm:

$$\begin{aligned} x_0 &\in C, \\ y_k &= \alpha_k x_k + (1 - \alpha_k)[\delta x_k + (1 - \delta)Tx_k], \\ C_k &= \{z \in C : \|z - y_k\| \leq \|z - x_k\|\}, \\ Q_k &= \{z \in C : \langle z - x_k, x_0 - x_k\rangle \leq 0\}, \\ x_{k+1} &= P_{C_k \cap Q_k}(x_0), \end{aligned} \tag{1.11}$$

where $\delta \in (\gamma, 1)$. They proved that if $T : C \to C$ is a γ-strict pseudo-contraction for some $0 \leq \gamma < 1$ with $F(T) \neq \emptyset$ and the sequence $\{\alpha_k\}$ is chosen so that $\alpha_k < 1$ for all $k \geq 0$, then $\{x_k\}$ defined by (1.11) converges strongly to $P_{F(T)}(x_0)$.

It is clear that (1.10) is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \frac{1 - \gamma}{2}\|(I - T)x - (I - T)y\|^2$$

for all $x, y \in C$. So, the mapping $A := I - T$ satisfies the following condition

$$\langle Ax - Ay, x - y\rangle \geq \lambda\|Ax - Ay\|^2, \quad \lambda = \frac{1 - \gamma}{2}, \tag{1.12}$$

and usually, a mapping A satisfying (1.12) is called λ-inverse strongly monotone. Moreover, it is well-known [11] that $p \in F(T)$ if and only if $p \in \Omega_A$, the solution set of the following variational inequality problem: find an element $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0 \quad \forall x \in C,$$

and if T is nonexpansive ($\gamma = 0$), then $I - T$ is $(1/2)$-inverse strongly monotone.

Further, Takahashi and Toyota [31] considered the problem of finding a solution of the variational inequality which is also a fixed point of some mapping. More precisely, given a nonempty closed convex subset C of H, a nonexpansive mapping T on C and an λ-inverse strongly monotone mapping $A : C \to H$, in order to find an element $p \in F(T) \cap \Omega_A$, they introduced the following iterative scheme:

$$\begin{aligned} x_0 &\in C, \\ x_{k+1} &= \alpha_k x_k + (1 - \alpha_k)TP_C(x_k - \lambda_k Ax_k), \end{aligned} \tag{1.13}$$

for all $k \geq 0$, where $\{\alpha_k\}$ is a sequence in $(0, 1)$ and $\{\lambda_k\}$ is a sequence in $(0, 2\lambda)$. They showed that if $F(T) \cap \Omega_A \neq \emptyset$, then the sequence $\{x_k\}$ defined by (1.13) converges weakly to some point $p \in F(T) \cap \Omega_A$. Later on, in order to achieve strong convergence to an element $p \in F(T) \cap \Omega_A$ under the same assumptions, by using a hybrid method, Iiduka and Takahashi [11] modified the iterative scheme as follows:

$$\begin{aligned} x_0 &\in C, \\ y_k &= \alpha_k x_k + (1-\alpha_k) T P_C(x_k - \lambda_k A x_k), \\ C_k &= \{z \in C : \|z - y_k\| \le \|z - x_k\|\}, \\ Q_k &= \{z \in C : \langle z - x_k, x_0 - x_k\rangle \le 0\}, \\ x_{k+1} &= P_{C_k \cap Q_k}(x_0), \end{aligned} \tag{1.14}$$

for all $k \ge 0$, where $0 \le \alpha_k \le c < 1$ and $0 < a \le \lambda_k \le b < 2\lambda$. They proved that if $F(T) \cap \Omega_A \ne \emptyset$, then the sequence $\{x_k\}$ defined by (1.14) converges strongly to $P_{F(T)\cap\Omega_A}(x_0)$. To overcome the restriction of the above methods to the class of λ-inverse strongly monotone mappings, by using the extragradient method of Korpelevich [14], Nadezhkina and Takahashi [19] were able to show the weak convergence result of the following method:

$$\begin{aligned} x_0 &\in C, \\ y_k &= P_C(x_k - \lambda_k A x_k), \\ x_{k+1} &= \alpha_k x_k + (1-\alpha_k) T P_C(x_k - \lambda_k A y_k), \end{aligned}$$

for mappings A that are only supposed to be monotone and Lipschitz continuous.

Recently, for finding an element of $F(T) \cap \Omega_A$, by combining a hybrid-type method with an extragradient method, Nadezhkina and Takahashi [20] introduced the following iterative method:

$$\begin{aligned} x_0 &\in C, \\ y_k &= P_C(x_k - \lambda_k A x_k), \\ z_k &= \alpha_k x_k + (1-\alpha_k) T P_C(x_k - \lambda_k A y_k), \\ C_k &= \{z \in C : \|z - y_k\| \le \|z - x_k\|\}, \\ Q_k &= \{z \in C : \langle z - x_k, x_0 - x_k\rangle \le 0\}, \\ x_{k+1} &= P_{C_k \cap Q_k}(x_0), \end{aligned} \tag{1.15}$$

for all $k \ge 0$. Under the following conditions: $\{\alpha_k\} \subset [a, b]$ for some $a, b \in (0; 1/L)$ and $\{\lambda_k\} \subset [0, c]$ for some $c \in [0, 1)$, the sequences $\{x_k\}, \{y_k\}$ and $\{z_k\}$ defined by (1.15) converge strongly to the same point $p = P_{F(T)\cap\Omega_A}(x)$.

Very recently, Ceng, Hadjisavvas and Wong [4] considered the following method:

$$\begin{aligned} x_0 &\in C, \\ y_k &= (1-\gamma_k) x_k + \gamma_k P_C(x_k - \lambda_k A x_k), \\ z_k &= (1-\alpha_k-\beta_k) x_k + \alpha_k y_k + \beta_k T P_C(x_k - \lambda_k A y_k), \\ C_k &= \{z \in C : \|z - z_k\| \le \|z - x_k\| + (3 - 3\gamma_k + \alpha_k) b^2 \|A x_k\|^2\}, \\ Q_k &= \{z \in C : \langle x_k - z, x_0 - x_k\rangle \ge 0\}, \\ x_{k+1} &= P_{C_k \cap Q_k}(x_0), \end{aligned} \tag{1.16}$$

for all $k \geq 0$, where $\{\lambda_k\} \subset [a, b]$ with $a > 0$ and $b < 1/(2L)$, and $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$ are three sequences in $[0, 1]$ satisfying the conditions:

(i) $\alpha_k + \beta_k < 1$ for all $k \geq 0$;

(ii) $\lim_{k\to\infty} \alpha_k = 0$;

(iii) $\liminf_{k\to\infty} \beta_k > 0$;

(iv) $\lim_{k\to\infty} \gamma_k = 1$ and $\gamma_k > 3/4$ for all $k \geq 0$;

Let $\{T(s) : s > 0\}$ be a nonexpansive semigroup on a closed convex subset C, that is,

(1) for each $s > 0, T(s)$ is a nonexpansive mapping on C;

(2) $T(0)x = x$ for all $x \in C$;

(3) $T(s_1 + s_2) = T(s_1) \circ T(s_2)$ for all $s_1, s_2 > 0$;

(4) for each $x \in C$, the mapping $T(.)x$ from $(0, \infty)$ into C is continuous.

Denote by $\mathcal{F} = \cap_{s>0} F(T(s))$. We know [21] that $\mathcal{F}$ is a closed convex subset in H and $\mathcal{F} \neq \emptyset$ if C is compact (see [6]).

For finding an element $p \in \mathcal{F}$, Shioji and Takahashi [26] introduced the implicit iteration method:

$$x_k = \alpha_k u + (1 - \alpha_k)\frac{1}{s_k}\int_0^{s_k} T(s)x_k ds, \quad k \geq 0,$$

where $\{\alpha_k\}$ is a sequence in (0,1) and $\{s_k\}$ is a positive real number divergent sequence. Further, Nakajo and Takahashi [21] introduced also an iteration procedure for the nonexpansive semigroup as follows:

$$\begin{aligned}
&x_0 \in C,\\
&y_k = \alpha_k x_k + (1 - \alpha_k)\frac{1}{s_k}\int_0^{s_k} T(s)x_k ds,\\
&C_k = \{z \in C : \|y_k - z\| \leq \|x_k - z\|\},\\
&Q_k = \{z \in C : \langle x_k - z, x_0 - x_k\rangle \geq 0\},\\
&x_{k+1} = P_{C_k\cap Q_k}(x_0)
\end{aligned} \tag{1.17}$$

for each $k \geq 0$. They showed that if $\alpha_k \in$ [0,a] for some $a \in$ [0,1) and $\{s_k\}$ is a positive real number divergent sequence, then the sequence $\{x_n\}$ defined by (1.17) converges strongly to $P_{\mathcal{F}}(x_0)$.

In 2007, He and Chen [10] considered an iteration procedure for any nonexpansive semigroup $\{T(s) : s > 0\}$ on C as follows:

$$\begin{aligned}
&x_0 \in C,\\
&y_k = \alpha_k x_k + (1 - \alpha_k)T(s_k)x_k,\\
&C_k = \{z \in C : \|y_k - z\| \leq \|x_k - z\|\},\\
&Q_k = \{z \in C : \langle x_k - z, x_0 - x_k\rangle \geq 0\},\\
&x_{k+1} = P_{C_k\cap Q_k}(x_0)
\end{aligned} \tag{1.18}$$

for $k \geq 0$, where $\alpha_k \in [0, a)$ for some $a \in [0, 1)$ and $s_k \geq 0, \lim_{k\to\infty} s_k = 0$, then the sequence $\{x_k\}$ in (1.18) converges to $P_{\mathcal{F}}(x_0)$. In 2008 Saejung [25] proved the result under new condition on s_k:

$$\liminf_k s_k = 0, \limsup_k s_k > 0 \quad \text{and} \quad \lim_k (s_{k+1} - s_k) = 0. \tag{1.19}$$

It is easy to see that if C is a proper subset of H, then C_k and Q_k are not two halfspaces. Then, a natural question is posed: how to construct the closed convex subsets C_k and Q_k for a fixed closed convex subset C and if we can express x_{k+1} in (1.8), (1.9), (1.11) and (1.14)-(1.18) in a similar form as (1.6) and (1.7). Obviously, the answer is positive, if C_k and Q_k in these methods are also two halfspaces. In this paper, this idea is used to solve a more general problem, the problem of finding an element

$$p \in \mathcal{F} \cap \Omega_{\mathcal{A}}, \tag{1.20}$$

assumed to be nonempty, for any monotone Lipschitz continuous mapping A and a nonexpansive semigroup $\{T(s) : s > 0\}$ on C.

Motivated by Alber's algorithm, (1.15) and (1.16), to solve (1.20) we consider the following algorithm:

$$\begin{aligned} x_0 &\in H, \\ y_k &= P_C(x_k - \lambda_k A P_C(x_k)), \\ z_k &= x_k - \mu_k[x_k - T_k P_C(x_k - \lambda_k A y_k)], \\ H_k &= \{z \in H : \|z_k - z\| \leq \|x_k - z\|\}, \\ W_k &= \{z \in H : \langle x_k - x_0, z - x_k\rangle \geq 0\}, \\ x_{k+1} &= P_{H_k \cap W_k}(x_0), k \geq 0, \end{aligned} \tag{1.21}$$

where $\{s_k\}$ is a sequence of positive real numbers satisfying condition (1.19) and $T_k x = T(s_k)x$ for $x \in C$. The strong convergence of algorithm (1.21) is proved in the next section. Some applications are showed in Section 3.

2. Main results

We formulate some facts needed in the proof of our results.

Definition 2.1. A Banach space E is said to satisfy Opial's condition [22] if whenever $\{x_k\}$ is a sequence in E which converges weakly to x, as $k \to \infty$, then

$$\limsup_{k\to\infty} \|x_k - x\| < \limsup_{k\to\infty} \|x_k - y\|, \quad \forall y \in E \quad \text{with } x \neq y.$$

It is well known that Hilbert space and $l^p (1 < p < \infty)$ satisfy Opial's condition [18].

Lemma 2.2 [16]. *Let H be a real Hilbert space H. There hold the following identities:*
(i) $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle$;
(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \forall t \in [0, 1], \forall x, y \in H$;

(iii) $\|x-y\|^2 \geq \|x-P_C(x)\|^2 + \|y-P_C(x)\|^2$ *for any* $x \in H$ *and for all* $y \in C$ *where* C *is a nonempty closed convex subset in* H.

Lemma 2.3 [16]. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *For any* $x \in H$, *there exists a unique* $z \in C$ *such that* $\|z-x\| \leq \|y-x\|$ *for all* $y \in C$, *and* $z = P_C(x)$ *if and only if* $\langle z-x, y-z\rangle \geq 0$ *for all* $y \in C$, *where* P_C *is the metric projection of* H *onto* C.

Lemma 2.4 [8]. *Every Hilbert space* H *has Randon-Riesz property or Kadec-Klee property, that is, for a sequence* $\{x_n\} \subset H$ *with* $x_n \rightharpoonup x$ *and* $\|x_n\| \to \|x\|$, *then there hodls* $x_n \to x$.

Theorem 2.5. *Let* C *be a nonempty closed convex subset in a real Hilbert space* H. *Let* $A : C \to H$ *be a monotone L-Lipschitz continuous mapping and let* $\{T(s) : s > 0\}$ *be a nonexpansive semigroup on* C *such that* $\mathcal{F} \cap \Omega_{\mathcal{A}} \neq \emptyset$. *Let* $\{\lambda_k\} \subset [a, b]$ *for some* $a, b \in (0; 1/L)$, $\{s_k\}$ *be a sequence of positive real numbers satisfying (1.19) and* $\{\mu_k\} \subset [c, 1]$ *for some* $c \in (0, 1)$. *Then, the sequences* $\{x_k\}, \{y_k\}$ *and* $\{z_k\}$ *defined by (1.21) converge strongly to the same point* $z_0 = P_{\mathcal{F}\cap\Omega_{\mathcal{A}}}(x_0)$.

Proof. First, note that $\|z_k - z\| \leq \|x_k - z\|$ is equivalent to

$$\langle z_k - x_k, x_k - z\rangle \leq -\frac{1}{2}\|z_k - x_k\|^2.$$

Thus, H_k is a halfspace. For each $u \in \mathcal{F} \cap \Omega_{\mathcal{A}}$, by putting $t_k = P_C(x_k - \lambda_k A y_k)$ and using (iii) and (i) in Lemma 2.2 with $x = x_k - \lambda_k A y_k$ and $y = u \in \Omega_A \subseteq C$ we have that

$$\begin{aligned}\|t_k - u\|^2 &\leq \|x_k - \lambda_k A y_k - u\|^2 - \|x_k - \lambda_k A y_k - t_k\|^2\\ &= \|x_k - u\|^2 - \|x_k - t_k\|^2 + 2\lambda_k\langle A y_k, u - t_k\rangle\\ &\leq \|x_k - u\|^2 - \|x_k - t_k\|^2 + 2\lambda_k\langle A y_k, y_k - t_k\rangle\\ &= \|x_k - u\|^2 - \|x_k - y_k\|^2 - \|y_k - t_k\|^2\\ &\quad + 2\langle x_k - \lambda_k A y_k - y_k, t_k - y_k\rangle.\end{aligned} \tag{2.1}$$

Next, by using Lemma 2.3 with $x = x_k - \lambda_k A P_C(x_k), z = y_k$ and $y = t_k$, we get

$$\begin{aligned}2\langle x_k - \lambda_k A y_k - y_k, t_k - y_k\rangle &= 2\langle x_k - \lambda_k A P_C(x_k) - y_k, t_k - y_k\rangle\\ &\quad + 2\lambda_k\langle A P_C(x_k) - A y_k, t_k - y_k\rangle\\ &\leq 2\lambda_k L\|P_C(x_k) - P_C(y_k)\|\|y_k - t_k\|\\ &\leq 2\lambda_k L\|x_k - y_k\|\|y_k - t_k\|,\end{aligned} \tag{2.2}$$

since $y_k \in C$. Therefore, from (2.1), (2.2) and the condition on λ_k we obtain the estimation

$$\begin{aligned}\|t_k - u\|^2 &\leq \|x_k - u\|^2 + (\lambda_k^2 L^2 - 1)\|x_k - y_k\|^2\\ &\leq \|x_k - u\|^2,\end{aligned} \tag{2.3}$$

and hence

$$\begin{aligned}\|z_k - u\|^2 &= \|(1-\mu_k)(x_k - u) + \mu_k(T_k t_k - u)\|^2 \\ &\le (1-\mu_k)\|x_k - u\|^2 + \mu_k\|T_k t_k - u\|^2 \\ &\le (1-\mu_k)\|x_k - u\|^2 + \mu_k\|t_k - u\|^2 \\ &\le (1-\mu_k)\|x_k - u\|^2 + \mu_k\|x_k - u\|^2 \\ &\le \|x_k - u\|^2.\end{aligned} \tag{2.4}$$

It means that $\mathcal{F} \cap \Omega_{\mathcal{A}} \subset \mathcal{H}_{\|}$ for all $k \ge 0$. On the other hand, when $k = 0$ we have $W_0 = H$. Consequently, $\mathcal{F} \cap \Omega_{\mathcal{A}} \subset \mathcal{W}_{\prime}$. We shall prove by mathematical induction that $\mathcal{F} \cap \Omega_{\mathcal{A}} \subset \mathcal{W}_{\|}$ for all $k \ge 0$. Assume that $\mathcal{F} \cap \Omega_{\mathcal{A}} \subset \mathcal{W}_{\rangle}$, we have to prove that $\mathcal{F} \cap \Omega_{\mathcal{A}} \subset \mathcal{W}_{\rangle+\infty}$. Indeed, since $\mathcal{F} \cap \Omega_{\mathcal{A}} \subset \mathcal{W}_{\rangle}$ there exists a unique element $x_{i+1} \in H_i \cap W_i$ such that $x_{i+1} = P_{H_i \cap W_i}(x_0)$ and for all $z \in H_i \cap W_i$ we have that $\langle x_{i+1} - z, x_0 - x_{i+1}\rangle \ge 0$. Hence, $z \in W_{i+1}$. Finally, we have that $\mathcal{F} \cap \Omega_{\mathcal{A}} \subset \mathcal{H}_{\|} \cap \mathcal{W}_{\|}$ for all $k \ge 0$.

Note that we also have from $x_{k+1} = P_{H_k \cap W_k}(x_0)$ that

$$\|x_{k+1} - x_0\| \le \|u - x_0\| \tag{2.5}$$

for all $u \in \mathcal{F} \cap \Omega_{\mathcal{A}}$. Thus, $\{x_k\}$ is bounded and hence $\{AP_C(x_k)\}$ is also bounded. From $x_k = P_{W_k}(x_0)$ and $x_{k+1} \in H_k \cap W_k$ it follows

$$\|x_k - x_0\| \le \|x_{k+1} - x_0\|.$$

This fact and the bounded property of $\{x_k\}$ imply that there exists $\lim_{k\to\infty} \|x_k - x_0\| = c$. Since $x_k = P_{W_k}(x_0)$ and $x_{k+1} \in W_k$, from (ii) in Lemma 2.2 we have

$$\begin{aligned}\|x_k - x_0\|^2 &\le \|(x_k + x_{k+1})/2 - x_0\|^2 \\ &= \|(x_k - x_0)/2 + (x_{k+1} - x_0)/2\|^2 \\ &= \|x_k - x_0\|^2/2 + \|x_{k+1} - x_0\|^2/2 - \|x_k - x_{k+1}\|^2/4.\end{aligned}$$

So, we get

$$\|x_k - x_{k+1}\|^2 \le 2(\|x_{k+1} - x_0\|^2 - \|x_k - x_0\|^2).$$

Since $\lim_{k\to\infty} \|x_k - x_0\| = c$, we obtain

$$\lim_{k\to\infty} \|x_k - x_{k+1}\| = 0. \tag{2.6}$$

From $x_{k+1} \in H_k$ it implies that

$$\|z_k - x_k\| \le \|x_k - x_{k+1}\| + \|x_{k+1} - z_k\| \le 2\|x_k - x_{k+1}\|.$$

By (2.6), we obtain

$$\lim_{k\to\infty} \|z_k - x_k\| = 0. \tag{2.7}$$

Further, from (2.4) it follows

$$\|z_k - x_k\|^2 + 2\langle z_k - x_k, x_k - u\rangle \le \mu_k(\|T_k t_k - u\|^2 - \|x_k - u\|^2) \le 0.$$

So, we have from (2.7) and the boundedness of $\{x_k\}$ that

$$\lim_{k\to\infty} \mu_k(\|T_k t_k - u\|^2 - \|x_k - u\|^2) = 0.$$

Since $\mu_k \in [c, 1]$, we get

$$\lim_{k\to\infty}(\|T_k t_k - u\|^2 - \|x_k - u\|^2) = 0.$$

Next, from (2.3), the last equality and

$$\lim_{k\to\infty}(\|T_k t_k - u\|^2 - \|x_k - u\|^2) \leq \lim_{k\to\infty}(\|t_k - u\|^2 - \|x_k - u\|^2) \leq 0$$

it implies that

$$\lim_{k\to\infty}(\|x_k - u\|^2 - \|t_k - u\|^2) = 0.$$

Therefore, we have from (2.3), the last equality and $1 - \lambda_k^2 L^2 > 1 - b^2 L^2 > 0$ that

$$\lim_{k\to\infty} \|x_k - y_k\| = 0. \tag{2.8}$$

By the similar argument, we also obtain from (2.1) and (2.2) that

$$\begin{aligned}\|t_k - u\|^2 &\leq \|x_k - u\|^2 + (\lambda_k^2 L^2 - 1)\|y_k - t_k\|^2\\ &\leq \|x_k - u\|^2,\end{aligned}$$

and hence

$$\lim_{k\to\infty} \|y_k - t_k\| = 0. \tag{2.9}$$

Further, we have from (2.8), (2.9) and the Lipschitz continuity of A that

$$\lim_{k\to\infty} \|Ay_k - At_k\| = \lim_{k\to\infty} \|x_k - t_k\| = 0.$$

Since $\{x_k\}$ is bounded, there exist an element $z \in H$ and a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that $\{x_{k_i}\}$ converges weakly to z as $i \to \infty$. Thus, $\{y_{k_i}\}$ and $\{t_{k_i}\}$ also converges weakly to z as $i \to \infty$. Since $\{t_{k_i}\} \subset C$ and C is a closed convex subset, we have $z \in C$. Now, we shall prove that $z \in \mathcal{F} \cap \Omega_{\mathcal{A}}$. First, we show that $z \in \Omega_A$. Set $Bv = Av + N_C v$ for $v \in C$ where

$$N_C v = \{w \in H : \langle v - y, w\rangle \geq 0 \quad \forall y \in C\}$$

and $Bv = \emptyset$ for $v \notin C$. Then, B is a maximal monotone mapping and $0 \in Bv$ if and only if $v \in \Omega_A$ (see [23]). Let $(v, w) \in G(B)$. Then we have $w \in Bv = Av + N_C v$ and $w - Av \in N_C v$ which is equivalent to

$$\langle v - y, w - Av\rangle \geq 0 \quad \forall y \in C.$$

Consequently, from $t_k = P_C(x_k - \lambda_k A y_k) \in C$ and Lemma 2.3, we have that

$$\langle t_k - v, x_k - \lambda_k A y_k - t_k\rangle \geq 0.$$

Therefore,

$$\langle v - t_k, (t_k - x_k)/\lambda_k + Ay_k\rangle \geq 0.$$

Hence,

$$\begin{aligned}\langle v - t_{k_i}, w\rangle &\geq \langle v - t_{k_i}, Av\rangle\\ &\geq \langle v - t_{k_i}, Av\rangle - \langle v - t_{k_i}, (t_{k_i} - x_{k_i})/\lambda_{k_i} + Ay_{k_i}\rangle\\ &\geq \langle v - t_{k_i}, Av - At_{k_i}\rangle + \langle v - t_{k_i}, At_{k_i} - Ay_{k_i}\rangle\\ &\quad + \langle v - t_{k_i}, (t_{k_i} - x_{k_i})/\lambda_{k_i}\rangle\\ &\geq \langle v - t_{k_i}, At_{k_i} - Ay_{k_i}\rangle + \langle v - t_{k_i}, (t_{k_i} - x_{k_i})/\lambda_{k_i}\rangle\end{aligned}$$

After passing $i \to \infty$ in the last inequality, we obtain that $\langle v - z, w\rangle \geq 0$ for all $v \in C$. So, $z \in B^{-1}0$. It means that $z \in \Omega_A$.

Next, we prove that $z \in \mathcal{F}$. From (1.21), (2.7) and the condition on $\{\lambda_k\}$ it follows

$$c \lim_{k\to\infty} \|T_k t_k - x_k\| \leq \lim_{k\to\infty} \mu_k \|T_k t_k - x_k\| = \lim_{k\to\infty} \|z_k - x_k\| = 0. \tag{2.10}$$

From (2.8)-(2.10), we get

$$\lim_{k\to\infty} \|T_k t_k - t_k\| = 0.$$

Without loss of generality, as in [25], let

$$\lim_{j\to\infty} s_{k_j} = \lim_{j\to\infty} \frac{\|T_{k_j} t_{k_j} - t_{k_j}\|}{s_{k_j}} = 0. \tag{2.11}$$

Now, we prove that $z = T(s)z$ for a fixed $s > 0$. It is easy to see that

$$\begin{aligned}\|t_{k_j} - T(s)z\| \leq &\sum_{l=0}^{[s - s_{k_j}]-1} \|T(l s_{k_j}) t_{k_j} - T((l+1)s_{k_j}) t_{k_j}\\ &+ \left\| T\left(\left[\frac{s}{s_{k_j}}\right]\right) t_{k_j} - T\left(\left[\frac{s}{s_{k_j}}\right]\right) z \right\| + \left\| T\left(\left[\frac{s}{s_{k_j}}\right]\right) z - T(s)z \right\|\\ \leq &\left[\frac{s}{s_{k_j}}\right] \|t_{k_j} - T(s_{k_j}) t_{k_j}\| + \|t_{k_j} - z\| + \left\| T\left(t - \left[\frac{s}{s_{k_j}}\right] s_{k_j}\right) z - z \right\|.\end{aligned}$$

Therefore,

$$\begin{aligned}\|t_{k_j} - T(s)z\| \leq &\frac{s}{s_{k_j}} \|t_{k_j} - T(s_{k_j}) t_{k_j}\|\\ &+ \|t_{k_j} - z\| + \sup\{\|T(s)z - z\| : 0 \leq s \leq s_{k_j}\}.\end{aligned}$$

This fact and (2.11) imply that

$$\limsup_{j\to\infty} \|t_{k_j} - T(s)z\| \leq \limsup_{j\to\infty} \|t_{k_j} - z\|.$$

As every Hilbert space satisfies Opial's condition, we have $T(s)z = z$. Therefore, $z \in \mathcal{F}$. Thus, (2.5) and the weakly lower semicontinuity of the norm guarantee that

$$\|x_0 - z_0\| \leq \|x_0 - z\| \leq \liminf_{j\to\infty} \|x_0 - x_{k_j}\| \leq \limsup_{j\to\infty} \|x_0 - x_{k_j}\| \leq \|x_0 - z_0\|.$$

Hence, we obtain

$$\lim_{j\to\infty} \|x_{k_j} - x_0\| = \|x_0 - z\| = \|x_0 - z_0\|.$$

By Lemma 2.4, we have that

$$\lim_{j\to\infty} x_{k_j} = z_0.$$

Since z_0 is a unique element, we have that all the sequence $\{x_k\}$ converges strongly to z_0 as $k \to \infty$. Therefore, the sequences $\{y_k\}$ and $\{z_k\}$ also converge strongly to z_0. Theorem is proved.

3. Applications

If one puts $T(s) = T$ for all $s > 0$ in Theorem 2.5, then one obtains an algorithm to find a common element of the set $F(T) \cap \Omega_A$ by

Theorem 3.1. *Let C be a nonempty closed convex subset in a real Hilbert space H. Let $A : C \to H$ be a monotone L-Lipschitz continuous mapping and T be a nonexpansive mapping on C such that $F(T) \cap \Omega_A \neq \emptyset$. Let $\{x_k\}, \{y_k\}$ and $\{z_k\}$ be sequences generated by:*

$$\begin{aligned} x_0 &\in H, \\ y_k &= P_C(x_k - \lambda_k A P_C(x_k)), \\ z_k &= x_k - \mu_k[x_k - T P_C(x_k - \lambda_k A y_k)], \\ H_k &= \{z \in H : \|z_k - z\| \le \|x_k - z\|\}, \\ W_k &= \{z \in H : \langle x_k - x_0, z - x_k\rangle \ge 0\}, \\ x_{k+1} &= P_{H_k \cap W_k}(x_0). \end{aligned} \tag{3.1}$$

for all $k \ge 0$, where $\{\lambda_k\} \subset [a, b]$ for some $a, b \in (0; 1/L)$, $\{\mu_k\} \subset [c, 1]$ for some $c \in (0, 1)$. Then, the sequences $\{x_k\}, \{y_k\}$ and $\{z_k\}$ defined by (3.1) converge strongly to the same point $z_0 = P_{F(T)\cap\Omega_A}(x_0)$.

Taking $T(s) = I$ for all $s > 0$ in Theorem 3.1, one finds the following theorem providing an algorithm to find the solution of a variational inequality.

Theorem 3.2. *Let C be a nonempty closed convex subset in a real Hilbert space H. Let $A : C \to H$ be a monotone L-Lipschitz continuous mapping such that $\Omega_A \neq \emptyset$. Let $\{x_k\}, \{y_k\}$ and $\{z_k\}$ be sequences generated by:*

$$\begin{aligned} x_0 &\in H, \\ y_k &= P_C(x_k - \lambda_k A P_C(x_k)), \\ z_k &= x_k - \mu_k(x_k - P_C(x_k - \lambda_k A y_k)), \\ H_k &= \{z \in H : \|z_k - z\| \le \|x_k - z\|\}, \\ W_k &= \{z \in H : \langle x_k - x_0, z - x_k\rangle \ge 0\}, \\ x_{k+1} &= P_{H_k \cap W_k}(x_0). \end{aligned} \tag{3.2}$$

for all $k \geq 0$, *where* $\{\lambda_k\} \subset [a, b]$ *for some* $a, b \in (0; 1/L)$, $\{\mu_k\} \subset [c, 1]$ *for some* $c \in (0, 1)$. *Then, the sequences* $\{x_k\}, \{y_k\}$ *and* $\{z_k\}$ *defined by (3.2) converge strongly to the same point* $z_0 = P_{\Omega_A}(x_0)$.

If one puts $A = 0$ and $T(s) = T$ for all $s > 0$ in Theorem 3.1, then one obtains an algorithm to find the fixed point of a nonexpansive mapping.

Theorem 3.3. *Let* C *be a nonempty closed convex subset in a real Hilbert space* H. *Let* T *be a nonexpansive mapping on* C *such that* $F(T) \neq \emptyset$. *Let* $\{x_k\}$ *and* $\{z_k\}$ *be sequences generalized by:*

$$\begin{aligned} x_0 &\in H, \\ z_k &= x_k - \mu_k(x_k - TP_C(x_k)), \\ H_k &= \{z \in H : \|z_k - z\| \leq \|x_k - z\|\}, \\ W_k &= \{z \in H : \langle x_k - z, x_0 - x_k \rangle \geq 0\}, \\ x_{k+1} &= P_{H_k \cap W_k}(x_0). \end{aligned} \tag{3.3}$$

for all $k \geq 0$, *where* $\{\mu_k\} \subset [c, 1]$ *for some* $c \in (0, 1)$. *Then, the sequences* $\{x_k\}$ *and* $\{z_k\}$ *defined by (3.3) converge strongly to the same point* $z_0 = P_{F(T)}(x_0)$.

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References

1. G.L. Acedo and H.K. Xu, *Iterative methods for strict pseudocontractions in Hilbert spaces,* Nonlinear Anal. **67** (2007), 2258-2271.
2. Ya.I. Alber, *On the stability of iterative approximations to fixed points of nonexpansive mappings,* J. Math. Anal. Appl. **328** (2007), 958-971.
3. E.F. Browder, *Fixed-point theorems for noncompact mappings in Hilbert spaces,* Proceed. Nat. Acad. Sci. USA **53** (1965), 1272-1276.
4. L.C Ceng, N. Hadjisavvas, and Ng. Ch. Wong, *Strong convergence theorem by hybrid extragradient-like approximation method for variational inequalities and fixed poit problems,* J. of Glob. Optim. **46**(4) (2010), 635-646.
5. C.E. Chidume, S.A. Mutangadura, *An example on the Mann iteration method for Lipschitz pseudocontraction,* Proc. Amer. Math. Soc. **129** (2001), 2359-2363. item [6.] R. DeMarr, *Common fixed points for commuting contraction mappings,* Pacific J. Math. **13** (1963), 1139-1141.
7. A. Genel, J. Lindenstrass, *An example concerning fixed points,* Israel J. Math. **22** (1975), 81-86.
8. K. Goebel and W.A. Kirk, *Topics in Metric Fixed Point Theory,* Cambridge Studies in Advanced Math., V. 28, Cambridge Univ. Press, Cambridge 1990.
9. O. Güler, *On the convergence of the proximal point algorithm for convex minimization,* SIAM J. Contr. Optim. **29** (1991), 403-419.
10. H. He and R. Chen, *Strong convergence theorems of the CQ method for nonexpansive semigroups,* FPTA 2007, DOI: 10.1155/2007/59735.
11. H. Iiduka, and W. Takahashi, *Strong convergence theorems for nonexpansive nonself mappings and inverse-strongly monotone mappings,* J. of Conv. Anal. **11**(1) (2004), 69-79.
12. S. Ishikawa, *Fixed point by new iteration method,* Proc. Amer. Math. Soc. **44** (1974), 147-150.

13. T.H. Kim, *Strong convergence of approximating fixed point sequences for relatively nonlinear mappings,* Thai J. Math. **6** (2008), 17-35.
14. G.M. Korpelevich, *The extragradient method for finding sadle points and other problems,* Matecon. **12** (1976), 747-756.
15. W.R. Mann, *Mean value methods in iteration,* Proc. Amer. Math. Soc. **4** (1953), 506-510.
16. G. Marino and H.K. Xu, *Weak and strong convergence theorems for stric pseudo-contractions in Hilbert spaces,* J. Math. Anal. Applic. **329** (2007), 336-346.
17. C. Martinez-Yanes and H.K. Xu, *Strong convergence of the CQ method for fixed iteration processes,* Nonlinear Anal. **64** (2006), 2400-2411.
18. R.E. Megginson, *An introduction to Banach space theory,* vol. 183 of Graduate texts in Mathematics, Springer, New York, NY, USA, 1998.
19. N. Nadezhkina, and W. Takahashi, *Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings,* J. Optim. Theory and Appl. **128**, 191-201 (2006), 191-201.
20. N. Nadezhkina, and W. Takahashi, *Strong convergence theorem by a hybrid method for nonexpansive mappings and Lipschitz continuous monotone mappings,* SIAM J. Optim. **16**(4) (2006), 1230-1241.
21. K. Nakajo and W. Takahashi, *Strong convergence theorem for nonexpansive mappings and nonexpansive semigroup,* J. Math. Anal. Applic. **279** (2003), 372-379.
22. Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings,* Bull. Amer. Math. Soc., **73** (1967), 591-597.
23. R.T. Rockafellar, *On the maximality of of sums of nonlinear monotone operators,* Trans. Amer. Math. Soc. **149** (1970), 75-88.
24. R.T. Rockafellar, *Monotone operators and proximal point algorithm,* SIAM J. Contr. Optim. **14** (1976), 877-897.
25. S. Saejung, *Strong convergence theorems for nonexpansive semigroups without Bochner integrals,* FPTA, 2008, DOI: 10.1155/2008/745010.
26. N. Shioji and W. Takahashi, *Strong convergence theorems for continuous semigroup in Banach spaces,* Math. Japon **50** (1999), 57-66.
27. M.V. Solodov, B.F. Svaiter, *Forcing strong convergence of proximal point iterations in Hilbert space,* Math. Progr. **87** (2000), 189-202.
28. Y. Su, M. Shang, and D. Wang, *Strong convergence on monotone CQ algorithm for relatively nonexpansive mappings,* Banach J. Math. Anal. **2** (2008), 1-10.
29. Y. Su and X. Qin, *Monotone CQ iteration processes for nonexpansive semigroups and maximal monotone operators,* Nonlinear Anal. **68** (2008), 3657-3664.
30. J. Sun, Y. Yu, and R. Chen, *Convergence theorems of CQ iteration processes for a finite family of averaged mappings in Hilbert spaces,* Int. J. Math. Anal. **2** (2008), 1045-1049.
31. W. Takahashi, and M. Toyoda, *Weak convergence theorem for nonexpansive mappings and monotone mappings,* J. Optim. Theory and Appl. **118**(2) (2003), 417-428.
32. Y. Yao and R. Chen, *Strong convergence theorems for strict pseudo-contractions in Hilbert spaces,* J. Appl. Math. Comput. DOI: 10.1007/s12190-009-0233-x.

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